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ON THE CONJECTURE OF ALON-TARSI

* CARLOS GAMAS

Abstract. The Alon-Tarsi Conjecture states that for even n, the number of even latin squares of order n differs from the number of odd latin squares. In this note we prove that this conjecture is true if and only if there exists a permutation $\zeta \in S_{n^2}$ and a spherical function, φ, such that $\varphi(\zeta) \neq 0$.

1. Introduction

A Latin square of order n is a $n \times n$ table filled with different symbols, which we may take to be $1, \cdots, n$, in such a way that each symbol occurs exactly once in each row and exactly once in each column. Each row and each column of a Latin square can be considered as the representation of a permutation.

The *sign* of a row or column of a Latin square, L, is its sign as a permutation of the set $\{1, \cdots, n\} = [n]$. A Latin square is column (row) even (odd) if the product of n column (row) *signs* is 1 (-1). The *sign*, $s(L)$, of L is the product of $2n$ column and row signs. L is even if $s(L) = 1$ and odd if $s(L) = -1$.

In [5] Alon and Tarsi conjectured the following

Conjecture 1.1. Let n be an even integer. Then $\Sigma s(L) \neq 0$, where the sum runs over all Latin squares L of order n.

There exists a vast literature on the Alon-tarsi conjecture and equivalent conjectures (see ,for example, [11], [12] and [15]). In this note we give a conjecture equivalent to the conjecture 2 (see below).

We write $CELS(n)$ and $COLS(n)$ to denote the number of column even and column odd Latin squares of order n, respectively.

In ([11], Conjecture 3) it was proven that conjecture 1 is equivalent to the following

Conjecture 1.2. If n is even, then $CELS(n) \neq COLS(n)$.

Let $a_{i,j}$, $i, j = 1, \cdots, n$ be distinct natural numbers. Consider the following table filled with the $a_{i,j}$

$$\begin{matrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \dots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{matrix} \tag{1.1}$$

Let ζ be the following permutation of S_{n^2}

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* Corresponding author.
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$$\zeta = (a_{1,1} \dots a_{n,1}) \cdots (a_{1,i} \dots a_{n,i})^i \dots a_{n,n-1})^{n-1}.$$

Let R_1 be the subgroup of S_{n^2}

$$R_1 = S_{\{}a_{1,1}, \dots, a_{1,n}\} \cdots S_{\{}a_{n,1}, \dots, a_{n,n}\}. \tag{1.2}$$

Let $\gamma = 1_{R_1}$ be the identity character of R_1 and λ an irreducible character of S_{n^2} corespondent to the partition $(n, \cdots, n)$ of n^2. Let

$$\varphi_{\lambda, 1_{R_1}}(g) = \frac{1}{|R_1|} \sum_{h \in R_1} \lambda(gh) \ \ , \ \ g \in S_{n^2}.$$

We prove that conjecture 2 is true if and only if

$$\varphi_{\lambda, 1_{R_1}}(\zeta) \neq 0.$$

2. Auxiliary Results

Let $\lambda = (\lambda_1, \cdots, \lambda_q)$, with $\lambda_1 \geq \cdots \geq \lambda_q > 0$, be a partition of n. We denote the partition and the irreducible character it induces by the same letter λ. With this partition we associate a table, denoted by D_λ, consisting of λ_1 boxes in the first row, λ_2 boxes in the second row and so on. We define the diagram $D_{\lambda,\rho}$, $\rho \in S_n$, as the array obtained obtained by filling the boxes of D_λ in such a way that the box in the row $i \in [q]$ and column $j \in [\lambda_i]$ is filled by the integer $\rho(\lambda_1 + \cdots + \lambda_{i-1} + j)$, with $\lambda_0 = 0$.

The set N_j, $1 \leq j \leq \lambda_1$, is the set of integers in the jth column of $D_{\lambda,\rho}$ and M_j is the set of integers in its jth row.

We define $R(D_{\lambda,\rho})$, $C(D_{\lambda,\rho})$ and $\xi(D_{\lambda,\rho})$ as follows:

$$R(D_{\lambda,\rho}) = \{g \in S_n : g(M_j) = M_j \ , j = 1, \cdots, q\},$$
$$C(D_{\lambda,\rho}) = \{g \in S_n : g(N_j) = N_j \ , j = 1, \cdots, \lambda_1\},$$
$$\xi(D_{\lambda,\rho}) = \textstyle\sum_{h \in R(D_{\lambda,\rho})} \sum_{g \in C(D_{\lambda,\rho})} sign(g)\ gh,$$

where $sign$ denotes the alternating character.

