Gulf Journal of Mathematics
Vol 4, Issue 4 (2016) 39-47

ON THE FLATNESS OF Int(D) AS A $D[X]$–MODULE

L. IZELGUE[1] AND A. TAMOUSSIT[2,*]

Abstract. Let D be an integral domain with quotient field K and X an indeterminate. We show that if D is either Krull or Noetherian, then $\text{Int}(D) := \{f \in K[X] : f(D) \subseteq D\}$ is flat over $D[X]$ if and only if $\text{Int}(D) = D[X]$. Then, we give several examples of domains D with $\text{Int}(D)$ not flat over $D[X]$. Also, we generalize our investigations to the case of $\text{Int}(E, D) := \{f \in K[X] : f(E) \subseteq D\}$, where E is a subset of D.

1. Introduction

Let D be an integral domain with quotient field K and X an indeterminate over K. The ring of *integer-valued polynomials over* D is known to be $\text{Int}(D) := \{f \in K[X] : f(D) \subseteq D\}$. Clearly, $D[X] \subseteq \text{Int}(D) \subseteq K[X]$. We will assume that D is not a field.

Integer-valued polynomial rings were first considered over rings of integers of a number field by G. Polya and A. Ostrowski [18],[17]. Particularly, they investigated the existence of bases of $\text{Int}(D)$ as a D–module. Then Integer-valued polynomials rings give rise a challenging area of research is the theory of commutative rings, remarkably from 1970. They possess a rich theory and provide an excellent source of examples and counter-examples, especially in the theory of non-Noetherian commutative rings. Literature abounds of studies covering different types of problems, in various areas of Commutative Algebra and Number Theory. A base reference is [6].

For instance, the ring $\text{Int}(\mathbb{Z})$ is known to be a non-Noetherian Prüfer domain of Krull dimension two that is contained in $\mathbb{Q}[X]$. Moreover, as a $\mathbb{Z}$–module, $\text{Int}(\mathbb{Z})$ is free and thus flat.

In the last decade the problem of flatness of $\text{Int}(D)$ as a D–module occupied a wide place in research. Precisely, works of J. Elliott [8], [9], in part, investigated the problem of flatness. Then, his works gave rise to many related open question. In particular, he conjectured that $\text{Int}(D)$ is not flat as a D–module for $D = \mathbb{F}_2[[T^2, T^3]]$ and for $D = \mathbb{F}_2 + T\mathbb{F}_4[[T]]$, which conjecture is still open.

Date: Accepted: Oct 24, 2016.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary: 13F20, 13C11; secondary: 13B25, 13F05, 13B30, 13F30, 13C15.

Key words and phrases. Integer-valued polynomials, Flat modules, (Locally) Free modules, Prime ideal, Maximal ideal, Residue field, Localization, Krull domain, Noetherian domain, Locally finite representation, Krull dimension.

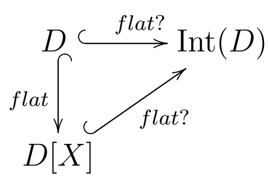
But till know we do not know of an example of a domain D such that $\mathrm{Int}(D)$ is not flat over D. On the other hand, since $D \subseteq D[X] \subseteq \mathrm{Int}(D)$ and $D[X]$ is always a flat D–module, one may asks about the flatness of $\mathrm{Int}(D)$ as a $D[X]$–module.

The aim of this paper is then to investigate this last question and then investigate the relationships with the problem of flatness of $\mathrm{Int}(D)$ as a D–module. Thus in Section 2 , we give a characterization of when $\mathrm{Int}(D)$ is flat over $D[X]$ in the case of domains that are either Noetherian or Krull. Also we establish that $\dim \mathrm{Int}(D) \leq \dim D[X]$, whenever $D[X] \hookrightarrow \mathrm{Int}(D)$ is flat. An affirmative answer to the [11, Conjecture $(\mathcal{C}_1)$]. Our study allows us, to construct examples of integral domains such that $\mathrm{Int}(D)$ is not flat over $D[X]$.

In Section 3, we investigate the case $\mathrm{Int}(E, D)$. Among other results, we establish that for a subset E of a Noetherian domain D, if $\mathrm{Int}(E, D)$ is not Noetherian, then it is not flat over $D[X]$. For instance this happens when D is a Dedekind domain with finite residue fields. Then we generalize some results about the behavior under localization established for $\mathrm{Int}(D)$ to the case of $\mathrm{Int}(E, D)$. So, we are led to characterize domains of locally finite representation D, such that $\mathrm{Int}(E, D)$ is D–flat, with a restricted hypothesis on the subset E.

