Gulf Journal of Mathematics
Vol 4, Issue 4 (2016) 28-32

ON SKEW DERIVATIONS IN 3-PRIME NEAR-RINGS

ABDELKARIM BOUA* AND LAHCEN TAOUFIQ‡

Abstract. In this paper, we study the relationship between the behavior of skew derivations satisfying certain local properties in near-rings. In particular our purpose is to extend the results of [1] and [5].

1. Introduction

Throughout this paper, $\mathcal{N}$ will denote a left near-ring. A near-ring $\mathcal{N}$ is called zero symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that in a left near ring $x0 = 0$ for all $x \in \mathcal{N}$). $\mathcal{N}$ is called 3-prime if $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. The symbol $Z(\mathcal{N})$ will represent the multiplicative center of $\mathcal{N}$, that is, $Z(\mathcal{N}) = \{x \in \mathcal{N} \mid xy = yx$ for all $y \in \mathcal{N}\}$. For any $x, y \in \mathcal{N}$; as usual $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie product and Jordan product respectively. Recall that $\mathcal{N}$ is called 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in \mathcal{N}$. For terminologies concerning near-rings we refer to G. Pilz [7]. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [8], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. Let g be an automorphism of $\mathcal{N}$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is called a skew derivation of N if $d(xy) = d(x)g(y) + xd(y)$ for all $x, y \in \mathcal{N}$. Obviously, any derivation is a skew derivations, but the converse is not true in general. Moreover, if g is the identity map of $\mathcal{N}$, then all skew derivations associated with g are certainly derivations of $\mathcal{N}$. An additive mapping d is said to be commuting if for all $x \in \mathcal{N}$, $[d(x), x] = 0$. There has been an ongoing interest concerning the relationship between the commutativity of a 3-prime near-ring $\mathcal{N}$ and the behavior of a derivation on $\mathcal{N}$. The study of commutativity of 3-prime near-rings by using derivations was initiated by H. E. Bell and G. Mason in 1987 and serval authors. Motivated by the result of Bell and Daif in [5] which proved that a 2-torsion free prime ring must be commutative if it admits a strong commutativity preserving derivation d, that is a derivation satisfying $[d(x), d(y)] = [x, y]$ for all $x, y \in \mathcal{R}$. Our aim in this paper is to generalize this result in two directions. First of all we will only assume that the commutativity condition is imposed on a 3-prime near-ring $\mathcal{N}$ instead of one ring $\mathcal{R}$. Secondly we will treat the case of two skew derivations instead of one derivation.

Date: Accepted: Oct 24, 2016.

* Corresponding author.

2010 *Mathematics Subject Classification.* 16N60, 16W25, 16Y30.

Key words and phrases. 3-prime near-rings, Commutativity, Skew derivations.

In 2002 Ashraf and Rehman [1], prove that if $\mathcal{R}$ is a 2-torsion free prime ring, I is a nonzero ideal of $\mathcal{R}$ and d is a nonzero derivation of $\mathcal{R}$ such that $d(x)\circ d(y) = x\circ y$ for all $x, y \in I$, then $\mathcal{R}$ is commutative. Our result is motivated by the previous results and we here generalized the result obtained in [1], [5]. Moreover, we continue this line of investigation by examining what happens if a 3-prime near-ring $\mathcal{N}$ satisfies the identity $d_1(x) \circ d_2(y) = x \circ y$ for all $x, y \in \mathcal{N}$ where d_1 and d_2 are two skew derivations.

2. Some preliminaries

In this paper, we include some well known results which will be used for developing the proof of our main result.

Lemma 2.1. [2, Lemma 1.5] *Let $\mathcal{N}$ be a 3-prime near ring. If $\mathcal{N} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.2. *A near-ring $\mathcal{N}$ admits a skew derivation d if and only if it is zero-symmetric.*

Proof. Let $\mathcal{N}$ be a zero-symmetric near-ring. Then the zero map is a skew derivation d on $\mathcal{N}$. Conversely, assume that $\mathcal{N}$ has an skew derivation d. Let x, y be two arbitrary elements of $\mathcal{N}$. By definition of d, we have

$$\begin{aligned} d(x0y) &= d(x(0y)) \\ &= d(x)g(0y) + xd(0y) \\ &= d(x)g(0)g(y) + xd(0y) \\ &= (d(x)0)y + x(d(0)g(y) + 0d(y)) \\ &= 0y + (xd(0))g(y) + (x0)d(y) \\ &= 0y + (x0)g(y) + (x0)d(y) \\ &= 0y + 0g(y) + 0d(y). \end{aligned}$$

On the other hand

$$\begin{aligned} d(x0y) &= d((x0)y)) \\ &= d(0y) \\ &= d(0)g(y) + 0d(y) \\ &= 0g(y) + 0d(y) \end{aligned}$$

By comparing the last two expressions, we find that $0y = 0$ for all $y \in \mathcal{N}$, and hence $\mathcal{N}$ is a zero-symmetric left near-ring. □ □

Remark 2.3. The above lemma is also true in the case of right near-ring.

