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ON SOME METABELIAN 3− GROUPS AND APPLICATIONS I

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Abstract. Let G be a $3-$class group of maximal class, and $\gamma_2(G) = [G, G]$ its derived group. Assume that the commutator factor group $G/\gamma_2(G)$ is of type $(3, 3)$ and the transfers $V_{\chi_2(G)\to\gamma_2(G)}$ and $V_{H\to\gamma_2(G)}$ are trivial, where $\chi_2(G)$ is the biggest subgroup of G such that $[\chi_2(G), \gamma_2(G)] \subseteq \gamma_4(G)$, and, H is one of its maximal normals subgroups different to $\chi_2(G)$. Then G is completely determined with the isomorphism class groups of maximal class defined by B.Nebelung in [24]. Moreover the group G is realised. At the end numerical examples illustrating the results are given.

1. Introduction

Let G be an metabelian $p-$group of order p^n, $n \geq 4$, with abelianisation $G|\gamma_2(G)$ is of type (p, p), where $\gamma_2(G) = [G, G]$ is the commutator group of G. According to the basis theorem of Burnside, the group G can thus generated by two elements, put $G =< x, y >$. We define the lower central series of G recursively by

$$\begin{cases} \gamma_1(G) = G, \\ \gamma_j(G) = [\gamma_{j-1}(G), G], \quad \text{for } j \geq 2.. \end{cases}$$

Then we have Kaloujnine's commutator relation $[\gamma_i(G), \gamma_j(G)] \subseteq \gamma_{i+j}(G)$ for $i, j \geq 1$ and for an index of nilpotence $m \geq 2$ the series

$$\gamma_1(G) \supset \gamma_2(G) \supset ..... \supset \gamma_{m-1}(G) \supset \gamma_m(G) = 1.$$

The centralisers

$$\chi_2(G) = \{g \in G | [g, u] \in \gamma_4(G) \text{ for all } u \in \gamma_4(G)\}$$

that is the biggest subgroups of G with $[\chi_j(G), \gamma_j(G)] \subset \gamma_{j+2}(G)$. Assume that G is of maximal class, then x and y are selected such that $x \in G \setminus \chi_2(G)$ and $y \in \chi_2(G) \setminus \gamma_2(G)$ and x, y satisfies the relations cited in Theorem 2.1.
The central results of this paper concern $3-$group having two specials normals maximal subgroups noted H_1, H_2 such that

$$V_{H_i\to\gamma_2(G)} : H_i|\gamma_2(H_i) \longrightarrow \gamma_2(G),\ i \in \{1, 2\}$$

are trivial, means that

$$\ker\left(V_{H\to\gamma_2(G)}\right) = H|\gamma_2(H),$$

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Where $H = H_1$ or $H = H_2$. And we develop a general and new method to determine all invariants of G, our investigation is based on papers [24, 3, 4, 20, 21, 22]. In this case we show hat the group G is isomorphic to $G^{(4)}(0,1,0)$, $G \sim G^{(4)}(0,1,0)$, where the presentation $G^{(m)}(\alpha,\beta,\gamma)$ is due to Nebelung [24], consequently the isomorphism invariant$k = k(G)$ defined by $[\chi_2(G), \gamma_2(G)] = \gamma_{m-k}(G)$ is equal to $k = 0$, moreover the group G is in isoclinism families Φ_3 developed by P.Hall in [10].
Note that, this work is a part of a project with [A.Derhem, MM.Talbi, A.Azizi, D.C.Mayer] in which we are interested in the determination of the $p-$group of class not necessarily of maximal class and the location of G in coclass graph in the sense of Eick and Leedham-Green, as in the paper of the author D.C.Mayer [22].
Furthermore we are interested in the problem:

"whether there exists a field k *with Galois group* $\mathrm{Gal}\left(\mathrm{k}_p^{(2)}/\mathbb{Q}\right)$ *isomorphic to* G*".*

Where $\mathrm{k}_p^{(2)}$ be the Hilbert $p-$class field of the Hilbert $p-$class field $\mathrm{k}_p^{(1)}$ of k.