2. Flatness of $\mathrm{Int}(D)$ as a $D[X]$–module

Since $D[X]$ is a flat D–algebra, and $\mathrm{Int}(D)$ is a $D[X]$–module, then by the transitivity of flatness (cf. [15, p. 46]), if $\mathrm{Int}(D)$ is flat over $D[X]$, then $\mathrm{Int}(D)$ is flat over D. That means that whenever flatness of $\mathrm{Int}(D)$ holds over $D[X]$, it necessarily holds over D. So we are led to investigate flatness in parallel to the following diagram of homomorphisms:



That gives rise to the following question:

(Q1) *Is there a domain D such that $\mathrm{Int}(D)$ is flat over D and $\mathrm{Int}(D)$ is not flat over $D[X]$?*

The answer is Yes as asserted by Example 2.3 bellow.

Next we give a complete characterization of flatness $\mathrm{Int}(D)$ as a $D[X]$–module when D is Noetherian:

Proposition 2.1. *Let D be a Noetherian domain and consider the following statements:*

(i) $\mathrm{Int}(D)$ *is flat over* $D[X]$.
(ii) $\mathrm{Int}(D) = D[X]$.
(iii) $\mathrm{Int}(D)$ *is Noetherian.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) . *If moreover, D is either one-dimensional or integrally closed, then* (iii) $\Leftrightarrow$ (ii) *and the three statements are equivalent.*

Proof. (i) $\Rightarrow$ (iii). Assume that $\mathrm{Int}(D)$ is flat over $D[X]$. Since $\mathrm{Int}(D)$ is an overring of $D[X]$ and $D[X]$ is Noetherian (because D is also Noetherian). Then,

by [20, Corollary of Theorem 3], Int(D) is Noetherian.
(i) $\Rightarrow$ (ii). If Int(D) is flat over $D[X]$, then, by the previous implication, Int(D) is Noetherian. Hence, Int(D) $\subseteq D'[X]$ (cf. [6, Proposition VI.2.4]). It follows that Int(D) is integral over $D[X]$, and since Int(D) is an overring of $D[X]$. So, by [20, Proposition 2], Int(D) $= D[X]$.
(ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii). Straightforward.
If D is one-dimensional or integrally closed, by [6, Corollary VI.2.6], we have Int(D) is Noetherian implies Int(D) $= D[X]$, and then (iii) $\Rightarrow$ (ii). □

Remark 2.2. In general, statements (ii) and (iii) are not equivalent (cf. [6, Exercise VI.15]).

Example 2.3. It is well-known that $\mathbb{Z}$ is one-dimensional Noetherian domain such that Int($\mathbb{Z}$) $\neq \mathbb{Z}[X]$. Int($\mathbb{Z}$) is a free, and hence is a flat, $\mathbb{Z}$–module. However, by Proposition 2.1, Int($\mathbb{Z}$) is not flat as a $\mathbb{Z}[X]$–module. An affirmative answer to Question **(Q1)**.

Example 2.4. Let k be a finite field, T be an indeterminate over k and $D = k[T^2, T^3]$. D is a one-dimensional Noetherian domain. Let $f = \prod_{a\in k}(X - a)$ and $g = f^3/T^4$, then Int(D) is not trivial, because g is lie in Int(D) but not in $D[X]$. So, by Proposition 2.1, Int(D) is not a flat $D[X]$–module.

Next we characterize Krull domains D for which Int(D) is flat over $D[X]$.

Theorem 2.5. *Let D be a Krull domain. The following statements are equivalent.*

(i) Int(D) *is flat over* $D[X]$*;*
(ii) Int(D) $= D[X]$*;*
(iii) Int(D) *is Krull;*
(iv) *For each* $\mathfrak{P} \in X^1(D[X])$*, either* $\text{Int}(D)_{\mathfrak{P}} = D[X]_{\mathfrak{P}}$ *or* $\mathfrak{P}\text{Int}(D) = \text{Int}(D)$.

Proof. Since D is a Krull domain then so is $D[X]$. (i) $\Rightarrow$ (iii). If Int(D) is flat over $D[X]$, then Int(D) is a generalized ring of fractions $D[X]$ [19, section 3]. By [19, Corollaire 1, p. 1225], Int(D) is Krull.
(iii) $\Leftrightarrow$ (ii). [7, Corollary 2.7].
(ii) $\Rightarrow$ (i). Straightforward.
(i) $\Leftrightarrow$ (iv). [22, Theorem 1.9]. □

Remark 2.6. 1. It is well-known that if D is an UFD (resp., Krull domain), the D-module Int(D) has a regular basis [6, Exercise II.23] (resp., flat [8, Corollary 3.6]). However, by Theorem 2.5, Int(D) is not necessarily flat over $D[X]$.

