TOPOLOGICAL DUALS AND KÖTHE TOEPLITZ DUALS OF SOME DOUBLE SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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Abstract. In the present paper, topological duals and Köthe-Toeplitz Duals of some double sequence spaces defined by a modulus function are studied. Also, we investigate perfectness, normality and monotonicity of some vector valued modulus sequences.

1. Introduction

Modulus Function was introduced by Nakano [1], in 1953, and used to solve some structural problems of the scalar FK-spaces theory. Ruckle [3] used the modulus function to construct the space $L(f)$ of complex sequences and solve some problems encountered in theoretical investigations of classical FK-spaces by $L(f)$. Later Maddox [5] introduced the class of sequences which are strongly Cesaro summable with respect to a modulus function. Since then a lot of sequence spaces have been defined by a modulus function for some special aims. Yılmaz [2] defined the sequence space $\lambda(X_k, r, f, s)$ by a modulus function and constructed its FK-structure under some conditions. Further, Yılmaz gave generalized Köthe-Toeplitz duals of some important vector-valued modulus sequence spaces, and by these results, Yılmaz [4] outlined the basics of operator matrix transformations of modulus sequence spaces in the sense of Maddox [5].

Gökhan and Çolak [6], [7] have proved that $M_u(t)$ and $C_p(t)$, $C_{bp}(t)$ are complete paranormed spaces of double sequences and given the $\alpha-$, $\beta-$, $\gamma-$duals of the spaces $M_u(t)$ and $C_{bp}(t)$. Zeltser [8] had essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [9] introduced the statistical convergent and strongly Cesaro summable double sequences. The authors in [10] and[11] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem. They also have introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{j,k})$ into one whose core is a subset of the $M-$core of $x$.

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Altay and Basar [12] have defined the spaces $BS, BS(t), CS_p, CS_{bp}, CS_r$ and $BV$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), C_p, C_{bp}, C_r$ and $\mathcal{L}_u$, respectively. They also examined some properties of those sequence spaces and determined the $\alpha$–duals of the spaces $BS, CS_{bp}, BV$ and the $\beta(\nu)$–duals of the spaces $CS_{bp}, CS_r$ of double series. In this study $\nu$ is a kind of convergence for double sequence.

In this study for simplicity in notation, here and after we abbreviate the summations $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty}$ and $\sum_{i,j=0}^{m,n}$ by $\sum_{i,j}$, respectively.

We denote the set of all real or complex valued double sequences by $\Omega$ which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of $\Omega$ is called as a double sequence space. A double sequence is bounded if

$$\|x\| = \sup_{m,n \geq 0} |x_{mn}| < \infty.$$  

The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u = \left\{ x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty \right\},$$

which is a Banach space with the norm given by

$$\|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty,$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$. The space $\mathcal{L}_u$ is defined as

$$\mathcal{L}_u = \left\{ x = (x_{mn}) \in \Omega : \sum_{m,n} |x_{mn}| < \infty \right\}$$

and it is a Banach space with the norm

$$\|x\|_1 = \sum_{m,n} |x_{mn}|.$$

Let us consider the sequence $x = (x_{mn}) \in \Omega$. If for all $\varepsilon > 0$, there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{C}$ such that

$$|x_{mn} - l| < \varepsilon,$$

for all $m, n > n_0$, then it is said that the double sequence $x$ is convergent in the Pringsheim’s sense to the limit $l$. Then we write

$$p - \lim x_{mn} = l,$$

where $\mathbb{C}$ denotes the complex field. By $C_p$, we denote the space of all convergent double sequences in the Pringsheim’s sense. It is well known that there are such sequences in the space $C_p$ but not in the space $\mathcal{M}_u$. So we may mention the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim’s sense and bounded, i.e. $\mathcal{C}_{bp} = C_p \cap \mathcal{M}_u$. Moricz [13] proved that $\mathcal{C}_{bp}$ is a Banach space with the norm $\|\|_{{\infty}, \infty}$. A sequence in the space $C_p$ is said to be regularly convergent, if it is a single convergent sequence with respect to each index. It is denoted by $C_r$. 