A diagram $D_{\lambda,\rho}$ is *standard* if the rows and columns are increasing sequences. If $D_{\lambda,\sigma}$, $D_{\lambda,\rho}$, $\sigma, \rho \in S_n$ are standard diagrams we define $D_{\lambda,\sigma} < D_{\lambda,\rho}$ if and only if $(\sigma(1), \cdots, \sigma(n)) < (\rho(1), \cdots, \rho(n))$ by the lexicographic order.

Lemma 2.1 (4, p 103)**.** *Let $x \in CS_n$ with $x = \sum_{g \in S_n} x(g)g$,$x(g) \in C$, where C denotes the complex numbers. If $\pi \in S_n$ we have*

$$(\pi x)(g) = x(\pi^{-1} g) \ \ , \ \ (x\pi)(g) = x(g\pi^{-1})$$

for all $g \in S_n$.

Lemma 2.2 (2)**.** *Let $D_{\lambda,\sigma_1} < \cdots < D_{\lambda,\sigma_p}$ be standard diagrams, with $\sigma_1 = 1_{S_n}, \cdots, \sigma_p \in S_n$ where 1_{S_n} denotes the identity element of S_n. For $i = 1, \cdots, p$ we denote $R(D_{\lambda,\sigma_i})$, $C(D_{\lambda,\sigma_i})$ and $\xi(D_{\lambda,\sigma_i})$ by R_i , C_i and ξ_{ii} respectively. Let $\rho \in S_n$. We have*

(a) $p = \lambda(1_{S_n})$.

(b) $\xi_{ii}(\sigma g) = sign(\sigma)\xi_{ii}(g)$,$\xi_{ii}(g\tau) = \xi_{ii}(g)$,*for all* $g \in S_n$, $\sigma \in C_i$, $\tau \in R_i$,$i \in [p]$.

(c) $\xi(D_{\lambda,\rho}) = \rho\xi_{11}\rho^{-1}$.

(d) *If $i < j$, $i, j \in [p]$, there exist two integers in the same row of D_{λ,σ_i} and in the same column of D_{λ,σ_j}.*

(e) $\xi_{ii}\xi_{ii} = \frac{n!}{\lambda(1_{S_n})}\xi_{ii}$ *and* $\xi_{ii}\xi_{jj} = 0$ *if* $i < j$, $i, j \in \{1, \cdots, p\}$.

(f) $\xi_{i1}(\sigma_i) = 1$, $i = 1, \cdots, p$ *and* $\xi_{j1}(\sigma_i) = 0$ *for* $i < j$ *and* $i, j \in [p]$.

(g) *The set of elements of CS_n*

$$\{\xi_{11}, \cdots, \xi_{p1}\}$$

is a basis for the minimal ideal $CS_n\xi_{11}$.

Proposition 2.3. *For $j = 2, \cdots, p$ we have*

$$\sum_{h\in R_1} h\xi_{j1} = 0.$$

Proof. Bearing in mind Lemma 2.1 it is enough to show that

$$\sum_{h\in R_1} \xi_{j1}(h^{-1}g) = 0, \tag{2.1}$$

for all $g \in S_n$. We have

$$\sum_{h\in R_1} \xi_{j1}(h^{-1}g) = \sum_{h\in R_1}(\sigma_j\xi_{11})(h^{-1}g)$$
$$= \sum_{h\in R_1}(\sigma_j\xi_{11})(h^{-1}g\sigma_j^{-1}\sigma_j)$$

by Lemma 2.1

$$= \sum_{h\in R_1}(\sigma_j\xi_{11}\sigma_j^{-1})(h^{-1}g\sigma_j^{-1})$$

by (c) of Lemma 2.2

$$= \sum_{h\in R_1}(\xi_{jj})(h^{-1}g\sigma_j^{-1}). \tag{2.2}$$

By (d) of Lemma 2.2 there exists $(a\,b) \in R_1 \cap C_j$. Let $H = \{1_{S_n}, (a\,b)\}$ and

$$R_1 = H\dot{\cup}H\tau_2\dot{\cup}\cdots\dot{\cup}H\tau_l,$$

where $\tau_2, \cdots, \tau_l \in R_1$.