Lemma 2.4. *Let d be an arbitrary skew derivation on the near ring $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive law*

$$\big(d(x)g(y) + xd(y)\big)z = d(x)g(y)z + xd(y)z \quad \text{for all } x, y, z \in \mathcal{N}.$$

Proof. By a simple calculation of $d(xyz)$ for all $x, y, z \in N$, we obtain

$$\begin{aligned} d(xyz) &= d(x)g(yz) + xd(yz) \\ &= d(x)g(y)g(z) + xd(y)g(z) + xyd(z) \quad \text{for all } x, y, z \in \mathcal{N}. \end{aligned}$$

By another way

$$\begin{aligned} d(xyz) &= d(xy)g(z) + xyd(z) \\ &= \big(d(x)g(y) + xd(y)\big)g(z) + xyd(z) \quad \text{for all } x, y, z \in \mathcal{N}. \end{aligned}$$

Comparing the last two results, we obtain

$$\big(d(x)g(y) + xd(y)\big)g(z) = d(x)g(y)g(z) + xd(y)g(z) \quad \text{for all } x, y, z \in \mathcal{N}.$$

Since g is an amorphism, we obtain the required result. □

3. Main Results

Theorem 3.1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring which admits nonzero skew derivations d_1, d_2 such that d_1 is commuting. Then the following assertions are equivalent:*

(i) $[d_1(x), d_2(y)] = [x, y]$ *for all* $x, y \in \mathcal{N}$.

(ii) $\mathcal{N}$ *is a commutative ring.*

Proof. It is easy to verify that $(ii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$. Suppose that $[d_1(x), d_2(y)] = [x, y]$ for all $x, y \in \mathcal{N}$. Replacing y by xy, we get

$$\begin{aligned} [d_1(yx), d_2(y)] &= [yx, y] \\ &= y[x, y] \\ &= y[d_1(x), d_2(y)] \quad \text{for all } x, y \in \mathcal{N} \end{aligned}$$

this expression implies that

$$d_1(yx)d_2(y) - d_2(y)d_1(yx) = yd_1(x)d_2(y) - yd_2(y)d_1(x) \quad \text{for all } x, y \in \mathcal{N} \tag{3.1}$$

Using Lemma 2.4 and (3.1), then for all $x, y \in \mathcal{N}$

$$yd_1(x)d_2(y) + d_1(y)g(x)d_2(y) - d_2(y)d_1(y)g(x) - d_2(y)yd_1(x) = yd_1(x)d_2(y) - yd_2(y)d_1(x)$$

which implies that

$$d_1(y)[g(x), d_2(y)] = -yd_2(y)d_1(x) + d_2(y)yd_1(x) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.2}$$

Using the fact that d_1 is commuting, then (3.2) becomes

$$d_1(y)g(x)d_2(y) = d_1(y)d_2(y)g(x) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.3}$$

Putting xt instead of x in (3.3) and using it again, we find that

$$\begin{aligned} d_1(y)g(x)g(t)d_2(y) &= d_1(y)d_2(y)g(x)g(t) \\ &= d_1(y)g(x)d_2(y)g(t) \quad \text{for all } x, y, t \in \mathcal{N}. \end{aligned}$$

The latter expression reduces to

$$d_1(y)g(N)[d_2(x), g(t)] = \{0\} \quad \text{for all } x, y, t \in \mathcal{N}. \tag{3.4}$$

Using the fact that g is an automorphism and the 3-primeness of $\mathcal{N}$, (3.4) implies that

$$d_1(x) = 0 \text{ or } \ d_2(x) \in Z(g(\mathcal{N})) \quad \text{for all } \ x \in \mathcal{N}.$$

If there is $x_0 \in \mathcal{N}$ such that $d_1(x_0) = 0$, then by applying the hypotheses of our theorem, we find that $[x, x_0] = 0$ for all $x \in \mathcal{N}$ so $x_0 \in Z(\mathcal{N})$.
In the same way, If there is $x_0 \in \mathcal{N}$ such that $d_2(x_0) \in Z(g(\mathcal{N}))$, since g is an automorphism, then $[x_0, y] = 0$ for all $y \in \mathcal{N}$ which implies that $x_0 \in Z(\mathcal{N})$. In all cases we arrive at $x \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$, then $\mathcal{N} \subseteq Z(\mathcal{N})$ and by Lemma 2.1, we conclude that $\mathcal{N}$ is a commutative ring. □

We now consider differential identities involving anti-commutators instead of commutators. Our result is of a different kind.