In [25] A.Numera has studied a similarly problem which noted by $P(F,\Gamma)$, formulated as follows:

$P(F,\Gamma)$: *For a given Galois extension $F/\mathbb{Q}$ and finite group Γ, does there exists a Galois extension $M/F/\mathbb{Q}$ satisfying the conditions:*

(1) $\mathrm{Gal}(M/F)$ *is isomorphic to* Γ*;*
(2) *M/F is unramified?*

In [19], F. Lemmermeyer has conjectured that for any $2-$group Γ there exists a quadratic field F such that the answer to the problem $P(F,\Gamma)$ is affirmative, but this has been disproved by Boston and Leedham-Green in [6].
Here we prove that for any $3-$group of maximal class of order 3^n, $n \geq 4$ are realised by cubic cyclic k of conductor is divisible by two primes p and q such that:

$$(*)\left(\frac{p}{q}\right)_3 = \left(\frac{q}{p}\right)_3 = 1 \text{ and } h_3(\mathrm{k}) = 9,$$

where $h_3(\mathrm{k})$ is the $3-$class number of k. Or If $p = 3$ or $q = 3$, (for example q), the formula $(*)$ will be replaced by

$$\left(\frac{p}{9}\right)_3 = \left(\frac{9}{p}\right)_3 = 1.$$

Therefore $G_3^{(2)} = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}|\mathrm{k}\right)$ is an immediate predecessor of G on the coclass graph $\mathcal{G}(3,1)$, it can only be isomorphic to the extra special $3-$group $G^{(3)}(0,0,0)$ of exponent 3 on the main line.
For $n = 3$, the group G is isomorphic to $G^{(3)}(0,0,0)$, and $G_3^{(2)}$ is abelian.
Then for all case the capitulation problem in this case is solved completely.

2. Preliminary

2.1. **On the 3−class group of maximal class.** In this subsection, we recall some results about 3−group of maximal class, the detail can be found in [24, 20, 21, 22, 32].
Let G be a metabelian 3−group of order $|G| = 3^n$, $n \geq 4$. Assume that the commutator factor group $G/\gamma_2(G)$ of G is of type $(3,3)$. Then G admis four maximal normal subgroups $H_1, \dots H_4$, which contain the commutator group $\gamma_2(G)$ as a normal subgroup of index 3. Since the integer $n \geq 4$, the centraliser $\chi_2(G) = \{g \in G | [g,u] \in \gamma_4(G) \text{ for all } u \in \gamma_4(G)\}$ is one of the groups H_i. We fix $\chi_2(G) = H_1$. Therefore the Frattini subgroup

$$\Phi(G) = \cap_{i=1}^{4} H_i = G^3\gamma_2(G).$$

According the basis theorem of Burnside [[1],p.29,Th1.12], the group $G =< s, s_1 >$ can thus be generated by two elements are selected such that $s \in G \backslash \chi_2(G)$ and $s_1 \in \chi_2(G) \setminus \gamma_2(G)$ and we have the following theorem

Theorem 2.1. *LetG be an metbelian* $3-$ *group of order* $3^n, n \geq 4$ *with the abelianisation* $G/\gamma_2(G)$ *is of type* $(3,3)$*. Suppose that* $\mathrm{cl}(G) = n-1$*, thenG can be generted by two elements* $G =< s, s_1 >$ *are selected such that* $s \in G \setminus \chi_2(G)$ *and* $s_1 \in \chi_2(G) \setminus \gamma_2(G)$ *satisfying the following proprieties :*

(1) $s_i^3 s_{i+1}^3 s_{i+2} = 1$ *if* $i \geq 2$,
(2) $s^3 = s_{n-1}^{\alpha}$ *with* $\alpha \in \{0,1\}$,
(3) $s_1^3 s_2^3 s_3 = s_{n-1}^{\beta}$ *with* $\beta \in \{-1,0,1\}$,
(4) $[s_i, s] = s_{i+1}$ *if* $i \in \mathbb{N}$,
(5) $[s_i, s_j] = 1$, *if* $i, j \geq 2$,
(6) $[s_2, s_1] = s_{n-1}^{\gamma}$ *with* $\rho \in \{-1,0,1\}$,
(7) $[s_i, s_1] = 1$, *if* $i \geq 3$.

Where $s_2 = [s_1, s] \in \gamma_2(G)$ *and* $s_j = [s_{j-1}, s] = s_{j-1}^{s-1} \in \gamma_j(G)$ *for* $j \geq 3$

Proof. Voir [[24],p58] □

The four maximal normal subgroups $H_1 \dots H_4$ are arranged as follows $H_1 =< s_1, \gamma_2(G) >= \chi_2(G)$, $H_2 =< s, \gamma_2(G) >$, $H_3 =< ss_1, \gamma_2(G) >$ and $H_4 =< ss_1^2, \gamma_2(G) >$.
Therefore its commutator subgroups are given by:

$$\gamma_2(H_1) = \begin{cases} 1, & \text{if } k = 0; \\ \gamma_{m-1}(G), & \text{if } k = 1. \end{cases}$$

And for all i, $2 \leq i \leq 4$, we have: $\gamma_2(H_i) = \gamma_3(G)$, where $k = k(G)$ is the isomorphism invariant of G defined by

$$[\chi_2(G) : G_2] \subset \gamma_{n-k}(G)$$

.