We have $\sum_{h\in R_1}(\xi_{jj})(h^{-1}g\sigma_j^{-1}) = \xi_{jj}(g\sigma_j^{-1}) + \xi_{jj}((a\,b)g\sigma_j^{-1}) + \cdots + \xi_{jj}(\tau_l g\sigma_j^{-1}) + \xi_{jj}((ab)\tau_l g\sigma_j^{-1})$

by (b) of Lemma 2.2 $= \xi_{jj}(g\sigma_j^{-1}) - \xi_{jj}(g\sigma_j^{-1}) + \cdots + \xi_{jj}(\tau_l g\sigma_j^{-1}) - \xi_{jj}(\tau_l g\sigma_j^{-1}) = 0.$

From this last equality and (2.2) we conclude that (2.1) is true □

Definition 2.4 (3). . For all $t = 1, \cdots, p$ let

$$\phi_t^t = \xi_{t1}, \phi_{t+x}^t = \phi_{t+x-1}^t - \phi_{t+x-1}^t(\sigma_{t+x})\xi_{t+x\,1},\ x = 1, \cdots, p-t.$$

We define

$$\Theta_{t1} = \phi_p^t.$$

Proposition 2.5 (3)**.** *For all* $t = 1, \cdots, p$ *we have*

(a) $\Theta_{t1} = \xi_{t1} + a^t_{t+1}\xi_{t+1\,1} + \cdots + a^t_p\xi_{p1}$, $a^t_{t+1}, \cdots, a^t_p \in C$.

(b) $\Theta_{t1}(\sigma_t) = 1$ *and* $\Theta_{t1}(\sigma_j) = 0$, *if* $t \neq j$, $j \in \{1, \cdots, p\}$.

(c) $\xi_{11}\sigma_t^{-1}\Theta_{t1} = \frac{n!}{\lambda(1_{S_n})}\xi_{11}$ *and* $\xi_{11}\sigma_t^{-1}\Theta_{t+1\,1} = \cdots = \xi_{11}\sigma_t^{-1}\Theta_{p1} = 0$.

(d) *The set* $\{\Theta_{11}, \cdots, \Theta_{p1}\}$ *is a basis for the minimal ideal* $CS_n\xi_{11}$.

(e) $\lambda(g) = \Theta_{11}(g^{-1}) + \cdots + \Theta_{pp}(g^{-1})$, *for all* $g \in S_n$, *where* $\Theta_{22} = \Theta_{21}\sigma_2^{-1}, \cdots, \Theta_{pp} = \Theta_{p1}\sigma_p^{-1}$.

Proposition 2.6. *For all* $g \in S_n$ *and* $j = 2, \cdots, p$ *we have*

(a) $\sum_{h\in R_1} \Theta_{11}(h^{-1}g) = \sum_{h\in R_1} \xi_{11}(h^{-1}g)$.

(b) $\sum_{h\in R_1} \Theta_{j1}(h^{-1}g) = 0$.

Proof. **(a)** From (a) of Proposition 2.5 we obtain

$$\sum_{h\in R_1} \Theta_{11}(h^{-1}g) = \sum_{h\in R_1} \xi_{11}(h^{-1}g) + a^1_2 \sum_{h\in R_1} \xi_{21}(h^{-1}g) + \cdots + a^1_p \sum_{h\in R_1} \xi_{p1}(h^{-1}g)$$

by (2.1)

$$= \sum_{h\in R_1} \xi_{11}(h^{-1}g).$$

(b) Is an immediate consequence of (2.1) and (a) of Proposition 2.5

□

Proposition 2.7. *For* $g \in S_n$ *let*

$$A_1(g) = \{(\sigma, \tau, \mu) \in C_1 \times R_1 \times R_1 : \sigma = \tau g \mu\}. \tag{2.3}$$

We have

$$\sum_{h\in R_1} \lambda(hg) = \sum_{(\sigma,\tau,\mu)\in A_1(g)} sign(\sigma)$$

where

$$\sum_{(\sigma,\tau,\mu)\in A_1(g)} sign(\sigma) = 0, \text{ if } A_1(g) = \emptyset.$$