Theorem 3.2. *Let $\mathcal{N}$ be a 3-prime near-ring with $Z(\mathcal{N}) \neq \{0\}$. $\mathcal{N}$ admits no nonzero skew derivations d_1, d_2 such that $d_1(x) \circ d_2(y) = x \circ y$ for all $x, y \in \mathcal{N}$.*

Proof. Suppose that $d_1(x) \circ d_2(y) = x \circ y$ for all $x, y \in \mathcal{N}$. For $y \in Z(\mathcal{N})$ and by 2-torsion freeness, we obtain $d_1(x)d_2(y) = xy$ for all $x \in \mathcal{N}$, $y \in Z(\mathcal{N})$. Replacing x by xt, we find that

$$xd_1(t)d_2(y) + d_1(x)g(t)d_2(y) = xty \quad \text{forall } \ x, t \in \mathcal{N}, y \in Z(\mathcal{N}).$$

Since $d_1(t)d_2(y) = ty$ for all $t \in \mathcal{N}, y \in Z(\mathcal{N})$ the last expression becomes $d_1(x)g(\mathcal{N})d_2(y) = \{0\}$ for all $x \in \mathcal{N}, y \in Z(\mathcal{N})$. Since g is an automorphism, by 3-primeness of $\mathcal{N}$, we conclude that $d_2(Z(\mathcal{N})) = \{0\}$. In this case, for $y \in Z(\mathcal{N})$, our hypothesis gives $2xy = 0$ for all $x \in \mathcal{N}, y \in Z(\mathcal{N})$ which implies that $(2x)\mathcal{N}y = \{0\}$ for all $x \in \mathcal{N}, y \in Z(\mathcal{N})$, by 3-primeness and 2-torsion freeness of $\mathcal{N}$, we conclude that $Z(\mathcal{N}) = \{0\}$. □

The following example shows that the 3-primeness is necessary in the hypotheses of the above theorems.

Example 3.3. Let $\mathcal{S}$ be a non abelian near-ring. Define the sets $\mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha, \beta \in \mathcal{S} \right\}.$$

It is obvious that $\mathcal{N}$ is a near-ring not 3-prime. Next, we define the maps $d_1, d_2 : \mathcal{N} \to \mathcal{N}$ by:

$$d_1 \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad d_2 \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that d_1 and d_1 are two skew derivations with ($g = id_{\mathcal{N}}$) such that:

(1) $[d_1(x), d_2(y)] = [x, y]$
(2) $d_1(x) \circ d_2(y) = x \circ y$ for all $x, y \in \mathcal{N}$.

However, $\mathcal{N}$ is not a commutative ring.

4. ACKNOWLEDGMENT

The authors would like to thank Sidi Mohamed Ben Abdellah University (USMBA), LSI and FP of Taza in MOROCCO for its valued support.

References

1. M. Ashraf and N. Rehman, On commutativity of rings with derivations, Results Math. 42 (1-2) (2002) 3-8.
2. H. E. Bell and G. Mason, On derivations in near-rings. North-Holland Mathematics Studies 137 (1987) 31-35.
3. H. E. Bell, On derivations in near-rings II, Near-rings, nearfields and K-loops (Hamburg, 1995), 191-197, Math. Appl., 426, Kluwer Acad. publ., Dordrecht, 1997
4. H. E. Bell, A. Boua and L. Oukhtite, Semigroup ideals and commutativity in 3-prime near rings, Comm. Alg., 43 (2015), 1757-1770.
5. H. E. Bell and M. N. Daif, On commutativity and strong commutativity-preserving maps, Can. Math. Bull., 37 (1994) 443-447.
6. A. A. M. Kamal and K. H. Al-Shaalan, Existence of derivations on near-rings, Math. Slovaca, 63 (2013), no. 3, 431-448.
7. G. Pilz, Near-rings, *The theory and its applications.* Second edition. North-Holland Mathematics Studies, 23. North-Holland Publishing Co., Amsterdam, (1983).
8. X. K. Wang, *Derivations in prime near-rings,* Proc. Amer. Math. Soc., 121 (2), 1994, 361-366.

* Universit Ibn Zohr, Facult des sciences, Dpartement de Mathematiques, Equipe dEquations Fonctionnelles et Applications (EEFA), B. P. 8106, Agadir; Maroc.

E-mail address: karimoun2006@yahoo.fr or abdelkarimboua@yahoo.fr

‡ Ibn Zohr University, National School of Applied Sciences, Agadir; Morocco.

E-mail address: lahcentaoufiq@gmail.com