2. The domain D in [13, Example 5.2], is Noetherian such that Int(D) is not Noetherian but $\text{Int}(D_{\mathfrak{m}})$ is Noetherian for each maximal ideal $\mathfrak{m}$ of D. It follows that Int(D) is not $D[X]$–flat even if $\text{Int}(D_{\mathfrak{M}})$ is $D_{\mathfrak{M}}[X]$–flat for some maximal ideal $\mathfrak{M}$ of D and $\text{Int}(D_{\mathfrak{m}}) = \text{Int}(D)_{\mathfrak{m}}$ for each $\mathfrak{m} \in \text{Max}(D)$.

Example 2.7. Let $D = \mathbb{C}[(T_j)_{j\in J}]$ where J is an infinite set. It is clear that D is a non-Noetherian Krull domain, and by [6, Exercise VII.24] Int(D) $= D[X]$. Thus Int(D) is flat over $D[X]$.

Next, we give an affirmative answer to [11, Conjecture $(\mathcal{C}_1)$], about Krull dimension of $\text{Int}(D)$, when $\text{Int}(D)$ is flat over $D[X]$.

Proposition 2.8. *Let D be an integral domain.*
If $\text{Int}(D)$ is flat over $D[X]$, then $\dim \text{Int}(D) \leqslant \dim D[X]$.

Proof. Assume that $\text{Int}(D)$ is flat over $D[X]$. Since $\text{Int}(D)$ is an overring of $D[X]$, then, by [20, Theorem 2], $\text{Int}(D)_{\mathfrak{M}} = D[X]_{\mathfrak{M}\cap D[X]}$ for each maximal ideal $\mathfrak{M}$ of $\text{Int}(D)$. Thus

$$\begin{aligned}\dim \text{Int}(D) &= \max\{\dim \text{Int}(D)_{\mathfrak{M}},\ \ \mathfrak{M} \in \text{Max}(\text{Int}(D))\}\\ &= \max\left\{\dim D[X]_{\mathfrak{M}\cap D[X]},\ \ \mathfrak{M} \in \text{Max}(\text{Int}(D))\right\}\\ &\leqslant \max\{\dim D[X]_{\mathfrak{P}},\ \ \mathfrak{P} \in \text{Spec}(D[X])\}\\ &= \dim D[X].\end{aligned}$$

□

Notice that if D is a Noetherian domain such that $\text{Int}(D)$ is not Noetherian, then $\text{Int}(D)$ is not flat over $D[X]$. But in this case $\dim \text{Int}(D) = \dim D[X]$. Hence the converse of Proposition 2.8 is not true in general.

We conclude this section by the following open problem:

(Q) *Is there an example of an integral domain D such that $\text{Int}(D)$ is flat over $D[X]$ and $\text{Int}(D)$ not trivial?*

3. Integer-valued polynomials over a subset

A *fractional subset* of D is any subset E of K such that $dE \subseteq D$ for some nonzero element $d \in D$. The ring of *integer-valued polynomials* on E is $\text{Int}(E, D) := \{f \in K[X] \mid f(E) \subseteq D\}$. If E is D itself, we simply write $\text{Int}(D)$ for $\text{Int}(D, D)$. If E is not contained in D then $X \notin \text{Int}(E, D)$ and for any $E \subseteq D$ one has $D[X] \subseteq \text{Int}(D) \subseteq \text{Int}(E, D)$ (cf. [4, Proposition 1.2]). When E is a domain with infinite residue fields and D containing E, then $\text{Int}(E, D) = \text{Int}(D) = D[X]$ (cf. [6, Corollary IV.1.21]).

A nonempty subset E of a domain D is *residually cofinite* with D if and only if it possesses the property that for each prime ideal $\mathfrak{p}$ of D, $|E/\mathfrak{p}| < \infty \Longrightarrow |D/\mathfrak{p}| < \infty$ (see [16, Lemma 2]). By [16, Lemma 3(i)], any residually cofinite subset with D remains residually cofinite with $S^{-1}D$, for any multiplicatively closed subset of D.

If $E \subseteq D$ is residually cofinite with D, and D has no finite residue field, then $\text{Int}(E, D) = \text{Int}(D) = D[X]$ (cf. [16, Lemma 4(ii)]).

In what follows, we assume $E \subseteq D$ and investigate the flatness of $\text{Int}(E, D)$ as a $D[X]$–module.

