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ON SOME METABELIAN 3-GROUPS REALIZABLE AND PRINCIPALIZATION

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ABSTRACT. Let G be some metabelian 3-group with abelianisation of type (3, 3). In this paper, we prove that G is realizable with some fields k which is the normal closure of a pure cubic field and we apply these results over G to study the capitulation problem of the 3-ideal classes of k.

1. Introduction

Let k be an algebraic number field. We denote by $\mathcal{O}_{\mathrm{k}}$, E_{k} and $\mathrm{Cl}(\mathrm{k})$, the ring of integers, the unit group and the ideal class group of k, respectively. For a prime number p, let $\mathrm{Cl}_p(\mathrm{k})$ be the p-class group and $\mathrm{k}_p^{(1)}$ the Hilbert p-class field of k. Further, we define $\mathrm{k}_p^{(n)}$, for an integer $n \geqslant 0$, by $\mathrm{k}_p^{(0)} = \mathrm{k}$ and $\mathrm{k}_p^{(n+1)} = (\mathrm{k}_p^{(n)})^{(1)}$. So we have the sequence

$$\mathrm{k} \subseteq \mathrm{k}_p^{(1)} \subseteq \ldots \subseteq \mathrm{k}_p^{(n)} \subseteq \ldots$$

that is called the p-class field tower of k. We know that it is finite if and only if there exists a finite p-extension of k whose p-class number is equal to 1.

We shall consider a number fields with $\mathrm{Cl}_3(\mathrm{k})$ is of type $(3,3)$. The second 3-class group noted by $G = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{k}\right)$ is metabelian 3-group with abelianisation $G/\gamma_2(G)$ of type $(3,3)$ where $\gamma_2(G)$ is the derived group of G. By the Galois correspondence and reciprocity law of Artin, it's known that

$$\gamma_2(G) = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{k}_3^{(1)}\right) \simeq \mathrm{Cl}_3(\mathrm{k}_3^{(1)})$$

is abelian. And

$$G/\gamma_2(G) = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{k}\right)/\mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{k}_3^{(1)}\right) \simeq \mathrm{Gal}\left(\mathrm{k}_3^{(1)}/\mathrm{k}\right) \simeq \mathrm{Cl}_3(\mathrm{k}).$$

The four maximal normal subgroups $H_1, \ldots, H_4$ of G are associated with the four unramified cyclic extensions $\mathrm{K}_1, \ldots, \mathrm{K}_4$ of k of relative degree 3, which are represented by the norm class groups $N_{\mathrm{K}_i/\mathrm{k}}(\mathrm{Cl}_3(\mathrm{K}_i))$ as subgroups of index 3 in

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the 3-class group $\mathrm{Cl}_3(\mathrm{k})$ of k and by the Galois correspondence we have $H_i = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{K}_i\right)$.

In the present paper we shall consider $\mathrm{k} = \mathbb{Q}(\zeta_3, \sqrt[3]{3q^2})$, the normal closure of the pure cubic field $\Gamma = \mathbb{Q}(\sqrt[3]{3q^2})$, where q is a prime number which verifies $q \equiv -1 \pmod 9$, and ζ_3 is the third root of unity. Those fields are of type II in the sense of Isamïli [12]. Ismaïli also proved that in this case, there are three type of capitulation $(0,0,0,0)$, $(0,4,3,2)$, $(0,4,4,4)$, and the relative genus field of k over k_0, where $\mathrm{k}_0 = \mathbb{Q}(\zeta_3)$, is one of the four cyclic cubic extension of k, we will noted by K_1.

We investigate the theory of groups of maximal class and the works [2, 4, 3, 5, 6, 7, 8, 12, 15, 16, 17, 18], we determine the structure of $G_1 = \mathrm{Gal}\left((\mathrm{K}_1)_3^{(1)}/\mathrm{k}_0\right)$, precisely we show that G_1 is of maximal class 4 and $G_1 = G^{(4)}(0,1,0)$ (with the same notation of Nebelung [18] and Mayer [15]).
Further, With the aid of the structure of G_1, we prove that also $G = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{k}\right)$ is of maximal class and there are only one type of capitulation possible which is $(0,0,0,0)$ and the class field tower is finite and terminate at $\mathrm{k}_3^{(2)}$.

2. On decomposition of ideal in number field

In this section we develop some results that we need in this paper. A more precision or proof can be found in [1] and [10].

Let a, b are integers such that ab is square free and $ab > 1$. Set $\Gamma = \mathbb{Q}(\sqrt[3]{ab^2})$. Then an integral basis for Γ is given by:

(1) $\{1, \sqrt[3]{ab^2}, \frac{\sqrt[3]{ab^2}}{b}\}$, if $a^2 - b^2 \not\equiv 0 \pmod 9$,
(2) $\{1, \sqrt[3]{ab^2}, \frac{b^2 \pm b^2\sqrt[3]{ab^2} + \sqrt[3]{ab^2}^2}{3b}\}$, if $a^2 - b^2 \equiv 0 \pmod 9$.