Lemma 2.2. *Let* G *be a* $3-$*group of order* $|G| = 3^n\, n \geq 4$*. Assume that the commutator group* $\gamma_2(G)$ *is abelian, and the commutator factor group* $G/\gamma_2(G)$ *is of type* $(3,3)$*. Then* G *is of maximal class if and only if* G *has at least three maximal normal subgroups with the order of the commutator factor groups is* 3^2.

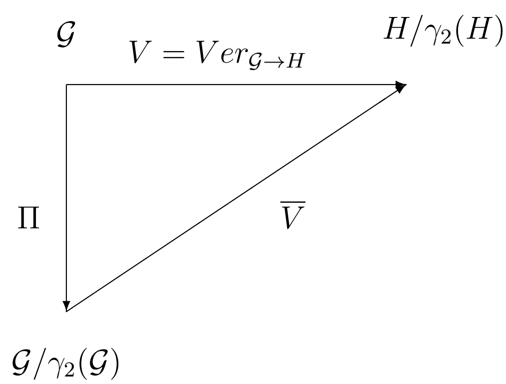
Proof. Assume that G is of maximal class, according to the maximal subgroup $H_2 \ldots H_4$, for i, $2 \leq i \leq 4$ the commutator factor group $\gamma_2(H_i) = \gamma_3(G)$ and we have $|H_i/\gamma_2(H_i)| = |H_i/\gamma_2(G)| = 9$, we conclude that G has three maximal normal subgroups with the order of commutator factor 9.
Conversely, Assume that $\mathrm{cc}(G) \geq 2$ where $\mathrm{cc}(G) = n - m + 1$, then the invariant $e = n - m + 2$ is greater than 3, and $m - k - 1 \geq 3$, where the invariant k is defined by $[\chi_s(G), \gamma_e(G)] = \gamma_{m-k}(G)$ [see [20]]. According to [[22], Theo 1.2 P.21], we deduce that the order of the factor commutator of its maximal normal subgroups is greater than 27. □

Lemma 2.3. *Let G be an metabelian $3-$group of maximal class, $m \geq 5$, with abelianisation $G/\gamma_2(G)$ is of type $(3,3)$. Let s, s_1 defined in the theorem 2.1 such that $G =< s, s_1 >$, then*

$$\gamma_3(G) =< s_1^3, \gamma_4(G) >$$

Proof. Voir [[24], P.120] □

2.2. **On the transfer concept.** Let $\mathcal{G}$ be a group and let $\mathcal{H}$ be a subgroup of $\mathcal{G}$.
The transfert from $\mathcal{G}$ to $\mathcal{H}$ can decomposed as follows: Also we note $\overline{V}$ by $V_{\mathcal{G}\to\mathcal{H}}$.



Definition 2.4. Let $\mathcal{G}$ be a group and let N a normal subgroup of $\mathcal{G}$, and let $g \in \mathcal{G}$ such that $\Theta(gN) = f$, $r = \frac{[G:N]}{f}$ and $g_1, \ldots, g_r$ be a representative system of $\mathcal{G}/N$, then the transfer from $\mathcal{G}$ to N, noted $V_{\mathcal{G}\to N}$, is defined by:

$$V_{\mathcal{G}\to N}: \begin{array}{ccc} \mathcal{G}/\gamma_2(\mathcal{G}) & \longrightarrow & N/\gamma_2(N) \\ g\gamma_2(\mathcal{G}) & \longmapsto & V_{\mathcal{G}\to N}(g\gamma_2(\mathcal{G})) = \prod_{i=1}^{r} g_i^{-1} g^f g_i \gamma_2(N) \end{array}$$

Special cases: Let $\mathcal{G}$ be a group and let N be an normal subgroup of $\mathcal{G}$.