Now, denote the space of all \( X \)-valued double sequences by \( \Omega(X) \), where \( X \) is a Banach space. The topology of \( (X) \) is locally convex topology produced by the family of seminorms, \( p_{ij}(x) = \|x_{ij}\| \). This topology is metrizable and total paranorm giving this metric is

\[
g(x) = \sum_{i,j} \frac{1}{2^i+j} \frac{\|x_{ij}\|}{1 + \|x_{ij}\|}.
\]

A modulus \( f \) is a function from \([0, \infty)\) to \([0, \infty)\) such that

1. \( f(x) = 0 \) if and only if \( x = 0 \),
2. \( f(x+y) \leq f(x) + f(y) \), for all \( x \geq 0, y \geq 0 \),
3. \( f \) is increasing,
4. \( f \) is continuous from the right at 0. Since \( |f(x) - f(y)| \leq f(|x - y|) \), it follows from condition (4) that \( f \) is continuous on \([0, \infty)\).

Let \( f \) be a modulus function and \( \Omega \) is the space of all scalar double sequences.

We present some double sequence spaces in the following:

\[
\mathcal{L}_u(f) = \{ x \in \Omega : \sum_{m,n} f(|x_{mn}|) < \infty \},
\]

\[
\mathcal{L}_u(X) = \{ x = (x_{mn}) \in \Omega(X) : \sum_{m,n} \|x_{mn}\| < \infty \},
\]

\[
\mathcal{M}_u(X) = \{ x = (x_{mn}) \in \Omega(X) : \sup_{m,n} |x_{mn}| < \infty \},
\]

\[
\Phi(X) = \{ x = (x_{mn}) \in \Omega(X) : \exists k_0 \exists \text{ for } m \geq k_0 \text{ or } n \geq k_0 \ x_{mn} = 0 \},
\]

\[
\mathcal{L}_u(X,f) = \{ x \in \Omega(X) : \sum_{m,n} f(\|x_{mn}\|) < \infty \},
\]

where \( \|,\| \) is the norm of \( X \). Further \( X' \) denotes the continuous dual of \( X \). \( \alpha \)-dual and \( \beta(\nu) \)-dual of some double sequence space \( E \) are given by

\[
E^\alpha = \{ (a_{ij}) \in \Omega : \sum |a_{ij}x_{ij}| < \infty, \text{ for all } x \in E \},
\]

\[
E^{\beta(\nu)} = \{ (a_{ij}) \in \Omega : \nu - \sum a_{ij}x_{ij} \text{ exists for all } x \in E \}.
\]

Now we will give some definitions for vector valued double sequence spaces. Let \( E \) be a non-empty subset of \( \Omega(X) \). Define \( \alpha \)-dual of \( E \) by

\[
E^\alpha = \{ (\rho_{nk}) \in \Omega(X') : \text{ for all } x \in E, \sum |\rho_{nk}(x_{nk})| < \infty \},
\]

and \( \beta(\nu) \)-dual of \( E \) by

\[
E^{\beta(\nu)} = \{ (\rho_{nk}) \in \Omega(X') : \text{ for all } x \in E, \nu - \sum \rho_{nk}(x_{nk}) \text{ exists} \}.
\]

In this study "\( \nu - \sum_{k,l} x_{kl} \text{ exists} \)" means that \( \nu - \lim_{m,n} \sum_{k=1}^{m} \sum_{l=1}^{n} x_{kl} \text{ exists} \).

Note that, always \( E^\alpha \subseteq E^{\beta(\nu)} \) and if \( E \subseteq F \subseteq \Omega(X) \), then \( E^\delta \supseteq F^\delta \), where \( \delta \in \{\alpha, \beta(\nu)\} \). Let \( E^{\alpha \alpha} = (E^\alpha)^\alpha \). Then \( E \) is called perfect whenever \( E = E^{\alpha \alpha} \).
Also, \( E^\alpha \subset \Omega (X') \) and hence \( E^{\alpha\alpha} \subset \Omega (X'') \). If \( X \) is a normed space, \( X' \) and \( X'' \) are Banach spaces and for each \( x \in X \), the function given by

\[
\hat{x} : X' \to \mathbb{C} \\
\hat{x} (f) = f(x)
\]

is linear and continuous. Then, the transformation defined by

\[
k : X \to X'' \\
x \to k(x) = \hat{x},
\]

is called embedding of \( X \) into \( X'' \). If \( k \) is onto then \( X \) is called reflexive. A reflexive normed space is complete, so it is a Banach space. We know that \( \lambda \subset \lambda'' \), for some scalar sequence space \( \lambda \).