And the discriminant of Γ is given by

$$d(\Gamma) = \begin{cases} -27a^2b^2 & \text{if } a^2 - b^2 \not\equiv 0 \pmod 9; \\ -3a^2b^2 & \text{if } a^2 - b^2 \equiv 0 \pmod 9. \end{cases}$$

Definition 2.1. Let $\Gamma = \mathbb{Q}\left(\sqrt[3]{ab^2}\right)$, be a pure cubic field. We say that Γ is of Kind 1 if $a^2 - b^2 \not\equiv 0 \pmod 9$ and of Kind 2 otherwise.

In the following proposition we summarize the results concerning the decomposition in pure cubic field.

Proposition 2.2. *Let $\Gamma = \mathbb{Q}\left(\sqrt[3]{ab^2}\right)$ be a pure cubic field, and $\mathcal{O}_\Gamma$ the ring of integer of Γ, and let $N_{\Gamma/\mathbb{Q}}$ be the absolute norm of Γ.*

(1) *If Γ is of Kind 1, then $3\mathcal{O}_\Gamma = \mathcal{P}^3$, where $\mathcal{P}$ is a prime ideal in $\mathcal{O}_\Gamma$.*
(2) *If Γ is of Kind 2, then $3\mathcal{O}_\Gamma = \mathcal{P}^2\mathcal{P}_1$, where $\mathcal{P}$ and $\mathcal{P}_1$ ($\mathcal{P} \neq \mathcal{P}_\infty$) are primes ideals in $\mathcal{O}_\Gamma$.*
(3) *If q is a prime number such that $q \nmid ab$ and $q \neq 3$, then q is unramified in Γ. More precisely, we have:*

(a) *If* $q \equiv -1 \pmod 3$ *then* $q\mathcal{O}_\Gamma = \mathcal{Q}\mathcal{Q}_1$, *with* $N_{\Gamma/\mathbb{Q}}(\mathcal{Q}) = q$ *and* $N_{\Gamma/\mathbb{Q}}(\mathcal{Q}_1) = q^2$, *where* $\mathcal{Q}$ *and* $\mathcal{Q}_1$ *are primes ideals in* $\mathcal{O}_\Gamma$.

(b) *If* $q \equiv 1 \pmod 3$, *then:*
- *if* $\left(\frac{ab^2}{q}\right)_3 = 1$, *then* $q\mathcal{O}_\Gamma = \mathcal{Q}\mathcal{Q}_1\mathcal{Q}_2$, *with* $N_{\Gamma/\mathbb{Q}}(\mathcal{Q}) = N_{\Gamma/\mathbb{Q}}(\mathcal{Q}_1) = N_{\Gamma/\mathbb{Q}}(\mathcal{Q}_2) = q$, *where* $\mathcal{Q}$, $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *are primes ideals in* $\mathcal{O}_\Gamma$,
- *if* $\left(\frac{ab^2}{q}\right)_3 \neq 1$, *then* $q\mathcal{O}_\Gamma = \mathcal{Q}$, *with* $N_{\Gamma/\mathbb{Q}}(\mathcal{Q}) = q^3$ *where* $\mathcal{Q}$ *is a prime ideal in* $\mathcal{O}_\Gamma$..

Theorem 2.3. *Let* L *be a number field whose contains the* ℓ*-th roots of units,* ℓ *a prime number, and* $\theta \in \mathrm{L}$ *such that* $\theta \neq \mu^\ell$ *for all* $\mu \in \mathrm{L}$. *Then*

(1) *The extension* $\mathrm{L}(\sqrt[\ell]{\theta})/\mathrm{L}$ *is cyclic of prime degree* ℓ.

(2) *Suppose that prime ideal* $\mathcal{P}$ *of* L *is above the principal ideal* (θ) *and define an integer* $a \in \mathbb{N}$ *by* $\mathcal{P}^a \parallel (\theta)$, *then we have:*

(a) *if* $a = 0$, *and* $\mathcal{P}$ *not divides* ℓ, *then* $\mathcal{P}$ *split completely (resp become prime) in* $\mathrm{L}(\sqrt[\ell]{\theta})$ *if and only if* $\theta \equiv \xi^\ell \pmod{\mathcal{P}}$ *is soluble (resp not soluble),*

(b) *if* a *is prime to* ℓ, *then* $\mathcal{P}$ *is totally ramified in* $\mathrm{L}(\sqrt[l]{\theta})$.

(3) *Let* $\mathcal{L}$ *denote a prime ideal above* ℓ *and define un integer* $a \in \mathbb{N}$ *by* $\mathcal{L}^a \parallel (1-\zeta_\ell)$, *where* ζ_ℓ *is the primitive* ℓ*-th root of unity. If* θ *is prime to* ℓ, *then* $\mathrm{L}(\sqrt[l]{\theta})/\mathrm{L}$ *is unramified at* $\mathcal{L}$ *if and only if* $\theta \equiv \xi^\ell \pmod{\mathcal{L}^{a\ell}}$.

Let p be a prime number such that $p \equiv 1 \pmod 3$, and let k_0 the cyclotomic field of third roots of unity. It follows from theorem 2.3 that (for more precision, you can see [13]),

(1) If $p = \pi_1\pi_2$ in k_0, π_1 and π_2 are conjugate, then $\pi_1 \equiv \pi_2 \equiv 1 \pmod{3\mathcal{O}_{k_0}}$.

(2) $\left(\frac{c}{\pi_1}\right)_3 = \left(\frac{c}{\pi_2}\right)_3^2$ for all $c \in \mathbb{Z}$ prime to p.