Proof. From the fact that λ is a complex character of S_n, we have

$$\sum_{h\in R_1} \lambda(hg) = \sum_{h\in R_1} \lambda(g^{-1}h^{-1})$$

by (e) of Proposition 2.5

$$= \sum_{h\in R_1} \Theta_{11}(hg) + \cdots + \sum_{h\in R_1} \Theta_{pp}(hg)$$

by Proposition 2.6

$$= \sum_{h\in R_1} \xi_{11}(hg) = \sum_{(\sigma,\tau,\mu)\in A_1(g)} sign(\sigma)$$

□

3. Spherical Functions

Let H be a subgroup of the finite group G and γ (respectively χ) be an irreducible complex character of H (respectively G). Suppose

$$c_{\chi,\gamma} = \frac{1}{|H|} \sum_{h\in H} \chi(h)\gamma(h^{-1}) \neq 0.$$

The spherical function $\varphi^H_{\chi,\gamma}$ is a complex valued function of G defined by

$$\varphi^H_{\chi,\gamma}(g) = \frac{\gamma(1_G)}{|H|c_{\chi,\gamma}} \sum_{h\in H} \chi(hg)\gamma(h^{-1}) \quad , g \in G \tag{3.1}$$

where 1_G denotes the identity element of G.

Let $G = S_{n^2}$ and $\lambda = \chi$, $R_1 = H$ be,respectively, the character of S_{n^2} and the subgroup of S_{n^2} defined in section 2.

It is well known that $c_{\lambda,1_{R_1}} = 1$ and so, from (3.1), we have

$$\varphi_{\lambda,1_{R_1}}(g) = \frac{1}{|R_1|}\sum_{h\in R_1}\lambda(hg) = \frac{1}{|R_1|}\sum_{h\in R_1}\lambda(gh)$$

This last equality is due to the fact that $\lambda(gh) = \lambda(hg)$ for all $g \in S_{n^2}$ and $h \in R_1$.

By Proposition 2.7 we have

$$\varphi_{\lambda,1_{R_1}}(g) = \frac{1}{|R_1|}\sum_{(\sigma,\tau,\mu)\in A_1(g)} sign(\sigma) \tag{3.2}$$

Suppose $A_1(g) \neq \emptyset$, $g \in S_n$. Let

$$F_1(g) = \{\pi_1, \cdots, \pi_x\}$$

be the set of all distinct elements $\pi \in C_1$ such that $(\pi, \mu, \tau) \in A_1(g)$.

From the definition of $F_1(g)$ we can write

$$\pi_j = \mu_j g\tau_j, \;\; \mu_j, \tau_j \in R_1, \;\; j = 1, \cdots, x. \tag{3.3}$$

For all $j = 1, \cdots, x$ let

$$A_1^j(g) = \{(\pi_j, \mu_j g\nu g^{-1}, \nu^{-1}\tau_j) \in S_n \times S_n \times S_n : g\nu g^{-1} \in gR_1g^{-1} \cap R_1\}.$$

Lemma 3.1. *For all $j = 1, \cdots, x$ we have*
$\pi_j = \mu g\tau$, $\mu, \tau \in R_1$ *if and only if* $(\pi_j, \mu, \tau) \in A_1^j(g)$.

Proof. Suppose

$$\pi_j = \mu g\tau$$

with $\mu = \mu_j\kappa$, $\tau = \nu^{-1}\tau_j$, $\kappa, \nu \in R_1$.
From this equality and (3.3) we get

$$\mu_j g\tau_j = \mu_j\kappa g\nu^{-1}\tau_j.$$

This equality implies

$$\kappa = g\nu g^{-1}.$$

From this last equality we get

$$\mu = \mu_j g\nu g^{-1}.$$

Thus we have

$$(\pi_j, \mu, \tau) = (\pi_j, \mu_j g\nu g^{-1}, \nu^{-1}\tau_j) \in A_1^j(g).$$

Conversely, if $(\pi_j, \mu_j g\nu g^{-1}, \nu^{-1}\tau_j) \in A_1^j(g)$, then

$$\mu_j g\nu g^{-1} g\nu^{-1}\tau_j = \pi_j$$

□

Proposition 3.2. *We have*

$$\varphi_{\lambda,1_{R_1}}(g) = \frac{1}{|R_1|}\Big(\sum_{\pi\in F_1(g)} sign(\pi)\Big) \;\; |gR_1g^{-1} \cap R_1|.$$