Similarly, for some \( E \subset \Omega (X) \), \( E \subset E^{\delta\delta} \), \( (\delta = \alpha, \beta \ (\nu)) \), providing that \( X \) is reflexive normed space. Further \( E^{\delta} = E^{\delta\delta} \), in this case.

Let \( E \subset \Omega (X) \) be a double sequence space and \( x = (x_{nk}) \in E. \) \( E \) is called normal, if for each \( (\rho_{nk}) \in \Omega (X') \),

\[
|\rho_{nk} (y_{nk})| \leq |\rho_{nk} (x_{nk})|,
\]

implies \( y = (y_{nk}) \in E. \)

If \( \lambda \) is a normal sequence space then \( \lambda (X) \subset \Omega (X) \) is normal. \( E \) is called monotone, if for each \( x = (x_{kl}) \in E \) and \( y = (y_{ij}) \in \{0, 1\}^{N \times N} \),

\[
x.y = (x_{kl}.y_{ij}) \in E.
\]

Note that normality of \( E \) implies its monotonicity.

### 2. Dual Spaces

**Lemma 2.1.** [4] Let \( A \) be a set and \( (X, P) \) be a locally convex space. Then for all \( x \in l (A, X) \)

\[
x = \sum_{a \in A} (I_a \circ x) (a),
\]

where \( P \) is a family of seminorms on \( X \) and

\[
I_a : X \to l (A, X) \\
b \neq a \Rightarrow y(b) = 0, y(a) = t \text{ and } I_a (t) = y.
\]

**Theorem 2.2.** If \( X \) is a normed space and \( f \in L_u (X) \), then \( f \) is represented by

\[
f (x) = \sum_{i,j} \Psi_{ij} (x_{ij}), \quad \Psi \in M_u (X'),
\]

and hence

\[
L_u (X)' = M_u (X').
\]

**Proof.** From Lemma 2.1, \( (S_F (x) : F \in \mathcal{F}) \) is the net of finite summations and it converges to \( x \), where

\[
x = \sum_{i,j \in N \times N} I_{ij} (x_{ij}) \in L_u (X).
\]
If $f \in \mathcal{L}_u(X)'$, then we get
\[ f(x) = \sum_{i,j \in \mathbb{N} \times \mathbb{N}} (f \circ I_{ij})(x_{ij}), \]
by the continuity of $f$. Let us define
\[ \Psi : \mathbb{N} \times \mathbb{N} \to X' \]
\[ \Psi_{ij} = f \circ I_{ij}. \]
Then, for each $(i, j) \in \mathbb{N} \times \mathbb{N},$
\[ \|\Psi(i, j)\| \leq \|f\| \|I_{ij}\| = \|f\| < \infty, \]
and so $\|\Psi\| = \text{Sup}\{\|\Psi(i, j)\| : (i, j) \in \mathbb{N} \times \mathbb{N}\} \leq \|f\|.$ Hence we obtain $\Psi \in \mathcal{M}_u(X').$ Further,
\[ |f(x)| \leq \left\| \sum_{i,j} \Psi_{ij}(x_{ij}) \right\|, \]
\[ \leq \sum_{i,j} \|\Psi_{ij}\| \|\!(x_{ij})\|, \]
\[ \leq \|\Psi\| \cdot \|x\|_u, \]
which gives $\|f\| \leq \|\Psi\|.$ So, we get $\|\Psi\| = \|f\|.$