(3) $\left(\frac{c}{\pi_1}\right)_3 = \left(\frac{c}{\pi_2}\right)_3 = 1$ if and only if c is cubic residue modulo p.

(4) $\left(\frac{\pi_1}{\pi_2}\right)_3 = \left(\frac{\pi_2}{\pi_1}\right)_3 = 1$

Let us mention that, $c \in \mathbb{Z}$ is cubic residue modulo p means that the congruence $X^3 \equiv c \pmod p$ has a solution in $\mathbb{Z}$.

Moreover, by applying theorem 2.3 we deduce the decomposition of a prime number q in the normal closure of some cubic pure field. This can be done in straightforward fashion by combining the factorization rules for his cubic pure field and the cyclotomic field contains the third roots of unity. Let $\mathrm{k} = \mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})$ the normal closure of Γ, then the decomposition in k is determined as follows:

Proposition 2.4. *If* q *divides* ab, *then* $q\mathcal{O}_\mathrm{k} = \mathcal{Q}^3$, *where* $\mathcal{Q}$ *is a prime ideal in* k. *And, if* q *not divides* ab, *then:*

(1) *Suppose that* $q \equiv 1 \pmod 3$, *then:*

(a) *if* (q) *become principal in* Γ, *then* $q\mathcal{O}_\mathrm{k} = \mathcal{P}_1\mathcal{P}_2$,

(b) *if* (q) *split completely in* Γ, *then* $q\mathcal{O}_\mathrm{k} = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3\mathcal{P}_4\mathcal{P}_5\mathcal{P}_6$.

(2) *If* $q \equiv -1 \pmod 3$, *then* $q\mathcal{O}_\mathrm{k} = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$.

These results are summarized in the following:
A rational prime factors in k $= \mathbb{Q}(\zeta_3, \sqrt[3]{ab^2})$ as $\mathcal{P}_1\mathcal{P}_2.....\mathcal{P}_r$, where r is given by:

$$r = \begin{cases} 3, & \text{if}\, q \equiv -1 \pmod 3; \\ 6, & \text{if}\, q \equiv 1 \pmod 3 \text{ and } x^3 - ab^2 \equiv 0 \pmod{(q)} \text{ is solvable;} \\ 2, & \text{Otherwise.} \end{cases}$$

3. The 3-Hilbert field of de genus field of k over k_0

Let any finite 3-group G is nilpotent, and $\gamma_2(G)$ his commutator subgroup. Assume that the commutator factor group $G/\gamma_2(G)$ is of type $(3,3)$, the subgroup G^3 of G generated by the 3-th powers is contained in the commutator group $\gamma_2(G)$, which therefore coincides with the Frattini subgroup

$$\Phi(G) = \bigcap_{j=1}^{j=4} M_j = G^3\gamma_2(G) = \gamma_2(G),$$

where M_j, $1 \leq j \leq 4$ are the maximal normals subgroups of G. According to the basis theorem of Burnside, the group G is generated by two elements.
We define the lower central series of G recursively by

$$\begin{cases} \gamma_1(G) = G, \\ \gamma_j(G) = [\gamma_{j-1}(G), G], \quad \text{for } j \geq 2, \end{cases}$$

then we have Kaloujnine's commutator relations

$$[\gamma_i(G), \gamma_j(G)] \subseteq \gamma_{i+j}(G), \text{ for } i, j \geq 1,$$

and for a certain index of nilpotence $m \geq 2$ the series

$$\gamma_1(G) \supsetneq \gamma_2(G) \supsetneq .... \supsetneq \gamma_{m-1}(G) \supsetneq \gamma_m(G) = 1$$

becomes stationary. The number of non-trivial factors $\gamma_j(G)/\gamma_{j+1}(G)$ is called the class of G noted by $\mathrm{cl}(G) = m - 1$.
The group G of order 3^n is of maximal class if and only if $n = m$. In this case G is of coclass $\mathrm{cc}(G) = n - \mathrm{cl}(G) = 1$. On the other hand, for $2 \leq j \leq m-1$, the centralisers

$$\chi_j(G) = \{g \in G \,|\, [g, u] \in \gamma_{j+2}(G), (\forall\, u \in \gamma_j(G))\},$$

of two-step factor groups $\gamma_j(G)/\gamma_{j+2}(G)$ of the lower central series,

$$\chi_j(G)/\gamma_{j+2}(G) = \mathrm{Centraliser}_{G/\gamma_{j+2}(G)}\left(\gamma_i(G)/\gamma_{i+2}(G)\right).$$

According to [15], [18] or [19], when $\mathrm{cc}(G) = 1$, we have

$$\gamma_2(G) \subsetneq \chi_2(G) = \chi_3(G) = ... = \chi_{m-2}(G) \subsetneq \chi_{m-1}(G) = G.$$