Proof. Lemma 3.1 and the definition of $A_1(g)$ lead to

$$A_1(g) = A_1^1(g)\dot{\cup}\cdots\dot{\cup}A_1^x(g).$$

From this equality we can conclude that Proposition 3.2 is true

□

Lemma 3.3 (2)**.** *Let $\sigma, \rho \in S_n$ and $\mu, \tau \in R_1$. We have $\sigma = \mu\rho\tau$ if and only if $|\sigma(M_i) \cap M_j| = |\rho(M_i) \cap M_j|$ for all $i, j = 1, \cdots, q$.*

Lemma 3.4. (a) *If $\rho = \mu\sigma\tau$, $\mu, \tau \in R_1$, then $|\sigma R_1\sigma^{-1} \cap R_1| = |\rho R_1\rho^{-1} \cap R_1|$.*

(b) *If $|\sigma R_1\sigma^{-1} \cap R_1| = 1 = |\rho R_1\rho^{-1} \cap R_1|$, then $\rho = \mu\sigma\tau$, with $\mu, \tau \in R_1$.*

Proof. (a) Let $\rho = \mu\sigma\tau$. Let $A = \rho R_1\rho^{-1} \cap R_1$ and $B = \sigma R_1\sigma^{-1} \cap R_1$. Let $\rho = \mu\sigma\tau$. Let

$$f : A \longrightarrow R_1$$

be such that $f(h) = \mu^{-1}h\mu$. f is an injective map and we are going to prove that $f(A) = B$. First, we prove that $f(A) \subseteq B$. Suppose $h \in A$. Then

$$\begin{aligned} h &= \rho s\rho^{-1},\ s \in R_1, \\ &= \mu\sigma\tau s\tau^{-1}\sigma^{-1}\mu^{-1}. \end{aligned}$$

This equality implies

$$\mu^{-1}h\mu \in \sigma R_1\sigma^{-1} \Rightarrow f(h) \in B.$$

Now, we prove that $B \subseteq f(A)$. Let $s \in B$. Then

$$\begin{aligned} s &= \sigma d\sigma^{-1},\ d \in R_1, \\ &= \mu^{-1}\rho\tau^{-1}d\tau\rho^{-1}\mu. \end{aligned}$$

This equality implies

$$\mu s\mu^{-1} \in A$$

We have

$$s = f(\mu s\mu^{-1}) \in f(A).$$

(b) Suppose

$$|\sigma R_1\sigma^{-1} \cap R_1| = 1 = |\rho R_1\rho^{-1} \cap R_1|. \tag{3.4}$$

For $M \subset [n]$, let S_M denote the subgroup of S_n consisting of the $|M|!$ elements that leave each point of $[n]\backslash M$ fixed. We have then

$$R_1 = S_{M_1} \times \cdots \times S_{M_q}\ , \quad \sigma R_1\sigma^{-1} = S_{\sigma(M_1)} \times \cdots \times S_{\sigma(M_q)}$$

and

$$\rho R_1\rho^{-1} = S_{\rho(M_1)} \times \cdots \times S_{\rho(M_q)}.$$

These equalities and (3.4) lead to

$$|\sigma(M_i) \cap M_j| = 1 = |\rho(M_i) \cap M_j|, \tag{3.5}$$

for all $i, j = 1, \cdots, q$.

From Lemma 3.3 we conclude

$$\rho = \mu\sigma\tau,$$

with $\mu, \tau \in R_1$

$\square$

4. Latin Squares

The table (1.1) is the table $D_{\lambda,1_{S_{n^2}}}$, with $\lambda = (n, \cdots, n)$. Let $C_1 = C(D_{\lambda,1_{S_{n^2}}})$ and $R_1 = R(D_{\lambda,1_{S_{n^2}}})$.

Lemma 4.1. *Let $\sigma \in C_1$. We have* $|\sigma(R_1) \cap R_1| = 1$ *if and only if, for all $i = 1, \cdots n$, $\sigma(a_{i,1}), \cdots, \sigma(a_{i,n})$ are in distinct rows of R_1.*

Proof. Suppose

$$|\sigma(R_1) \cap R_1| = 1. \tag{4.1}$$

If $\sigma(a_{i,k}) = a_{j,k}$ and $\sigma(a_{i,t}) = a_{j,t}$, with $k \neq t$, then $(a_{j,k}, a_{j,t}) \in \sigma(R_1) \cap R_1$ which contradicts (4.1).