We can define an isometry $T$ such that
\[ T : \mathcal{L}_u(X)' \to \mathcal{M}_u(X') \]
\[ Tf = \Psi, \]
for each $\Psi \in \mathcal{M}_u(X').$ Also consider a linear functional $f$ on $\mathcal{L}_u(X)$ given by
\[ f(x) = \sum_{i,j \in \mathbb{N} \times \mathbb{N}} \Psi_{ij}(x_{ij}). \]
In this case $Tf = \Psi.$ Since $\Psi_{ij} = f \circ I_{ij}$ and $T$ are continuous by the fact that
\[ |f(x)| \leq \|\Psi\| \cdot \|x\|_u, \]
then we obtain $Tf = \Psi.$ Moreover, $T$ is onto. Hence $T$ is an equivalence, i.e., $T$ is a linear isometry from $\mathcal{L}_u(X)'$ onto $\mathcal{M}_u(X').$ \(\square\)

**Lemma 2.3.** [4] Let $f$ be an unbounded modulus function which is not the identity on $[0, \infty).$ Then, for each $x \in [0, \infty),$ there exists a positive integer $n$ such that
\[ f\left(\frac{x}{n}\right) > \frac{1}{n} \]
[4].

**Theorem 2.4.** If $f$ is an unbounded modulus function which is not the identity on $[0, \infty),$ then $\mathcal{L}_u(X, f)$ is not locally convex.
Proof. Let us take $D = \{ x : P(x) \leq 1 \}$ in $L_u(X, f)$, where

$$P(x) = \sum_{i} \sum_{j} f(\|x_{ij}\|).$$

and $D$ is a neighborhood of zero in $L_u(X, f)$. Since $D$ contains the sphere

$$\{ x : P(x) \leq 1 \},$$

In order to show that $D$ does not contain any convex neighborhood of zero, let $N$ be a convex neighborhood of zero in $L_u(X, f)$. Then, for some $\delta > 0$, $N$ contains

$$\{ x : P(x) \leq \delta \}.$$

Since $f$ is unbounded, we can take an $\xi > 0$ such that $f(\xi) = \delta$. Now define

$$I_{ij} : X \to L_u(X, f)$$

$$I_{ij}(t) = \begin{bmatrix} 0 & 0 & \ldots & 0 & \ldots \\ 0 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \ldots & t & \ldots \\ 0 & 0 & \ldots & 0 & \ldots \end{bmatrix}.$$

If $U_{ij} \in S_X$, where $S_X = \{ x \in X : \|x\| = 1 \}$, then, $I_{ij}(\xi U_{ij}) \in N$, for each $(i, j)$.

Since

$$P(I_{ij}(\xi U_{ij})) = f(\|\xi U_{ij}\|) = f(\xi) = \delta,$$

and

$$I_{ij}(\xi U_{ij}) \in \{ x : P(x) \leq \delta \}.$$

from Lemma 2.3, we can find $m, n \in \mathbb{Z}^+$ such that

$$f\left(\frac{\xi}{mn}\right) > \frac{1}{mn}.$$

Using the convexity of $N$, we have

$$x = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} I_{ij}(\xi U_{ij}) = \begin{bmatrix} \xi_{11} & \xi_{12} & \ldots & \xi_{1n} & 0 \\ \xi_{m1} & \xi_{m2} & \ldots & \xi_{mn} & 0 \\ 0 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \ldots & 0 & \ldots \end{bmatrix}.$$

But, since we have

$$P(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(\|\xi U_{ij}\|/mn\right),$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(\|\xi/mn\|\right) = m.n.f\left(\frac{\xi}{mn}\right) > \frac{mn}{mn} = 1,$$
then we get \( x \notin D \), i.e., \( N \notin D \), which shows that \( D \), the neighbourhood of zero, does not contain a convex neighbourhood. Thus, \( \mathcal{L}_u(X,f) \) can not be locally convex.

\[ \square \]

**Theorem 2.5.** If \( X \) is a reflexive normed space and \( E \subset \Omega(X) \) is perfect, then the space \( E \) is normal.