Theorem 3.1. *With a prime $p \geq 2$, let G be a p-group of order $|G| = p^n$ and class $\mathrm{cl}(G) = n-1$, where $n \geq 2$. Suppose that the commutator group $\gamma_2(G)$ is abelian and the commutator factor group $G/\gamma_2(G)$ is of type (p,p). Let generators of $G = \langle x, y\rangle$ be selected such that $x \in G \setminus \chi_2(G)$, if $n \geq 4$, and $y \in \chi_2(G) \setminus \gamma_2(G)$. Assume that the order of the maximal normal subgroups $M_i = \langle g_i, \gamma_2(G)\rangle$ is defined by $g_1 = y$ and $g_i = xy^{i-2}$ for $2 \leq i \leq p+1$. Finally, let the invariant k of G be declared by $[\chi_2(G), \gamma_2(G)] = \gamma_{n-k}(G)$, where $k = 0$ for $n \leq 3$, $0 \leq k \leq n-4$*

for $n \geq 4$ *and* $0 \leq k \leq \min\{n-4, p-2\}$ *for* $n \geq p+1$.
Then the order of the commutator factor groups of $M_1, \ldots, M_{p+1}$ *is given by*

(1) $|M_i/\gamma_2(M_i)| = p$ *for* $1 \leq i \leq p+1$, *if* $n = 2$;
(2) $|M_i/\gamma_2(M_i)| = p^2$ *for* $2 \leq i \leq p+1$, *if* $n \geq 3$;
(3) $|M_1/\gamma_2(M_1)| = p^{n-k-1}$, *if* $n \geq 3$.

Proof. See [15, Theorem 3.1]. □

Theorem 3.2. *Let* G *be a metabelian* 3*-group of coclass* $cc(G) \geq 2$ *with order* $|G| = 3^n$, *class* $\mathrm{cl}(G) = m-1$, *and invariant* $e = n-m+2 \geq 3$, *where* $4 \leq m < n \leq 2m-3$. *Suppose that the commutator factor group* $G/\gamma_2(G)$ *is of type* $(3,3)$. *Let generators of* $G = \langle x, y\rangle$ *be selected such that*

$$\gamma_3(G) = \langle x^3, y^3, \gamma_4(G)\rangle,\ x \in G \setminus \chi_s(G),\ \text{if } s < m-1,\ \text{and } y \in \chi_s(G) \setminus \gamma_2(G).$$

Assume that the order of the maximal normal subgroups $M_i = \langle g_i, \gamma_2(G)\rangle$ *is defined by* $g_1 = y$, $g_2 = x$, $g_3 = xy$, $g_4 = xy^{-1}$. *Finally, let the invariant* k *of* G *be declared by* $[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G)$, *where* $k = 0$ *for* $m = 4$ *and* $0 \leq k \leq 1$ *for* $m \geq 5$.
Then the order of the commutator factor groups of $M_1, \ldots, M_4$ *is given by*

(1) $|M_1/\gamma_2(M_1)| = 3^{m-k-1}$,
(2) $|M_2/\gamma_2(M_2)| = 3^e$,
(3) $|M_i/\gamma_2(M_i)| = 3^3$, *for* $3 \leq i \leq 4$.

Proof. See [15, Theorem 3.3] □

Lemma 3.3. *Let* G *be a* 3*-group of order* $|G| = 3^n$, $n \geq 3$. *Assume that the commutator group* $\gamma_2(G)$ *is abelian, and the commutator factor group* $G/\gamma_2(G)$ *is of type* $(3,3)$. *Then* G *is of maximal class if and only if* G *has at least three maximal normal subgroups with the order of the commutator factor groups is* 3^2.

Proof. Assume that G is of maximal class, then by theorem 3.1, we conclude that G has three maximal normal subgroups with the order of commutator factor is 9 if $n \geq 4$, and has four when $n = 3$.
Conversely, Assume that $\mathrm{cc}(G) \geq 2$, the invariant $e = n-m+1$, where $\mathrm{cc}(G) = m$, is greater than 3, and $m-k-1 \geq 3$, where the invariant k is defined by $[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G)$ (see [15]). According to theorem 3.2 we deduce that the order of the factor commutator of its maximal normal subgroups is greater than 27. □

The groups of maximal class are parameterized as follows.

Theorem 3.4. *Let* G *be a* 3*-group of maximal class, then* G *is one of the parametric groups* $G^{(m)}(\alpha, \beta, \gamma) = \langle x, y\rangle$, *where* x, y *are selected as theorem 3.1, and* α, β, γ *are defined as follows:*
We define the commutator $s_2 = [y, x] \in \gamma_2(G)$ *and the higher commutators*

$$s_j = [s_{j-1}, x] = s_{j-1}^{x-1} \in \gamma_j(G) \text{ for } j \geq 3.$$

The group G *satisfies the following relations:*

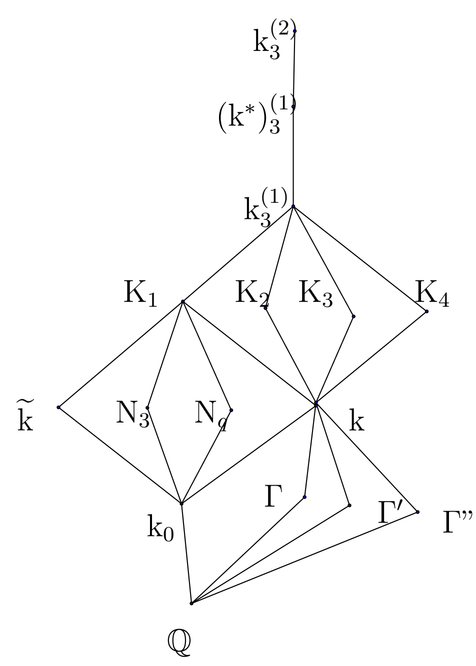
(1) $s_i^3 s_{i+1}^3 s_{i+2} = 1$, *if* $i \geq 2$,