Since any flat overring of a Noetherian domain is also a Noetherian domain [20, Corollary of Theorem 3], then we have:

Proposition 3.1. *Let D be a Noetherian domain and E be a subset of D. If $\text{Int}(E, D)$ is not Noetherian, then $\text{Int}(E, D)$ is not flat over $D[X]$.*

Proof. It follows from [20, Corollary of Theorem 3]. □

So, as a consequence of the previous proposition, we have:

Corollary 3.2. *Let D be a Dedekind domain with finite residue fields and let E be a subset of D. Then* $\text{Int}(E, D)$ *is not flat over* $D[X]$.

Proof. It follows from the fact $\text{Int}(E, D)$ is a two-dimensional non-Noetherian Prüfer domain [6, Proposition V.1.8 and Exercise VI.4(iv)]. Thus, by Proposition 3.1, $\text{Int}(D)$ is not flat over $D[X]$. □

Example 3.3. The ring $\mathbb{Z}$ of integers is Dedekind. So if $\mathbb{P}$ is the set of all prime integers, by Corollary 3.2, $\text{Int}(\mathbb{P}, \mathbb{Z})$ is not flat as a $\mathbb{Z}[X]$–module. However, $\text{Int}(\mathbb{P}, \mathbb{Z})$ is a free and hence a flat $\mathbb{Z}$–module (cf. [3]).

On the other hand, it was established,[5], that, for any Dedekind domain D, the D–module $\text{Int}(D)$ is locally free, hence faithfully flat. Next we generalize this result to $\text{Int}(E, D)$.

Proposition 3.4. *Let D be a Dedekind domain and E be an infinite subset of D. Then the D–module* $\text{Int}(E, D)$ *is locally free, and hence faithfully flat.*

Proof. Since D is Noetherian, for each prime ideal $\mathfrak{p}$ of D, $\text{Int}(E, D)_\mathfrak{p} = \text{Int}(E, D_\mathfrak{p})$, [6, Proposition I.2.7(i)]. Since D is Dedekind, then $D_\mathfrak{p}$ is a discrete valuation ring. By [3, Proposition 2.2], $\text{Int}(E, D_\mathfrak{p})$ is a free $D_\mathfrak{p}$–module. Thus, $\text{Int}(E, D)$ is locally free as a D–module. □

As a consequence of [4, Theorem 4.3]:

Corollary 3.5. *Let D be a Noetherian domain and E be a subset of D such that there exists a height-one prime ideal of D' with finite residue field. Then* $\text{Int}(E, D)$ *is not flat over* $D[X]$.

In [9] Elliott conjectured that if either $D = \mathbb{F}_2[[T^2, T^3]]$ or $D = \mathbb{F}_2 + T\mathbb{F}_4[[T]]$, then $D \hookrightarrow \text{Int}(D)$ is not flat. The conjecture is still open. The following example shows that as a $D[X]$–module, $\text{Int}(D)$ is not flat.

Example 3.6. If either $D = \mathbb{F}_2[[T^2, T^3]]$ or $D = \mathbb{F}_2 + T\mathbb{F}_4[[T]]$, we have D is a Noetherian domain and its integral closure D' has a height-one maximal $\mathfrak{m}$ of finite residue field. Hence, by Corollary 3.5, $\text{Int}(E, D)$ is not flat over $D[X]$, for each subset E of D.

Corollary 3.7. *Let D be a Noetherian domain and E be a finite subset of D. Then* $\text{Int}(E, D)$ *is not flat over* $D[X]$.

Proof. It follows easily from Proposition 3.1 and [4, Lemma 4.1]. □

Remark 3.8. (1) Let D be a Noetherian domain with quotient field K. Then $D + XK[X]$ is not flat as a $D[X]$–module. Just take $E = \{0\}$ in Corollary 3.7. However, by [1, Lemma 3.6], $D + XK[X]$ is a flat D–module.

(2) Now, let D be a Noetherian domain with infinite residue fields and let $E \subseteq D$ be a residually cofinite with D. By [16, Lemma 4(ii)], $\text{Int}(E, D) = \text{Int}(D) = D[X]$. Thus, $\text{Int}(E, D)$ is flat over $D[X]$. On the other hand, $\text{Int}(E, D) = \bigcap_{a \in E} \text{Int}(\{a\}, D)$ and, by Corollary 3.7, $\text{Int}(\{a\}, D)$ is not flat over $D[X]$ for each $a \in E$.