**Proof.** Since \( E \) is perfect, \( E = E^{\alpha} \). For \( x = (x_{nk}) \in E \) and for each \((g_{nk}) \in \Omega(X')\), let us take

\[ |g_{nk}(y_{nk})| \leq |g_{nk}(x_{nk})|. \tag{2.1} \]

By perfectness of \( E \), if \( x \in E \), then \( x \in E^{\alpha} \subseteq \Omega(X'' \cap \Omega(X)). \) For each \((g_{nk}) \in E^{\alpha} \), we get

\[ \sum_{n,k} |g_{nk}(x_{nk})| < \infty. \tag{2.2} \]

From (2.1) and (2.2) we obtain

\[ \sum_{n,k} |g_{nk}(y_{nk})| < \infty, \]

which shows \( y = (y_{nk}) \in E^{\alpha} = E \). So, \( E \) is a normal space. \( \square \)

**Theorem 2.6.** Normality implies monotonicity for any double sequence space \( E \subset \Omega(X) \).

**Proof.** Let \( z = (z_{nk}) \) be a double sequence defined by \( z_{nk} = x_{nk} \cdot y_{nk} \), for \( x = (x_{nk}) \in E \) and \( y = (y_{nk}) \in \{0,1\}^{\mathbb{N} \times \mathbb{N}} \). By definition of \( \{0,1\}^{\mathbb{N} \times \mathbb{N}} \), the finite terms which are different from each other in sequence \( y \) are repeated infinity times. Let \( \mu_1, \mu_2, ..., \mu_n \) be such terms and say that \( \mu = \max\{|\mu_i|: 1 \leq i \leq n\} \). It is clear that \( |y_{nk}| \leq \mu \). Also, for each \((g_{nk}) \in \Omega(X')\), we have

\[ |g_{nk}(z_{nk})| = |y_{nk}g_{nk}(x_{nk})| = |y_{nk}| |g_{nk}(x_{nk})| \leq \mu \cdot |g_{nk}(y_{nk})|. \]

Thus, using the normality of \( E \), we get \( z \in E \), which implies that \( E \) is monotone. \( \square \)

**Theorem 2.7.** If \( E \subset \Omega(X) \) is a monotone double sequence space, then we have \( E^{\alpha} = E^{\beta(\nu)} \).

**Proof.** If \((g_{nk}) \in E^{\beta(\nu)} \), then for each \((x_{nk}) \in E \),

\[ \nu - \sum_{n,k} g_{nk}(x_{nk}) \]

exists. Now, we define a double sequence \((z_{nk})\) by

\[ z_{nk} = \begin{cases} \frac{|g_{nk}(x_{nk})|}{g_{nk}(x_{nk})} x_{nk}, & g_{nk}(x_{nk}) \neq 0, \\ 0, & g_{nk}(x_{nk}) = 0. \end{cases} \]

Since \( E \) is monotone, then we have \( z = (z_{nk}) \in E \). Thus

\[ \nu - \sum_{n,k} g_{nk}(z_{nk}) \]
exists,

\[
\nu - \sum_{n,k=1}^{\infty} g_{nk} (z_{nk}) = \sum_{n,k=1}^{\infty} g_{nk} \left( \frac{|g_{nk} (x_{nk})|}{g_{nk} (x_{nk})} x_{nk} \right),
\]

\[
= \sum_{n,k=1}^{\infty} \frac{|g_{nk} (x_{nk})|}{g_{nk} (x_{nk})} g_{nk} (x_{nk}),
\]

\[
= \sum_{n,k=1}^{\infty} |g_{nk} (x_{nk})|.
\]

Thus \( (g_{nk}) \in E^\alpha \).

\[\square\]

**Corollary 2.8.** \( \alpha \)-duals and \( \beta (\nu) \) duals of \( L_u (X) \), \( M_u (X) \), \( \Phi (X) \) and \( \Omega (X) \) are overlap.