(2) $x^3 = s_{m-1}^{\alpha},$ *with* $\alpha \in \{0,1\}$,
(3) $y^3 s_2^3 s_3 = s_{m-1}^{\beta},$ *with* $\beta \in \{0,1\}$,
(4) $[s_i, x] = s_{i+1},$ *if* $i \geq 2$,
(5) $[s_i, s_j] = 1,$ *if* $i, j \geq 2$,
(6) $[s_2, y] = s_{m-1}^{\gamma},$ *with* $\gamma \in \{0,1\}$,
(7) $[s_i, y] = 1,$ *if* $i \geq 3$.

The following notation will be used throughout the remainder of this paper:

- For a number field F, $\mathrm{F}_3^{(1)}$ denote the 3-Hilbert class field of F, $\mathrm{Cl}_3(\mathrm{F})$ his 3-class group and $\mathrm{h}_3(\mathrm{F})$ his 3-class number;
- q a prime number such that $q \equiv -1 \pmod 9$;
- $\mathrm{k}_0 := \mathbb{Q}(\zeta_3)$ where ζ_3 is the primitive 3-th root of unity;
- $\Gamma := \mathbb{Q}(\sqrt[3]{3q^2})$ be a pure cubic field;
- $\Gamma' := \mathbb{Q}(\zeta_3 \sqrt[3]{3q^2})$ and $\Gamma'' := \mathbb{Q}(\zeta_3^2 \sqrt[3]{3q^2})$;
- $\mathrm{k} := \mathbb{Q}(\zeta_3, \sqrt[3]{3q^2})$ the normal closure of Γ;
- $\widetilde{\mathrm{k}} := \mathbb{Q}(\zeta_3, \sqrt[3]{3q})$;
- $\mathrm{N}_3 := \mathrm{k}_0(\sqrt[3]{3})$; $\mathrm{N}_q := \mathrm{k}_0(\sqrt[3]{q})$;
- $\mathrm{K}_1 := (\mathrm{k}/\mathrm{k}_0)^{(*)}$ the genus field of k over k_0;
- $\mathrm{H} := \mathrm{Gal}\left((\mathrm{K}_1)_3^{(1)}/\mathrm{k}\right)$; $\widetilde{\mathrm{H}} := \mathrm{Gal}\left((\mathrm{K}_1)_3^{(1)}/\widetilde{\mathrm{k}}\right)$;
- $\mathrm{H}_3 := \mathrm{Gal}\left((\mathrm{K}_1)_3^{(1)}/\mathrm{N}_3\right)$; $\mathrm{H}_q := \mathrm{Gal}\left((\mathrm{K}_1)_3^{(1)}/\mathrm{N}_q\right)$.

Let Γ, k and $\widetilde{\mathrm{k}}$ as above. And suppose that k and $\widetilde{\mathrm{k}}$ has 3-class group of type $(3,3)$, thus $\mathrm{k}_3^{(1)}/\mathrm{k}$ possessing four unramified cyclic cubic extensions noted $\mathrm{K}_1, \ldots, \mathrm{K}_4$ where K_1 is the relative genus field of k over k_0. And we have the figure of fields as following



Let σ be a generator of $\mathrm{Gal}\,(\mathrm{k}/\mathrm{k}_0)$, then $\mathrm{Gal}\,(\mathrm{K}_1/\mathrm{k}) \simeq \mathrm{Cl}_3(\mathrm{k})/\mathrm{Cl}_3(\mathrm{k})^{1-\sigma}$, where $\mathrm{Cl}_3(\mathrm{k})$ is the 3-class group of k, and $\mathrm{Cl}_3(\mathrm{k})^{1-\sigma} = \{\mathfrak{a}^{1-\sigma} \mid \mathfrak{a} \in \mathrm{Cl}_3(\mathrm{k})\}$. Furthermore $\mathrm{Cl}_3(\mathrm{k})/\mathrm{Cl}_3(\mathrm{k})^{1-\sigma}$ is an elementary abelian 3-group, because $\mathrm{Cl}_3(\mathrm{k})^3 \subseteq \mathrm{Cl}_3(\mathrm{k})^{1-\sigma}$.

In Hilbert's theory, an ambiguous ideal class $\mathfrak{a}$ of k/k_0 is one for which $\mathfrak{a}^\sigma = \mathfrak{a}$. Let $\mathrm{Cl}_3(\mathrm{k})^{(\sigma)} = \{\mathfrak{a} \in \mathrm{Cl}_3(\mathrm{k}) \mid \mathfrak{a}^\sigma = \mathfrak{a}\}$, the set of all ambiguous ideal classes, which is subgroup of $\mathrm{Cl}_3(\mathrm{k})$. Since the 3-class group of k_0 is trivial by [6], the rank of $\mathrm{Cl}_3(\mathrm{k})/\mathrm{Cl}_3(\mathrm{k})^{1-\sigma}$ is the rank of $\mathrm{Cl}_3(\mathrm{k})^{(\sigma)}$. Moreover is an elementary abelian 3-group.