In what follows, we establish some usefull results about localizations of $\mathrm{Int}(E, D)$. Then we prove that $\mathrm{Int}(E, D)$ behaves well under localization, at prime ideals of a domain D of locally finite representation.

Lemma 3.9. *Let $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$, where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, be an integral domain and E a subset of D. Then $\mathrm{Int}(E, D) = \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathrm{Int}(E, D_{\mathfrak{p}})$.*

Proof. Since $\mathrm{Int}(E, D) \subseteq \mathrm{Int}(E, D)_{\mathfrak{p}} \subseteq \mathrm{Int}(E, D_{\mathfrak{p}})$ for each $\mathfrak{p} \in \mathcal{P}$, then $\mathrm{Int}(E, D) \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathrm{Int}(E, D_{\mathfrak{p}})$. On the other hand, since $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$, then $f \in \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathrm{Int}(E, D_{\mathfrak{p}})$ satisfies $f(E) \subseteq D_{\mathfrak{p}}$, for each $\mathfrak{p} \in \mathcal{P}$. Thus $f(E) \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}} = D$, that is, $f \in \mathrm{Int}(E, D)$. □

In the spirit of [9, Lemma 2.5] and [21, Proposition 2.3] we prove the following result.

Lemma 3.10. *Let D be a domain, and let $E \subseteq D$ be residually cofinite with D. Then $\mathrm{Int}(E, D_{\mathfrak{p}})_{\mathfrak{q}} = (D_{\mathfrak{p}})_{\mathfrak{q}}[X]$ for each pair of prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of D with $\mathfrak{p} \neq \mathfrak{q}$, and thus $\mathrm{Int}(E, D_{\mathfrak{p}})_{\mathfrak{q}} = \mathrm{Int}(E, D_{\mathfrak{q}})_{\mathfrak{p}}$.*

Proof. Let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(D)$ with $\mathfrak{p} \neq \mathfrak{q}$. So if $\mathfrak{p} \subset \mathfrak{q}$, then $\mathfrak{p}$ is not maximal and thus $D/\mathfrak{p}$ is infinite. So $\mathrm{Int}(E, D_{\mathfrak{p}})_{\mathfrak{q}} = (D_{\mathfrak{p}}[X])_{\mathfrak{q}} = (D_{\mathfrak{p}})_{\mathfrak{q}}[X] = D_{\mathfrak{p}}[X]$.
If $\mathfrak{p} \not\subseteq \mathfrak{q}$ we set $\mathcal{B} = \{\mathfrak{p}' \in \mathrm{Spec}(D);\ \mathfrak{p}' \subset \mathfrak{p} \cap \mathfrak{q}\}$. Notice that each $\mathfrak{p}' \in \mathcal{B}$ is a non-maximal ideal of D. Hence, $\mathrm{Int}(E, D_{\mathfrak{q}}) \subseteq \mathrm{Int}(E, D_{\mathfrak{p}'}) = D_{\mathfrak{p}'}[X]$. It follows that $\mathrm{Int}(E, D_{\mathfrak{q}}) \subseteq \bigcap_{\mathfrak{p}'\in\mathcal{B}} D_{\mathfrak{p}'}[X] = (D_{\mathfrak{p}})_{\mathfrak{q}}[X] = (D_{\mathfrak{p}}[X])_{\mathfrak{q}} \subseteq \mathrm{Int}(E, D_{\mathfrak{p}})_{\mathfrak{q}}$, and hence $\mathrm{Int}(E, D_{\mathfrak{q}})_{\mathfrak{p}} \subseteq (D_{\mathfrak{p}})_{\mathfrak{q}}[X] \subseteq \mathrm{Int}(E, D_{\mathfrak{p}})_{\mathfrak{q}}$. The result follows by a symetry argument. □

Next we generalize [9, Theorem 2.6] to the case of $\mathrm{Int}(E, D)$.

Theorem 3.11. *Let $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$, where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, be a domain of locally finite representation and let $E \subseteq D$ be infinite and residually cofinite with D. Then $\mathrm{Int}(E, D)_{\mathfrak{p}} = \mathrm{Int}(E, D_{\mathfrak{p}})$, for each prime ideal $\mathfrak{p} \in \mathrm{Spec}(D)$.*