(1) If $g \in N$, then $f = 1$ and $r = [\mathcal{G} : N]$ and $V_{\mathcal{G}\to N}(g\gamma_2(\mathcal{G}))$ is given by:

$$V_{\mathcal{G}\to N}(g\gamma_2(\mathcal{G})) = g^{\sum_{i=1}^{r} g_i}\gamma_2(\mathcal{G})$$

(2) If $\mathcal{G}/N$ is cyclic and if $g \in \mathcal{G}$ such that $\mathcal{G}/N =< gN >$, then $f = [\mathcal{G} : N]$ therefore also

$$V_{\mathcal{G}\to N}(g\gamma_2(\mathcal{G})) = g^f \gamma_2(N)$$

(3) In the case, when $g \in Z(\mathcal{G})$, we obtain

$$V_{\mathcal{G}\to N}(g\gamma_2(\mathcal{G})) = g^{[\mathcal{G}:N]}\gamma_2(N)$$

(4) Assume that $\mathcal{G}/N$ is cyclic group of order 3 and $\mathcal{G} =< h, N >$, then the transfer $V_{\mathcal{G}\to N}$ is defined
   (a) If $g \in N$, then $V_{\mathcal{G}\to N}\left(g\gamma_2(\mathcal{G})\right) = g^{1+h+h^2}\gamma_2(N) = g^3[g,h]^3[[g,h],h]\gamma_2(N)$
   (b) $V_{\mathcal{G}\to N}\left(h\gamma_2(G)\right) = h^3\gamma_2(N)$

Theorem 2.5. *Let $G =< s, s_1 >$ be a metabelian $3-$group of maximal class of order $|G| = 3^n$, where $m \geq 3$. Suppose that the generators of G are selected as in 2.1. Let $H_1, \ldots, H_4$ its four maximal normal subgroups ordered $H_1 =< s_1, \gamma_2(G) >$ and $H_i =< ss_1^{i-2} >$, $2 \leq i \leq 4$. Assume that the cosets $g\gamma_2(G) \in G/\gamma_2(G)$ are represented in the shape $g \equiv x^j y^l (\mathrm{mod}\gamma_2(G))$ avec $0 \leq j, l \leq 2$, then the images of the transfers are generally given by:*

$$V_{G\to H_i}\left(s^j s_1^l\right) = s_{n-1}^{\alpha j+\gamma l}\gamma_2(H_i) \text{ for } 1 \leq i \leq 4.$$

From the explicit form of the commutator groups $\gamma_2(H_i)$, they are given by:

$$V_{G\to H_1}\left(s^j s_1^l\right) = \begin{cases} s_{n-1}^{\alpha j+\gamma l}\gamma_2(H_i).1, & \text{if } [\chi_2(G),\gamma_2(G)] = 1, m \geq 3; \\ \gamma_{n-1}(G), & \text{if } [\chi_2(G),\gamma_2(G)] = \gamma_{n-1}(G), k = 1, n \geq 5. \end{cases}$$

And for $2 \leq i \leq 4$ we have:

$$V_{G\to H_i}\left(s^j s_1^l\right) = \begin{cases} s_2^{\alpha j+\gamma l}\gamma_2(H_i).1, & \text{if } , n = 3; \\ \gamma_3(G), & \text{if } , n \geq 4. \end{cases}$$

Proof. See [[22],Th 2.2, P 4] for $p = 3$. □

In Figure 1, vertices of coclass graph $\mathcal{G}(3;1)$ are classified according to their defect k by using different symbols:

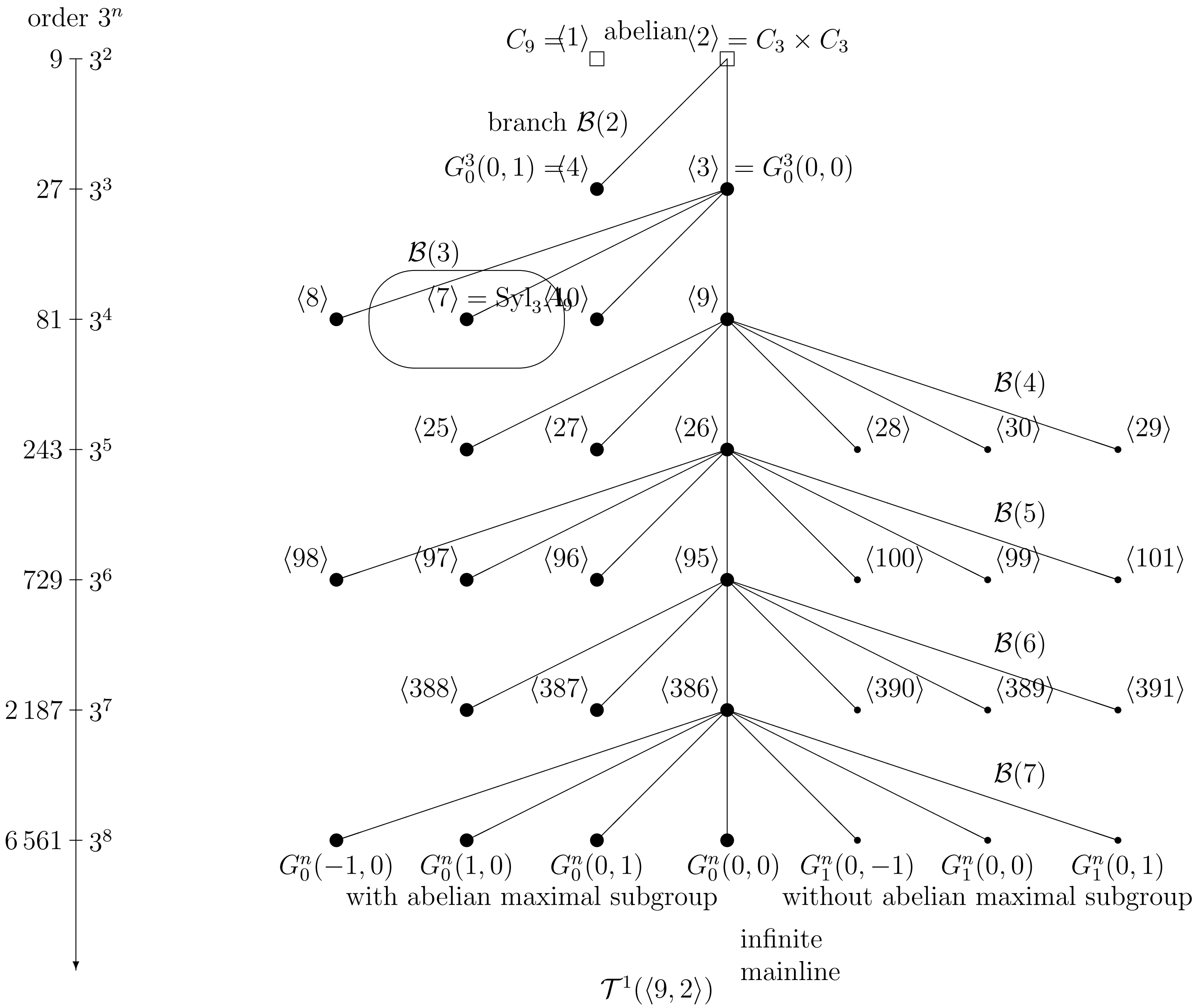
(1) large contour squares denote abelian groups,
(2) big full discs denote metabelian groups with abelian maximal subgroup and $k = 0$,
(3) small full discs denote metabelian groups with defect $k = 1$.

3. Main Results

In this section we begin with a purely group theoretic statement concerning the determination of the invariants of metabelian $3-$group of maximal class developed in the proof of the theorem 2.1 which illustrates the new method to solve capitulation problem in several cases. As a special case we applied this theory to cubic cyclic with conductor is divisible by two prime p and q satisfying

$$\left(\frac{p}{q}\right)_3 = \left(\frac{q}{p}\right)_3 = 1 \text{ and } \mathrm{h}_3(\mathrm{k}) = 9$$

where $\left(\frac{\cdot}{\cdot}\right)_3$ means the residue cubic symbol, and $\mathrm{h}_3(\mathrm{k})$ is the $3-$class number of k.



TKT	a.3	a.3	a.2	a.1	a.1	a.1	a.1
$\varkappa =$	(2000)	(2000)	(1000)	(0000)	(0000)	(0000)	(0000)

FIGURE 1. Roots $C_3 \times C_3$ and branches $\mathcal{B}_j, 2 \leq j \leq 7$, on the coclass graph $\mathcal{G}(3,1)$

In this case we consider the group $G = \mathrm{Gal}\left((\mathrm{k}^*)_3^{(1)}|\mathbb{Q}\right)$, where $\mathrm{k}^* = (\mathrm{k}/\mathrm{Q})^*$ is the genus absolute field.
We show that G is of maximal class and $G = \mathrm{Gal}\left(\mathrm{k}_3^{(2)}|\mathbb{Q}\right) \sim G^4(1,0,0) \sim Syl_3 A_9$.