For a number field F, with $\mathrm{Cl}_3(\mathrm{F}) = \{1\}$, and K be an extension of F. In [9], Hasse specifies rank $\mathrm{Cl}_3(\mathrm{K})^{(\sigma)}$, as follows:

$$\mathrm{rank}\mathrm{Cl}_3(\mathrm{K})^{(\sigma)} = d + q^* - (r + 1 + o).$$

Where

- $d =$ number of ramified primes in K/F,
- $r =$ rank of the free abelian part of the group of units E_{F} of F,
- $o = 1$ or 0 according as F contains a primitive 3-th root o f unity or not,
- q^* is defined by $[V_{\mathrm{F}^*} : E_{\mathrm{F}}^3] = 3^{q^*}$, where $V_{\mathrm{F}^*} = \{x \in E_{\mathrm{F}} \mid x = N_{\mathrm{K}/\mathrm{F}}(y), y \in \mathrm{K} - \{0\}\}$. Here $N_{\mathrm{K}/\mathrm{F}}$ is the relative norm from K to F.

Theorem 3.5. *Let* k *and* $\widetilde{\mathrm{k}}$ *as above. And suppose that* k *and* $\widetilde{\mathrm{k}}$ *has* 3*-class group of type* $(3,3)$*. Then*

1. *The extension* k/k_0 *and* $\widetilde{\mathrm{k}}/\mathrm{k}_0$ *are the same genus field noted by* K_1*. And* $\mathrm{K}_1 = (\mathrm{k}/\mathrm{k}_0)^{(*)} = \mathrm{k}(\sqrt[3]{3}) = \mathrm{k}(\sqrt[3]{q})$ *is bicubic bicyclic over* k_0*.*
2. *Let denote* $G_1 = \mathrm{Gal}\left((\mathrm{K}_1)_3^{(1)}/\mathrm{k}_0\right)$*, then* G_1 *is metabelian* 3*-group of maximal class* m*,* $m \geq 3$*, i.e.,* $G_1 \in ZEF(m,m)$*.*
3. $\chi_2(G_1) = H_3$*, where* $\chi_2(G_1)$ *is the carcateristic group of* G_1*.*
4. *The transfers from* H_3 *and* H *to* $\gamma_2(G_1)$ *is trivial.*

Proof.

(1) First we calculate the rank of ambiguous classes under the action of $\mathrm{Gal}(\mathrm{k}/\mathrm{k}_0) = \langle\sigma\rangle$, we have

$$\mathrm{rank}(\mathrm{Cl}_3(\mathrm{k})^{(\sigma)}) = d + q^* - (r + 1 + o) = 1,\ (d = 2,\ q^* = 1,\ r = 1 \text{ and } o = 0).$$

Since the ambiguous classes in this situations are in

$$\mathrm{Cl}_3(\mathrm{k})^- = \{\mathfrak{a} \in \mathrm{Cl}_3(\mathrm{k}) | \mathfrak{a}^\tau = \mathfrak{a}^{-1}\},$$

by Duality theory (See [7, 8]), we deduce that the generator of the genus field of $\mathrm{K}_1 = (\mathrm{k}/\mathrm{k}_0)^{(*)}$ is an element of $(\mathrm{k}_0/\mathrm{k}_0^3)^+$ and K_1 is a cyclic cubic extension given by

$$\mathrm{K}_1 = \mathrm{k}(\sqrt[3]{q}) = \mathrm{k}(\sqrt[3]{3}),$$

and $\mathrm{K}_1/\mathrm{k}_0$ is a bicubic bicyclic extension. In the same way as above , we show that the genus field of $\widetilde{\mathrm{k}}$ over k_0 is K_1.

(2) Put $G_1 = \mathrm{Gal}((\mathrm{K}_1)_3^{(1)}/\mathrm{k}_0)$. Then we have $\gamma_2(G_1) = \mathrm{Gal}((\mathrm{K}_1)_3^{(1)}/\mathrm{K}_1)$, thus $G/\gamma_2(G) \simeq \mathrm{Gal}(\mathrm{K}_1/\mathrm{k}_0) \simeq (3,3)$. and $\gamma_2(G_1)$ is abelian, So $G_1 \in ZEF(m,n)$.

On the other hand, H is a maximal normal subgroup of G_1, and $\gamma_2(H) = \mathrm{Gal}\left((\mathrm{K}_1)_3^{(1)}/\mathrm{k}_3^{(1)}\right)$, then it follows that $H/\gamma_2(H) = \mathrm{Gal}(\mathrm{k}_3^{(1)}/\mathrm{k})$. Moreover by class field theory $\mathrm{Gal}(\mathrm{k}_3^{(1)}/\mathrm{k})$ is isomorphic to $\mathrm{Cl}_3(\mathrm{k})$ which is of type $(3,3)$ (By assumption), thus $|H/\gamma_2(H)| = 9$. The lemma 3.3 implies that $G_1 \in ZEF(m,m)$, means G_1 is of maximal class.

(3) We have G_1 is of maximal class and by assumption we have $\mathrm{h}_3(\mathrm{k}) = \mathrm{h}_3(\widetilde{\mathrm{k}}) = 9$. Then $\chi_2(G_1) = H_3$.

(4) Since the class number of N_3 is one, then the transfers from H_3 to $\gamma_2(G_1)$ is trivial.

