CHARACTERIZATION OF THE SOLUTION OF THE DIOPHANTINE EQUATION $X^2 + Y^2 = 2Z^2$

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Abstract. In this paper, we are interested to characterize the solutions of the diophantine equation $x^2 + y^2 = 2z^2$ by using the arithmetic technicals.

1. Introduction

The diophantine equation is interested by a lot of Mathematicians. As Bennett characterize the solutions of the diophantine equation $x^{2n} + y^{2n} = z^n$ [2] in 2004, Frits Beukers studied the diophantine equation $Ax^p + By^q = Cz^r$ [1] in 1998 and Nils Bruin [3] search the solution of the diophantine equation $x^9 + y^8 = z^3$ and $xyz = 0$ in 1999. In our paper we are interested to search the solution of the diophantine equation $x^2 + y^2 = 2z^2$.

2. Characterization of The Solution of The Diophantine Equation $x^2 + y^2 = 3z^2$

In this section we show the following result which characterizes the solution of the diophantine equation $x^2 + y^2 = 3z^2$.

Theorem 2.1. The diophantine equation $x^2 + y^2 = 3z^2$ has only one solution $(0, 0, 0)$ in $\mathbb{Z}^3$.

Proof. Consider the set of triples $(x_0, y_0, z_0)$ solutions of the equation $x^2 + y^2 = 3z^2$ such that $x_0 y_0 \neq 0$. Assume that $d = x_0 \wedge y_0$, then $x_0 = dx_1$, $y_0 = y_1d$ and $x_1 \wedge y_1 = 1$ and since $(x_0, y_0, z_0)$ is the solution of the equation $x^2 + y^2 = 3z^2$ then $z_0 = z_1d$ and the triplet $(x_1, y_1, z_1)$ is solution of the diophantine equation $x^2 + y^2 = 3z^2$. So we have two cases, the first one is $x_1 \equiv 0 \text{mod} 3$ and the second is $x_1 \equiv 1 \text{mod} 3$ or $x_1 \equiv -1 \text{mod} 3$. The first case implies that $x_1^2 \equiv 0 \text{mod} 3$ and since $x_1 \wedge y_1 = 1$ then $y_1 \equiv 1 \text{mod} 3$ or $y_1 \equiv -1 \text{mod} 3$ which is implies that $y_1^2 \equiv 1 \text{mod} 3$. We deduce that $x_1^2 + y_1^2 \equiv 1 \text{mod} 3$ which is absurd because $x_1^2 + y_1^2 = 3z_1^2$. And the second case implies that $x_1^2 \equiv 1 \text{mod} 3$ We have $y_1^2 \equiv 1 \text{mod} 3$ or $y_1^2 \equiv 0 \text{mod} 3$ this implies that $x_1^2 + y_1^2 \equiv 1 \text{mod} 3$ or $x_1^2 + y_1^2 \equiv 2 \text{mod} 3$ which is absurd because $x_1^2 + y_1^2 = 3z_1^2$.

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3. Characterization of The Solution of The Diophantine Equation \(x^2 + y^2 = 2z^2\)

In this paragraph we solve the diophantine equation \(x^2 + y^2 = 2z^2\) in \(\mathbb{Z}^3\).

**Theorem 3.1.** Let the diophantine equation \((E) : x^2 + y^2 = 2z^2\) then the following properties are equivalents:

(i) \((x, y, z)\) is the solution of \((E)\) and \(x \land y = 1\)

(ii) \(|x| = |−b^2 + 2bc + c^2|, |y| = |b^2 + 2bc - c^2|, |z| = |b^2 + c^2|\) or \(|x| = \frac{|−b^2 + 2bc + c^2|}{2}, |y| = \frac{|b^2 + 2bc - c^2|}{2}, |z| = \frac{|b^2 + c^2|}{2}\)

Proof. (ii) \(\implies\) (i) We have:

\[
x^2 + y^2 = (-b^2 + 2bc + c^2)^2 + (b^2 + 2bc - c^2)^2
= (2bc - (b^2 - c^2))^2 + (2bc + (b^2 - c^2))^2
= 2(4(bc)^2 + (b^2 - c^2)^2)
= 2(b^2 + c^2)^2
= 2z^2
\]

(i) \(\implies\) (ii)

If \(y\) is even and \(x \land y = 1\) then \(x\) is odd. Therefore \(y^2\) is even and \(x^2\) is odd then \(x^2 + y^2 = 2z^2\) is odd contradiction.