**Theorem 2.9.** The space \( L_u (f) \) is normal.

**Proof.** If \( x = (x_{nk}) \in L_u (f) \) then we have \( \sum f (|x_{nk}|) < \infty \). Assume that \( |y_{nk}| \leq |x_{nk}| \), then we write

\[
f (|y_{nk}|) < f (|x_{nk}|),
\]

which implies

\[
\sum_{n,k} f (|y_{nk}|) < \sum_{n,k} f (|x_{nk}|) < \infty.
\]

So we get

\[
\sum_{n,k} f (|y_{nk}|) < \infty
\]

which means that \( y = (y_{nk}) \in L_u (f) \).

\[\square\]

**Corollary 2.10.** The space \( L_u (f) \) is monotone. Thus, \( L_u (f)^\alpha = L_u (f)^{\beta (\nu)} = M_u \).

**Corollary 2.11.** Since \( L_u (f) \) is normal, then \( L_u (X, f) \) becomes normal. Accordingly \( L_u (X, f) \) is monotone in the same time, \( \alpha \) and \( \beta (\nu) \) duals of it are overlap, i.e. \( L_u (X, f)^\alpha = L_u (X, f)^{\beta (\nu)} \).

**Theorem 2.12.** If \( \lambda \) is a normal scalar double sequence space, then we have

\[
\lambda (X)^\alpha = \lambda^\alpha (X').
\]

**Proof.** If \((f_{nk}) \in \lambda^\alpha (X')\), then \((\|f_{nk}\|) \in \lambda^\alpha\). In this case, for each \((u_{nk}) \in \lambda\). We have

\[
\sum_{n,k} \|f_{nk}\| \cdot |u_{nk}| < \infty.
\]
On the other hand, for each $x = (x_{nk}) \in \lambda (X)$, $\left(\|x_{nk}\|\right) \in \lambda$, we obtain
\[
\sum_{n,k} |f_{nk}(x_{nk})| \leq \sum_{n,k} \|f_{nk}\| \|x_{nk}\| < \infty,
\]
that is, $(f_{nk}) \in \lambda (X)^\alpha$.

Conversely, suppose that $(f_{nk}) \in \lambda (X)^\alpha$. From definition of $(\|f_{nk}\|)$, for each $k$, there exists a $(y_{nk}) \in X$, such that $\|f_{nk}\| \leq 2 |f_{nk}(y_{nk})|$ and $\|y_{nk}\| \leq 1$. For each $(u_{nk}) \in \lambda$, define sequence $(z_{nk})$ by $z_{nk} = u_{nk} \cdot y_{nk}$. From normality of $\lambda$, since we have $\|z_{nk}\| = |u_{nk}| \|y_{nk}\| < |u_{nk}|$ and $(\|z_{nk}\|) \in \lambda$, then we get $z \in \lambda (X)$. Thus we write
\[
\sum_{n,k} \|f_{nk}\| |u_{nk}| \leq 2 \sum_{n,k} |u_{nk}| |f_{nk}(y_{nk})| = 2 \sum_{n,k} |f_{nk}(u_{nk} \cdot y_{nk})| = 2 \sum_{n,k} |f_{nk}(z_{nk})| < \infty,
\]
which implies $(\|f_{nk}\|) \in \lambda^\alpha$ and $(f_{nk}) \in \lambda^\alpha (X')$.

Remark 2.13. In the scalar case, perfectness of a sequence space is needed normality; normality is needed monotonicity. This valid for vector valued sequence space when $X$ is a reflexive normed space.

Corollary 2.14. $\mathcal{L}_u (X)$, $\mathcal{M}_u (X)$, $\Phi (X)$ and $\Omega (X)$ are normal and monotone.

Corollary 2.15. Let $X$ be a reflexive normed space. Then. we have
\[
\mathcal{L}_u (X)^\alpha = \mathcal{L}_u (X)^\beta (\nu) = \mathcal{M}_u (X').
\]

Theorem 2.16. If $\lambda$ is a normal double sequence space, then, we have
\[
\lambda (X, f)^\alpha = \lambda (X, f)^\beta (\nu) = (\lambda (f))^\alpha (X').
\]

Corollary 2.17. In above Theorem, if we take $\lambda = \mathcal{L}_u$ then we have
\[
\mathcal{L}_u (X, f)^\alpha = \mathcal{L}_u (X, f)^\beta (\nu) = (\mathcal{L}_u (f))^\alpha (X') = \mathcal{M}_u (X').
\]

References


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