For the transfers from H to $\gamma_2(G_1)$. By hypothesis the group $\mathrm{Cl}_3(\mathrm{k})$ is of type $(3,3)$, and generated by $\{\mathfrak{I}, \mathfrak{Q}\}$, where $\mathfrak{I}$ and $\mathfrak{Q}$ are ambiguous class under the action of K/k_0, (you can found this in [12]). Moreover all ambiguous classes capitulates in the genus field, which prove that we have total capitulation in K_1, means that the transfers from H to $\gamma_2(G_1)$ is trivial.

□

Corollary 3.6. *Let* $G_1 = \mathrm{Gal}\left((\mathrm{K}_1)_3^{(1)}/\mathrm{k}_0\right)$ *and assume that* $\mathrm{Cl}_3(\mathrm{k})$ *is of type* $(3,3)$*, then* $G_1 = G^{(4)}(0,1,0)$.

Proof. If $m \geq 5$, by [18] we conclude that $\gamma_3(G_1) = \langle y^3, \gamma_4(G_1)\rangle$, but the transfers from $\chi_2(G_1) = \langle y, \gamma_2(G_1)\rangle$ to $\gamma_2(G_1)$ is trivial then $y^3 = 1$ and we deduce that $\gamma_3(G_1) = \gamma_4(G_1)$ means that $m \leq 4$. And we conclude that $m = 4$.

On the other hand, according the same theorem 3.5, the transfers from H_3 to $\gamma_2(G_1)$ is trivial. But $V = V_{H_3/\gamma_2(G_1)}$ is given by $V(y) = y^3$, and $y^3 = s_2^{-3}s_3^{-1}s_3^{\beta}$. Since $s_2^{-3} \in \gamma_4(G) = 1$, we conclude that $1 = y^3 = s_3^{-1}s_3^{\beta}$ then $\beta = 1$.
We have $H = \langle x, \gamma_2(G)\rangle$, by theorem 3.4, the transfers from H to $\gamma_2(G)$ is trivial then $1 = V(x\gamma_2(H)) = x^3$. But the relation $(3): \ x^3 = s_3^{\alpha}$ of characterization of groups of maximal class imply that $\alpha = 0$, which terminate the proof. □

Remark 3.7.

(1) The transfers from G_1 to H, $\widetilde{H}$ and H_q is trivial. Because, we have G_1 is of maximal class, and $\chi_2(G_1) = H_3$.

(2) We can prove that the caracteristic group is H_3 by proving that $|\mathrm{Cl}_3(\mathrm{K}_1)^{(\sigma_1)}| = 9$. To prove this, we use the decomposition of ideals in Γ and k (see section 2).

4. Applications (Capitulation and class field tower)

Let k and $\widetilde{\mathrm{k}}$ as above. In this section we are interested in type of capitulation of ideals classes of $\mathrm{Cl}_3(\mathrm{k})$ in the 3-class groups $\mathrm{Cl}_3(\mathrm{K}_i)$ $1 \leq i \leq 4$ where $\mathrm{K}_1, \ldots, \mathrm{K}_4$ are the four unramified cyclic cubic extension of k.

Let $i \in \{1, \ldots, 4\}$, We say that an ideal class of k capitulates in K_i if it is in the kernel of the homomorphism

$$j_{\mathrm{K}_i/\mathrm{k}} : \mathrm{Cl}_3(\mathrm{k}) \to \mathrm{Cl}_3(\mathrm{K}_i)$$

induced by the extension of ideals from k to K_i. We define the multiplet $\chi \in [0,4]^4$ of capitulation types of k, for $i \in \{1, \dots, 4\}$ by

$$\ker(j_{\mathrm{K}_i/\mathrm{k}}) = N_{\mathrm{K}_{\chi(i)}/\mathrm{k}}(\mathrm{Cl}_3(\mathrm{K}_{\chi(i)})),$$

if $1 \leq i \leq 4$, that is, for partial capitulation (Or principalization), and we put $\chi(i) = 0$ for total capitulation, $\ker(j_{\mathrm{K}_i/\mathrm{k}}) = \mathrm{Cl}_3(\mathrm{k})$.

Proposition 4.1. *Let H, G_1 be groups as in theorem 3.5. Then The group H has fours normals maximus subgroups, noted by U_1, U_2, U_3 and U_4. Furthermore if K_1, K_2, K_3 and K_4 the associate fields by the correspondence of Galois group, where $\mathrm{K}_1 = (\mathrm{k}|\mathrm{k}_0)^*$, then the group G_1/H permutes cyclically the fields K_2, K_3 and K_4.*

Proof. We have $G_1 = \langle s, s_1 \rangle$ and $H = \langle s, \gamma_2(G_1) \rangle$.
The abelianaised $H/\gamma_2(H)$ is of type $(3,3)$ and the fours normals maximus subgroups of H are ordered as $U_1 = \gamma_2(G_1) = \langle s_2, \gamma_3(G_1) \rangle$, $U_2 = \langle s, \gamma_3(G_1) \rangle$, $U_3 = \langle ss_2, \gamma_3(G_1) \rangle$ and $U_4 = \langle ss_2^3, \gamma_3(G_1) \rangle$. (Note that $\gamma_3(G_1) = \langle s_3 \rangle$ cyclic of order 3.) By Galois theory these subgroups corresponds of the fields $\mathrm{K}_1, \dots, \mathrm{K}_4$ respectively. Moreover we have

$$s^{s_1} = s_1^{-1} s s_1 = [s_1, s]s = s_2 s$$

and

$$(s_2 s)^{s_1} = s_1^{-1} s_2 s s_1 = s_2 s_1^{-1} s s_1 s^{-1} s = s_2 [s_1, s] s = s_2^2 s$$

and

$$(s_2^2 s)^{s_1} = s_1^{-1} s_2^2 s s_1 = [s_1, s_2] s_2^{-1} s_2 s = s_3 s \in U_2$$