Proof. We argue mimicking the proof of [9, Theorem 2.6]. So let $\mathfrak{q} \in \mathrm{Spec}(D)$, we always have $\mathrm{Int}(E, D)_{\mathfrak{q}} \subseteq \mathrm{Int}(E, D_{\mathfrak{q}})$. For the inverse inclusion, let $f \in \mathrm{Int}(E, D_{\mathfrak{q}})$. Clearly, there exists $d \in D \setminus \{0\}$, such that $df \in D[X]$. Since $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ is a locally finite intersection, one has $d \notin \mathfrak{p}$, and therefore $f \in D_{\mathfrak{p}}[X]$, for all but finitely many $\mathfrak{p} \in \mathcal{P}$, say, $\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_n$. By Lemma 3.10, we have $\mathrm{Int}(E, D_{\mathfrak{q}}) \subseteq \mathrm{Int}(E, D_{\mathfrak{p}_i})_{\mathfrak{q}}$ for each $i = 1, ..., n$. So there exists $s_i \in D \setminus \mathfrak{q}$ such that $s_i f \in \mathrm{Int}(E, D_{\mathfrak{p}_i})$ for each i. Then $s = s_1 s_2 ... s_n \in D \setminus \mathfrak{q}$ and $sf \in \mathrm{Int}(E, D_{\mathfrak{p}_i})$ for each $i = 1, ..., n$. But $sf \in D_{\mathfrak{p}}[X] \subseteq \mathrm{Int}(E, D_{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathcal{P} \setminus \{\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_n\}$. Hence $sf \in \mathrm{Int}(E, D_{\mathfrak{p}})$ for each $\mathfrak{p} \in \mathcal{P}$. It follows that $sf \in \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathrm{Int}(E, D_{\mathfrak{p}})$. By Lemma 3.9, $sf \in \mathrm{Int}(E, D)$ and then $f \in \mathrm{Int}(E, D)_{\mathfrak{q}}$, since $s \in D \setminus \mathfrak{q}$. □

To avoid repetition we fix some notations :
Let $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$ be a domain of locally finite representation, where $\mathcal{P} \subseteq \mathrm{Spec}(D)$. Set $\mathcal{M}_0 := \{\mathfrak{m} \in \mathcal{P} \bigcap \mathrm{Max}(D) \mid \mathrm{Card}(D/\mathfrak{m}) < \infty\}$ and $\mathcal{M}_1 := \mathrm{Max}(D) \backslash \mathcal{M}_0$.

Proposition 3.12. *With the previous notations, if $E \subseteq D$ be infinite and residually cofinite with D. Then, $\mathrm{Int}(E, D)_{\mathfrak{m}} = \mathrm{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$, for each $\mathfrak{m} \in \mathcal{M}_1$.*

Proof. Let $\mathfrak{m} \in \mathrm{Max}(D)$. By Theorem 3.11, $\mathrm{Int}(E, D)_{\mathfrak{m}} = \mathrm{Int}(E, D_{\mathfrak{m}})$. So we only need to prove that $\mathrm{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$.
We note that:

$$D_{\mathfrak{m}} = \bigcap_{\mathfrak{p}\in\mathcal{P}} (D_{\mathfrak{p}})_{(D\setminus\mathfrak{m})} = \left(\bigcap_{\mathfrak{p}\in\mathcal{P},\mathfrak{p}\subseteq\mathfrak{m}} D_{\mathfrak{p}}\right) \bigcap \left(\bigcap_{\mathfrak{p}\in\mathcal{P},\mathfrak{p}\nsubseteq\mathfrak{m}} (D_{\mathfrak{p}})_{(D\setminus\mathfrak{m})}\right), \tag{3.1}$$

is a locally finite representation of $D_{\mathfrak{m}}$ [12, Proposition 43.5]. Now, let $\mathfrak{m} \in \mathcal{M}_1$.

If $\mathfrak{m} \notin \mathcal{P}$, then for each $\mathfrak{p} \in \mathcal{P}$ with $\mathfrak{p} \subset \mathfrak{m}$, $D/\mathfrak{p}$ is infinite and then $\mathrm{Int}(E, D_{\mathfrak{p}}) = D_{\mathfrak{p}}[X]$. If $\mathfrak{p} \in \mathcal{P}$ with $\mathfrak{p} \nsubseteq \mathfrak{m}$, then $(D_{\mathfrak{p}})_{(D\setminus\mathfrak{m})}$ has infinite residue fields and thus $\mathrm{Int}(E, (D_{\mathfrak{p}})_{(D\setminus\mathfrak{m})}) = (D_{\mathfrak{p}})_{(D\setminus\mathfrak{m})}[X]$. By (3.1) and Lemma 3.9,