We deduce that \(x, y\) are odds We have \(x = 2k_1 + 1\) and \(y = 2k_2 + 1\) then

\[
x^2 + y^2 = (2k_1 + 1)^2 + (2k_1 + 1)^2
= 4k_1^2 + 4k_1 + 1 + 4k_2^2 + 4k_2 + 1
= 4(k_1^2 + k_1 + k_2^2 + k_2) + 2
= 2z^2
\]

which is implies that \(z^2 = 2(k_1^2 + k_1 + k_2^2 + k_2) + 1\)

Then \(z\) is odd.

Assume that \(x^2 > z^2 > y^2\) then \(|x| > |y| > |z|\). Let set \(z \land x = d\). Since \(y^2 = 2z^2 - x^2\) then \(d^2\) divide \(y^2\) so \(y = kd\) which is implies that \(d\) divides \(x \land y = 1\) then \(x \land z = 1\).

We can deduce that \(z \land x = 1\) and \(z \land y = 1\) \((a_0)\)

It is obvious that \(\frac{|x| - |z|}{2}, \frac{|x| + |z|}{2}, \frac{|z| - |y|}{2}, \frac{|z| + |y|}{2} \in \mathbb{Z}\)

Let set \(\frac{|x| - |z|}{2} \land \frac{|x| + |z|}{2} = d\) then \(d\) divides \(\frac{|x| - |z|}{2} + \frac{|x| + |z|}{2} = |x|\) and \(d\) divides \(-\frac{|x| - |z|}{2} + \frac{|x| + |z|}{2} = |z|\) which is implies that \(d\) divides \(|x \land |z|| = 1\)

so \(\frac{|x| - |z|}{2} \land \frac{|x| + |z|}{2} = 1\) \((a_1)\)

Let set \(\frac{|x| - |y|}{2} \land \frac{|x| + |y|}{2} = d\) then \(d\) divides \(\frac{|x| - |y|}{2} + \frac{|x| + |y|}{2} = |z|\) and \(d\) divides \(-\frac{|x| - |y|}{2} + \frac{|x| + |y|}{2} = |y|\) which is implies that \(d\) divides \(|z \land |y|| = 1\)

so \(\frac{|x| - |y|}{2} \land \frac{|x| + |y|}{2} = 1\) \((a_2)\)

We put
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$\frac{|x|-|z|}{2} \land \frac{|z|-|y|}{2} = a > 0$ ($b_1$)

$\frac{|x|+|z|}{2} \land \frac{|z|+|y|}{2} = b > 0$ ($b_2$)

$\frac{|x|-|z|}{2} \land \frac{|z|-|y|}{2} = c > 0$ ($b_3$)

$\frac{|x|+|z|}{2} \land \frac{|z|+|y|}{2} = d > 0$ ($b_4$)

Since $x^2 - z^2 = z^2 - y^2$, $(a_1)$ and $(a_2)$ show that:

$\frac{|x|-|z|}{2} = ab$ ($a_3$)

$\frac{|x|+|z|}{2} = cd$ ($a_4$)

$\frac{|z|-|y|}{2} = ac$ ($a_5$)

$\frac{|z|+|y|}{2} = bd$ ($a_6$)

We have $|x| > |z| > |y|$ then $\frac{|x|+|z|}{2} > \frac{|z|+|y|}{2}$ so $cd > bd$ therefore $c > b$ it means $c - b > 0$ ($b_5$)

$\frac{|x|}{2} = ab + cd$

$\frac{|y|}{2} = bd - ac$

$\frac{|z|}{2} = ac + bd$

then $cd - ab = ac + bd \implies cd - bd = ac + ab$ then

$d(c - b) = a(b + c)$

1st case

$c - b$ is odd since $(a_1)$, $(a_3)$, $(a_4)$ then $a \land d = 1$ and $b \land c = 1$ then $c - b \land c + b = 1$ because $c - b$ is odd since $c - b > 0$, $a > 0$, $c > b > 0$ we deduce that

$d = b + c$

$a = c - b$

then

$\frac{|x|}{2} = ab + cd = -b^2 + 2cb + c^2$

$\frac{|y|}{2} = bd - ac = b^2 + 2cb - c^2$

$\frac{|z|}{2} = c^2 + b^2$

2nd case

$c - b$ is even then $\frac{c-b}{2} \land \frac{c+b}{2} = 1$ because $b \land c = 1$ and $a \land d = 1$. Since $c - b > 0$, $a > 0$, $c > b > 0$ we deduce that

$d = \frac{c + b}{2}$

$a = \frac{c - b}{2}$

then

$\frac{|x|}{2} = ab + cd = -\frac{b^2 + 2cb + c^2}{2}$

$\frac{|y|}{2} = bd - ac = \frac{b^2 + 2cb - c^2}{2}$

$\frac{|z|}{2} = \frac{c^2 + b^2}{2}$
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References


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