3.1. **The invariants** α, β, γ, m, k **of metabelian** $3-$**class group of** $cc(G) =$ **1.**. Let G be $3-$group of maximal class of order 3^n $(n \geq 4)$ and nilpotency class

$\mathrm{cl}(G) = n-1$, with commutator factor group $G/\gamma_2(G)$ of G is of type $(3,3)$. Let $H_1 = \chi_2(G), \ldots, H_4$ its four maximal normal subgroups, which contain the commutator group $\gamma_2(G)$ as a normal subgroup of index 3, in the following theorem we understand how the transfer from $H_1 = \chi_2(G)$ and H_i, $i \neq 1$, to $\gamma_2(G)$ allows to determine the invariants of G.

Theorem 3.1. *Let G be a metabelian $3-$group of maximal class with commutator factor group $G/\gamma_2(G)$ is of type $(3,3)$ and α, β, γ, m, k its invariants. Let $H_1 \ldots H_4$ the four maximal normals subgroups of G which contain the commutator group $\gamma_2(G)$ as a normal subgroup of index 3 ordered as in theorem 2.1. Assume that $V_{H_1\to\gamma_2(G)}$ et $V_{H_2\to\gamma_2(G)}$ are trivial then: $\alpha = 0, \beta = 1, \gamma = 0, m = n = 4$ and $k = 0$, i.e $G \sim G^4(1,0,0) \sim Syl_3 A_9$. And G is a stem group of the isoclinism families Φ_3.*

Proof. Let G be an metabelian $3-$ group of maximal class, and let $\gamma_2(G)$ its commutator group such that $G/\gamma_2(G) \simeq (3,3)$. According to theorem 2.1, the generators of $G =< s, s_1 >$ are selected so that $s \in G \setminus \chi_2(G)$ and $s_1 \in \chi_2(G) \setminus \gamma_2(G)$. On the other hand the maximal maximal groups of G are given by: $\chi_2(G) = H_1 =< s_1, \gamma_2(G) >$, $H_2 =< s, \gamma_2(G) >, H_3 =< ss_1, \gamma_2(G) >$ et $H_4 =< ss_1^2, \gamma_2(G) >$.
Let the main commutator $s_2 = [s, s_1]$ and $\mu(u) = 3 + 3\delta_u + \delta_u^2 = 1 + u + u^2$ where $\delta_u = 1-u$.
Since $H_1 =< s_1, \gamma_2(G) >$ we deduce

$$V_{H_1\to\gamma_2(G)}(s_1) = s_1^3 = 1 \text{ and } V_{H_1\to\gamma_2(G)}(s_2) = s_2^{\mu(s_1)} = 1,$$

because the transfert from H_1 to $\gamma_2(G)$, $V_{H_1\to\gamma_2(G)}$, is trivial.
Means that $s_1^3 = 1$ and $s_2^3 s_3^{3\gamma} s_4^{\gamma} = 1$, ρ is the parameter declared in theorem 2.1
Also, since $H_2 =< s, \gamma_2(G) >$, and $V_{H_2\to\gamma_2(G)}$ is trivial, whence

$$V_{H_2\to\gamma_2(G)}(s) = s^3 = 1 \text{ and } V_{H_2\to\gamma_2(G)}(s_2) = s_2^{\mu(s)} = 1,$$

thus $s^3 = 1$ and $s_2^3 s_3^3 s_4 = 1$, note this assertion is always verified.

Assume that $n \geq 5$, according to lemma, it follows that

$$\gamma_3(G) =< s_1^3, \gamma_4(G) >= \gamma_4(G),$$

which is impossible. so $n = 4$, consecontly $\gamma_4(G) = 1$. Moreover, for $n \geq 4$, according to [[21],Theorem 3.1, P3], we have $0 \leq k \leq n-4$, then the invariant $k = k(G) = 0$.

For all $i \geq 2$, the higher commutators satisfies the relations $s_i^3 s_{i+1}^3 s_{i+2} = 1$, then $V_{H_3\to}G_2 = s_i^3 s_{i+1}^3 s_{i+2} = 1$, also $s^3 = s_{n-1}^{\alpha} = s_3^{\alpha}(n = 4)$, so $s^3 = s_3^{\alpha}$. On the other hand $s_1^3 s_2^3 s_3 = s_3^{\beta}$, since the order of $\gamma_2(G)$ is $|\gamma_2(G)| = 9$ with exponent 3 it follows that $s_2^3 = 1$. observing that $s_1^3 = 1$ we conclude that $s_3 = s_3^{\beta}$. Then we obtain

$$s_3^{\beta} = s_3 \text{ et } s^3 = s_3^{\alpha}.$$

Also

$$s_3^{\gamma} = [s_2, s_1] \in \gamma_4(G) = 1,$$

then $s_3^\gamma = 1$ and $\gamma = 0$. Since $s_3^\beta = s_3$ imply that $\beta = 1$.
Moreover $G \in \Phi_3$ isoclinism families.
Furthermore, since $s^3 = 1$, we deduce that $\alpha = 0$.
Finally $G \sim G^{(4)}(0, 1, 0)$. □