$$\begin{aligned}
\mathrm{Int}(E, D_{\mathfrak{m}}) &= \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathrm{Int}(E, (D_{\mathfrak{p}})_{(D\setminus\mathfrak{m})}) \\
&= \left(\bigcap_{\mathfrak{p}\in\mathcal{P},\mathfrak{p}\subseteq\mathfrak{m}} \mathrm{Int}(E, D_{\mathfrak{p}})\right) \bigcap \left(\bigcap_{\mathfrak{p}\in\mathcal{P},\mathfrak{p}\nsubseteq\mathfrak{m}} \mathrm{Int}(E, (D_{\mathfrak{p}})_{(D\setminus\mathfrak{m})})\right) \\
&= \left(\bigcap_{\mathfrak{p}\in\mathcal{P},\mathfrak{p}\subseteq\mathfrak{m}} D_{\mathfrak{p}}[X]\right) \bigcap \left(\bigcap_{\mathfrak{p}\in\mathcal{P},\mathfrak{p}\nsubseteq\mathfrak{m}} (D_{\mathfrak{p}})_{(D\setminus\mathfrak{m})}[X]\right) \\
&= D_{\mathfrak{m}}[X].
\end{aligned}$$

Now, if $\mathfrak{m} \in (\mathrm{Max}(D) \bigcap \mathcal{P}) \setminus \mathcal{M}_0$, then $D/\mathfrak{m}$ is infinite, and hence $\mathrm{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$. Thus, $\mathrm{Int}(E, D)_{\mathfrak{m}} = \mathrm{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$ and the proof is complete. □

Now, we pursue the study of flatness of $\mathrm{Int}(E, D)$ as a D–module, in fact we complete some results established in [14]. So, applying the local aspect of flatness to the case of $\mathrm{Int}(E, D)$, we have:

Theorem 3.13. *Let $D = \bigcap_{\mathfrak{p}\in\mathcal{P}} D_{\mathfrak{p}}$, where $\mathcal{P} \subseteq \mathrm{Spec}(D)$, be a domain of locally finite representation, let $E \subseteq D$ be infinite and residually cofinite with D and set $\mathcal{M}_0$ and $\mathcal{M}_1$ as previously cited. The following statements are equivalent:*

(i) $\mathrm{Int}(E, D)$ *is flat as a D–module;*
(ii) $\mathrm{Int}(E, D)_{\mathfrak{m}}$ *is flat as a $D_{\mathfrak{m}}$–module for each $\mathfrak{m} \in \mathrm{Max}(D)$;*
(iii) $\mathrm{Int}(E, D)_{\mathfrak{m}}$ *is flat as a $D_{\mathfrak{m}}$–module for each $\mathfrak{m} \in \mathcal{M}_0$.*
(iv) $\mathrm{Int}(E, D_{\mathfrak{m}})$ *is flat as a $D_{\mathfrak{m}}$–module for each $\mathfrak{m} \in \mathcal{M}_0$.*

Proof. (i) ⇔ (ii) . By [2, Proposition 3.10, p. 41].
(ii) ⇒ (iii) . Straightforward.
(iii) ⇒ (ii) . It is clear that if $\mathfrak{m} \in \mathcal{M}_1$, then, by Proposition 3.12, $\mathrm{Int}(E, D)_{\mathfrak{m}} = \mathrm{Int}(E, D_{\mathfrak{m}}) = D_{\mathfrak{m}}[X]$.

By Theorem 3.11, for each prime ideal $\mathfrak{p}$ of D, $\mathrm{Int}(E, D)_{\mathfrak{p}} = \mathrm{Int}(E, D_{\mathfrak{p}})$, thus (iii) ⇔ (iv) is obvious, and this completes the proof. □

It is known that, for any Krull domain D, the D–module $\mathrm{Int}(D)$ is locally free [8, Corollary 3.6]. Next we generalize the same result to the case of $\mathrm{Int}(E, D)$:

Proposition 3.14. *Let D be a Krull domain and $E \subseteq D$ be infinite and residually cofinite with D. Then, $\mathrm{Int}(E, D)$ is a locally free D–module and hence faithfully flat.*

Proof. Let $\mathfrak{m} \in \mathcal{M}_0$. Since D is a Krull domain and $\mathfrak{m} \in X^1(D)$, then $D_\mathfrak{m}$ is a discrete valuation domain. By [3, Proposition 2.2], we have $\operatorname{Int}(E, D_\mathfrak{m})$ is a free $D_\mathfrak{m}$–module. Thus, by Theorem 3.11, the D–module $\operatorname{Int}(E, D)$ is locally free. □

4. ACKNOWLEDGMENT

The authors would like to thank Sidi Mohamed Ben Abdellah Univercity (USMBA), LSI and FP of Taza in MOROCCO for their valued supports.