Suppose that for all $i = 1, \cdots n$, $\sigma(a_{i,1}), \cdots, \sigma(a_{i,n})$ are in distinct rows of R_1. Then $|\sigma(M_i) \cap M_j| = 1$, for all $j = 1, \cdots, n$, which implies (4.1).

$\square$

Let $\rho_1, \cdots, \rho_n \in S_n$. Consider the table E_ρ

$$\begin{matrix} \rho_1(1) & \dots & \rho_n(1) \\ \vdots & \cdots & \vdots \\ \rho_1(n) & \dots & \rho_n(n) \end{matrix} \tag{4.2}$$

(4.2) is a latin square if and only for all $i = 1, \cdots, n$,

$$\rho_s(i) \neq \rho_t(i), \tag{4.3}$$

for all $s, t = 1, \cdots, n$ with $s \neq t$. The Latin square is even if $sign(\rho = \rho_1 \dots \rho_n) = 1$ and is odd if $sign(\rho) = -1$. Let E be the set of the tables E_ρ where ρ verifies (4.3). Then E is the set of all latin squares. Let

$$F = \{\sigma \in C_1 : |\sigma(R_1) \cap R_1| = 1\}.$$

For $i = 1, \cdots, n$ and $\sigma \in F$, let $\sigma(a_{i,1}) = a_{j_i^1,1}, \cdots, \sigma(a_{i,n}) = a_{j_i^n,n}$. Let $\rho_1, \cdots, \rho_n$ be defined by

$$\rho_1(i) = j_i^1, \cdots, \rho_n(i) = j_i^n,$$

for all $i = 1, \cdots, n$.

From Lemma 4.1 we can conclude that $\rho = \rho_1 \cdots \rho_n \in E$. It is not difficult to see that the correspondence defined above is a bijection between E and F and, if $\sigma \longmapsto \rho$, then $sign(\sigma) = sign(\rho)$. We have then

Lemma 4.2. *Let n be an even integer. Conjecture 2 is true if and only if*

$$\sum_{\rho \in E} sign(\rho) \neq 0 \neq \sum_{\sigma \in F} sign(\sigma).$$

Proposition 4.3. *Let n be an even integer and $\sigma \in F$. Conjecture 2 is true if and only if $\varphi_{\lambda,1_{R_1}}(\sigma) \neq 0$.*

Proof. It is an immediate consequence of the definition of F, Proposition 3.2, Lemma 3.4 and Lemma 4.2

□

Proposition 4.4. *We have*

$$\zeta = (a_{1,1} \dots a_{n,1}) \cdots (a_{1,i} \dots a_{n,i})^i \cdots (a_{1,n-1} \cdots a_{n,n-1})^{n-1} \in F$$

Proof. As $\zeta \in C_1$, we have only to prove that, for all $i = 1, \cdots, n$,

$$\zeta(a_{i,1}), \cdots, \zeta(a_{i,n})$$

are in distinct rows of R_1. Suppose that $\zeta(a_{i,t}) = \zeta(a_{i,s}) =$. We have then

$$\zeta(a_{i,t}) = (a_{i,1}, \cdots, a_{i,n})^t (a_{i,t}) = a_{i+t,t}$$

and

$$\zeta(a_{i,s}) = (a_{i,1}, \cdots, a_{i,n})^s (a_{i,s}) = a_{i+s,s}$$

which implies $i + t = i + s$ and $t = s$

□

From Proposition 4.3 and Proposition 4.4, Conjecture 2 is equivalent to the following

Conjecture 4.5. Let n be an even integer. Then $\varphi_{\lambda,1_{R_1}}(\zeta) \neq 0$.

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[1]DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COIMBRA, COIMBRA, PORTUGAL.
E-mail address: gamas@mat.uc.pt

A. CHILLALI, SIDI MOHAMED BEN ABDELLAH UNIVERSITY, FP, LSI, TAZA, MOROCCO.
E-mail address: abdelhakim.chillali@usmba.ac.ma