4. Application

Along this section section we adopt the following notation:

(1) p and q two different primes numbers,
(2) k is a cyclic cubic fields with conductor divided by pq,
(3) $\widetilde{\mathrm{k}}$ is the second cyclic cubic field with the same conductor as k,
(4) $\mathrm{k}^* = (\mathrm{k}/\mathbb{Q})^*$ the absolute genus field, that is the maximal unramified $3-$extension over k which is abelian over $\mathbb{Q}$,
(5) $(\mathrm{k}*)_3^{(1)}$ the absolute Hilbert class field of k^*, that is the maximal unramified $3-$ extension of k^*.

Proposition 4.1. *Let* k *be a cubic cyclic field with conductor* pq *and* k^* *its absolute genus field and let* $(\mathrm{k}*)_3^{(1)}$ *the Hilbert class field of* k^*. *Then*

(1) k^*/k *is a cyclic cubic extension, i.e* $[\mathrm{k}^* : \mathrm{k}] = 3$
(2) $\left(\frac{p}{q}\right)_3 = \left(\frac{q}{p}\right)_3 = 1$ *if and only if* $\mathrm{k}^* \neq (\mathrm{k}^*)_3^{(1)}$
(3) $\mathrm{h}_3(\mathrm{k}_p) = \mathrm{h}_3(\mathrm{k}_q) = 1$, *where* $\mathrm{h}_3(F)$ *denote* $3-$*class number of* F.

Proof. (1) Let $a_{\mathrm{k}|\mathbb{Q}}$ be the number of ambigous classes for the cyclic extension $\mathrm{k}|\mathbb{Q}$. By [Theorm 13 [9]], we have

$$a_{\mathrm{k}|\mathbb{Q}} = \frac{3^{t-1}}{[E_{\mathbb{Q}} : E_{\mathbb{Q}} \cap N_{\mathbf{k}|\mathbb{Q}}(\mathrm{k}^\times)]},$$

where t is the number of primes ideals of $\mathbb{Q}$ which are ramified in k. In our case, since the conductor of k is divided by two primes, then we obtain that $a_{\mathrm{k}|\mathbb{Q}} = 3$.
On the other hand, since $[\mathrm{k}^* : \mathrm{k}] = a_{\mathrm{k}|\mathbb{Q}}$, we conclude that $[\mathrm{k}^* : \mathrm{k}] = 3$.

(2) We apply the ambigous class formula when $F = \mathrm{k}_l$, where $l = p$ or q. If $\left(\frac{p}{q}\right)_3 = 1$ and $\left(\frac{q}{p}\right)_3 = 1$, we have p split completely in k_q, and also q split completely in k_p, then the ideals of k_p which ramified in k^* are those lying above q and the ideals of k_q which ramified in k^* are those lying above p, then $t = 3$, this imply that 3 divide $a_{\mathrm{k}^*|\mathrm{k}_l}$, where $l = p$ or q so 3 divide the class number of k^* and $\mathrm{k}^* \neq (\mathrm{k}*)_3^{(1)}$.
Conversely if $\mathrm{k}^* \neq (\mathrm{k}*)_3^{(1)}$, then 3 divide $a_{k^*|k_l}$, so that the integer $t \geq 3$. This imply that p split completely in k_q and also the the prime q split completely in k_q, which efforces that p and q satisfies $\left(\frac{p}{q}\right)_3 = \left(\frac{q}{p}\right)_3 = 1$.

(3) Let l be a prime number, $l = p$ or $l = q$. Assume that the $3-$ class numbers $\mathrm{h}_3(\mathrm{k}_l) \neq 1$, then there exists a unramified cyclic cubic extension of k_l, noted by F,. This extension is abelian over $\mathbb{Q}$ because his degree over $\mathbb{Q}$ is 9. Then F is contained in $(\mathrm{k}_l|\mathbb{Q})^*$, but by the ambigous formula we conclude that $(\mathrm{k}_l|\mathbb{Q})^* = \mathrm{k}_l$, which is impossible.