References

1. D.F. Anderson, S. El Baghdadi and S. Kabbaj; *On the Class Group of* $A+XB[X]$ *Domains*, in: D.E. Dobbs, M. Fontana, S. Kabbaj (Eds.), *Advances in Commutative Ring Theory*, Lect. Notes in Pur. Appl. Math., vol. 205, Dekker, New York (1999), 73–85.
2. M.F. Atiyah and I.G. McDonald; *Introduction to Commutative Algebra*, Addition–Wesley (1969).
3. J. Bourlanger, J.-L. Chabert, S.E. Evrard and G. Gerboud; *The Characteristic Sequence on Integer-Valued Polynomials on a Subset*, in: *Advances in Commutative Ring Theory*, (Fez 1997), Lect. Notes in Pur. Appl. Math., 205 Dekker, New York (1999), 161–174.
4. P.-J. Cahen; *Integer-Valued Polynomials on a Subset*, Proc. Amer. Math. Soc., **117** (1993), 919–929.
5. P.-J. Cahen and J.-L. Chabert; *Coefficients et Valeurs d'un Polynôme*, Bull. SC. Math. Série 2, **95** (1971), 295–304.
6. P.-J. Cahen and J.-L. Chabert; *Integer-Valued Polynomials*, Math. Surveys Monogr., vol. 48, Amer. Math. Soc. (1997).
7. P.-J. Cahen, S. Gabelli and E. Houston; *Mori Domains of Integer-Valued Polynomials*, J. Pur. Appl. Algebra **153** (2000), 1–15.
8. J. Elliott; *Some new Approaches to Integer-Valued Polynomial Rings*, in: *Commutative Algebra and its Applications*, Proceedings of the Fifth Interational Fez Conference on Commutative Algebra and Applications, Eds. Fontana, Kabbaj, Olberding, and Swanson, de Gruyter, New York, 2009, 223–237.
9. J. Elliott; *Integer-Valued Polynomial Rings,* t*–Closure, and Associated Primes*, Comm. Algebra **39** (11)(2011), 4128–4147.
10. M. Fontana, L. Izelgue, S. Kabbaj and F. Tartarone; *Polynomial Closure in Essential Domains and Pullbacks*, Lect. Notes in Pur. Appl. Math. Dekker **205** (1999), 307–321.
11. M. Fontana and S. Kabbaj; *Essential Domains and two Conjectures in Dimension Theory*, Proc. Amer. Math. Soc. **132** (2004), 2529–2535.
12. R. Gilmer; *Multiplicative Ideal Theory*, Queen's Papers in Pure and Appl. Math., vol. 90, Queen's University, Kingston, Ontario, 1992.
13. R. Gilmer, W. Heinzer and D. Lantz; *The Noeterian Property in Rings of Integer-Valued Polynomials*, Trans. Amer. Math. Soc., **338** (1993), 187–199.
14. L. Izelgue and A. Tamoussit; *On the Flatness of* $\operatorname{Int}(D)$ *as a* D*–Module*, Submitted.
15. H. Matsumura; *Commutative Ring Theory*, Cambridge University Press, (1986).
16. S.B. Mulay; *On Integer-Valued Polynomials*, in: *Zero-Dimensional Commutative Rings*, Lect. Notes Pur. Appl. Math., Vol **171**, Dekker, New York, (1995), 331–345.
17. A. Ostrowski; *Über Ganzwertige Polynome in Algebraischen Zahlkörpern*, J. Reine Angew. Math., **149**(1919), 117–124.
18. G. Polya; *Über Ganzwertige Polynome in Algebraischen Zahlkörpern*, J. Reine Angew. Math., **149** (1919), 79–116.
19. J. Querré; *Sur les Anneaux Réflexifs*, Can. J. Math., **27** (1975), 1222–1228.
20. F. Richman; *Generalized Quotient Rings*, Proc. Amer. Math. Soc., **16** (1965), 794–799.
21. F. Tartarone; *Integer-Valued Polynomials over Krull-type Domains and Prüfer* v*–Multiplication Domains*, Proc. Amer. Math. Soc., **128** (6)(1999), 1617–1625.

22. K. Yoshida; *On Flat over-rings of a Krull Domain*, J. Math. Soc. Japan **27** (2)(1975), 258–263.

[1]Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University, P.O. Box 2390, Marrakech 40 000, Morocco.
E-mail address: izelgue@uca.ac.ma

[2]Department of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University, P.O. Box 2390, Marrakech 40 000, Morocco.
E-mail address: tamoussit2009@gmail.com