Hence we have $U_2^{s_1} = U_3$, $U_3^{s_1} = U_4$ and $U_4^{s_1} = U_2$. We conclude that G_1/H permutes cyclicly the fields K_2, K_3 and K_4. □

Theorem 4.2. *Let k and $\widetilde{\mathrm{k}}$ as above, and suppose that k and $\widetilde{\mathrm{k}}$ has 3-class group of type $(3,3)$. Let $\mathrm{k}_3^{(2)}$ be the second 3-Hilbert class field of k and $\mathrm{K}_1 = (\mathrm{k}|\mathrm{k}_0)^*, \dots, \mathrm{K}_4$ are the four unramified cyclic cubic extensions of k. Then*

(1) $\mathrm{Gal}(\mathrm{k}_3^{(2)}/\mathrm{k})$ *is of maximal class. And $\mathrm{k}_3^{(2)} = \mathrm{k}_3^{(3)}$, where $\mathrm{k}_3^{(3)}$ the 3-Hilbert class field of $\mathrm{k}_3^{(2)}$.*
(2) *The principalization of k in $\mathrm{K}_1, \dots, \mathrm{K}_4$ is total, i.e., there are one type of capitualtion which is $(0,0,0,0)$.*

Proof.

(1) The Group $M = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{K}_1\right)$ is a maximal normal subgroup of $G = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{k}\right)$, with

$$\begin{cases} \gamma_2(M) = \gamma_2\left(\mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{K}_1\right)\right) = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{K}_1^{(1)}\right), \\ M/\gamma_2(M) = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{K}_1\right)/\mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{K}_1^{(1)}\right) \simeq \mathrm{Gal}\left(\mathrm{K}_1^{(1)}/\mathrm{K}_1\right). \end{cases}$$

Then $|M/\gamma_2(M)| = 9$, and we conclude by lemma 3.3 that $G = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{k}\right)$ is of maximal class.

On the other hand, put $R = \mathrm{Gal}\left(\mathrm{k}_3^{(3)}/\mathrm{k}\right)$ then $\gamma_2(R) = \mathrm{Gal}\left(\mathrm{k}_3^{(3)}/\mathrm{k}_3^{(1)}\right)$ and $R'' = \gamma_2(\gamma_2(R)) = \mathrm{Gal}\left(\mathrm{k}_3^{(3)}/\mathrm{k}_3^{(2)}\right)$. Hence $R/\gamma_2(R) \simeq \mathrm{Gal}\left(\mathrm{k}_3^{(1)}/\mathrm{k}\right) \simeq (3,3)$, according to [3], R can be generated by too elements.
Moreover

$$\gamma_2(R)/R'' \simeq \mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{k}_3^{(1)}\right) \simeq \mathrm{Cl}_3(\mathrm{k}_3^{(1)}),$$

since G is of maximal class, the rank $(\gamma_2(R)/R'') = \mathrm{rank}\ (\mathrm{Cl}_3(\mathrm{k}_3^{(1)})) \leq 2$, consequently $\gamma_2(R)$ can be generated by too elements. According to [4], we conclude that $\gamma_2(R)$ is abelian and $R'' = 1$ which means that $\mathrm{k}_3^{(3)} = \mathrm{k}_3^{(2)}$.

(2) Since $\mathrm{Gal}\left(\mathrm{k}_3^{(2)}/\mathrm{k}\right)$ is of maximal class, according to [17], we conclude that the types of principalization are $(0,0,0,0)$, $(1,0,0,0)$, $(2,0,0,0)$ or $(1,1,1,1)$ when his nilpotence class is 3. On the other hand the assertions (3) of theorem 3.4 prove that $(1,1,1,1)$ is not possible. Moreover, By proposition 4.1, G_1/H permutes cyclically the fields K_2, K_3 and K_4, This allows us to say, that the three fields posses the same number of classes which capitulate. Since in K_1 we have total capitulation, we conclude that the single possible type of principalization is $(0,0,0,0)$.

□

Example 4.3.

q	$h_3(\Gamma)$	$h_3(\mathrm{k})$	$h_3(\widetilde{\mathrm{k}})$	$h_3(\mathrm{K}_1)$	$\mathrm{Cl}_3(\mathrm{k})$	$\mathrm{Cl}_3(\widetilde{\mathrm{k}})$	$\mathrm{Cl}_3(\mathrm{K}_1)$
89	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
431	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
449	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
593	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
647	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
683	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
719	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
773	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
1151	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
1277	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
1367	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
1493	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
1583	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
1709	3	9	9	9	[3, 3]	[3, 3]	[3, 3]
1997	3	9	9	9	[3, 3]	[3, 3]	[3, 3]

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