□

Theorem 4.2. *Let* k *be a cyclic cubic field with conductor* pq *and* $\widetilde{k}$ *the second cyclic cubic field with the same conductor as* k.
Let $G = Gal\left((k^*)_3^{(1)}/\mathbb{Q}\right)$, *where* $(k^*)_3^{(1)}$ *is the* 3−*Hilbert class field absolute of the genus field of* k.
Assume that the 3−*class number* $h_3(k)$ *is* $h_3(k) = 9$, *then we have the following assertions*

1. *The fields* k *and* $\widetilde{k}$ *are the same* 3−*class number which eguals to* 9. *And* $k^{(2)} = (k^*)_3^{(1)}$
2. *The group* G *is a metabelian group of maximal class. Therefore* $\chi_2(G) = H_p$ *or* $\chi_2(G) = H_q$ *where* H_p *and* H_q *are the normals subgroups of* G *defined previously.*
3. *The group* $Gal(k^{(2)}/\mathbb{Q}) \simeq G^{(4)}(0,1,0)$ *and* $Gal(k^{(2)}/k)$ *is the extraspecial group of order* 27.

Proof.
(1) For the first statement See [Prop 3.2, P 27 [3]], and for the second see [Theorem 3.2, P36,[3]].

(2) The commutator group of $\gamma_2(G) = [G, G]$ of $G = \text{Gal}\left((k^*)_3^{(1)}|\mathbb{Q}\right)$, correspond by Galois theory to the genus field, because it's the maximal unramified extension of k, which is abelian over $\mathbb{Q}$. Thus $\gamma_2(G) = \text{Gal}\left((k^*)_3^{(1)}|k^*\right)$, and we have

$$G|\gamma_2(G) \sim \text{Gal}\,(k^*|\mathbb{Q}) \sim (3,3).$$

Then the group G has four normals subgroups ordered as $H_q = \text{Gal}\left((k^*)_3^{(1)}|k_q\right)$, $H_p = \text{Gal}\left((k^*)_3^{(1)}|k_p\right)$, $H = \text{Gal}\left((k^*)_3^{(1)}|k\right)$ and $\widetilde{H} = \text{Gal}\left((k^*)_3^{(1)}|\widetilde{k}\right)$.
Since the 3−class numbers $h_3(k) = h_3(\widetilde{k}) = 9$, means that the orders of the factors commutators groups are $|H|\gamma_2(H)| = 9$, and $|\widetilde{H}|\gamma_2(\widetilde{H})| = 9$. Then by 2.2, we conclude that G is of maximal class.
On the other hand, the nilpotency class $n \neq 3$, and the defect $k = k(G) = 0$, we conclude that one factor commutator of maximal normals subgroups of G, have the order divisible by 27, this subgroup is different to H and $\widetilde{H}$. This subgroup coincides with the characteristic subgroup $\chi_2(G)$ of G. then $\chi_2(G) = H_p$ or $\chi_2(G) = H_q$.

(3) Assume $\chi_2(G) = H_q$, We now turn our attention to transfert concept. Let $\mathcal{U}_{k_q}$ and $\mathcal{U}_{k_p}$ two modulus which contain respectively the primes p and q, then the extension of ideals coprimes to p and q of k_l, $l = p \,\text{or}\, q$ to k^*,

$$j_{k^*|k_l} : I_{k_l, f_{k_l}}/\mathcal{U}_{k_l} \longrightarrow I_{k^*, f_{k^*}}/\mathcal{U}_{k^*}$$

is trivial, because the 3− class number $h_3(k_p) = h_3(k_p) = 1$.
By Artin reciprocity low, we deduce that the transfer from $V_{H_q \to \gamma_2(G)}$ and $V_{H_p \to \gamma_2(G)}$ are trivials. Then by applying 3.1, and the fact that $k^{(2)} = k_1^*$ we conclude that the group $Gal(k^{(2)}/\mathbb{Q}) \simeq G^{(4)}(0,1,0)$. Furthermore the second 3−group class $G_3^{(2)}(k) = \text{Gal}\left(k^{(2)}|k\right)$ is of order 27 of exponent 3 it

can only be isomorphic to the extra special 3−group $G^3(0,1,0) = G_0^3(0,1)$ on coclass graph $\mathcal{G}(3,1)$ on the main line.

□

Example 4.3. The cyclic cubic field k with conductor 2439, satisfies:

1. (1) The conductor is product of two prime $2439 = 3^2 \times 271$ such that $\left(\frac{9}{271}\right)_3 = \left(\frac{271}{9}\right)_3 = 1$.
2. (2) The group $G \sim G^{(4)}(0,1,0))$.

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