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THE REPRESENTATION RING OF SL_d

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ABSTRACT. Let k be a field, GL_d be the algebraic linear group of order d over the field k and SL_d be the subgroup of matrices with a determinant of 1. The aim of this paper is to describe the representation rings of those groups and their associated completed rings.

1. Introduction and preliminaries

The representation theory group was created by Frobenius, Schur and developed by many others like Borel and Serre. The representation theory group has so many applications to the resolution of the heat equations, to theoretical physics, to algebraic geometry and algebraic topology. In [2] we will exploit our calculations of the representations rings of the linear group GL_d and the special group SL_d and the various morphisms between them to deduce morphisms between their associated completed rings. From [5], where Riou proves a motivic Atiyah isomorphism for linear group GL_d which generalizes Atiyah isomorphism, for finite group [1], between the completed representation ring and the ring of the K-theory of the classifying space, we prove a motivic Atiyah isomorphism for the special group SL_d.

Definition 1.1. Let d be an integer.

(1) The linear algebraic group GL_d is the group of invertible $d \times d$ matrices together with the operation of ordinary matrix multiplication. Let i_d be the morphism:

$$\begin{array}{ccc} GL_d & \longrightarrow & GL_{d+1} \\ M & \longmapsto & \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} \end{array} \tag{1.1}$$

Then the system $\{GL_d, i_d\}$ is direct.

(2) We recall that the infinite linear group GL is the direct limit of the system $\{GL_d, i_d\}$.

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2. Representation ring of the linear group

Denote by $\mathfrak{G}_m^d$ the maximal torus of GL_d and $j_d : \mathfrak{G}_m^d \to GL_d$ the embedding. Let $R_k(GL_d)$, $R_k(\mathfrak{G}_m^d)$ be their k-representation rings equipped with direct sum and tensor product of representations and $j_d^* : R_k(GL_d) \to R_k(\mathfrak{G}_m^d)$ be the restriction morphism. Recall $X_1, .., X_d, X_1^{-1}, .., X_d^{-1}$ the irreducible characters of $\mathfrak{G}_m^d$. Thanks to [6], we have:

Lemma 2.1. *The ring $R_k(\mathfrak{G}_m^d)$ and the ring $\mathbb{Z}\left[X_1, .., X_d, X_1^{-1}, .., X_d^{-1}\right]$ are isomorphic.*

$$R_k(\mathfrak{G}_m^d) = \mathbb{Z}\left[X_1, .., X_d, X_1^{-1}, .., X_d^{-1}\right] \tag{2.1}$$

Let S_d be the symmetric group and σ a permutation of S_d : $\sigma.(a_{ii}) = b_{ii}$ where $b_{ii} = a_{\sigma(i)\sigma(i)}$ defines a natural action of S_d on the torus $\mathfrak{G}_m^d$. From this follows an action of S_d on the k-representation ring $R_k(\mathfrak{G}_m^d)$. Let $(R_k(\mathfrak{G}_m^d))^{S_d}$ be the sub-ring of $R_k(\mathfrak{G}_m^d)$ invariant under S_d action. It follows from [6] that:

Lemma 2.2. *The morphism j_d induces an isomorphism*

$$j_d^* : R_k(GL_d) \to (R_k(\mathfrak{G}_m^d))^{S_d}. \tag{2.2}$$

Let $t_i = X_i - 1$ and s_i be the i^{th} symmetric polynomial in the t_i's. Since $X_1 \times X_2 \times ... \times X_d$ is invertible in $\mathbb{Z}\left[X_1, ..., X_d, , (X_1)^{-1}, ..., (X_d)^{-1}\right]$ so is $(1+t_1) \times (1+t_2) \times ... \times (1+t_d) = 1 + s_1 + s_2 + ... + s_d$ and we have:

Proposition 2.3. *The isomorphism j_d^* in the Lemma 2.2 becomes:*

$$R_k(GL_d) \cong \mathbb{Z}\left[s_1, ..., s_d, (1 + s_1 + ... + s_d)^{-1}\right] \tag{2.3}$$

Proof. We have $\mathbb{Z}[X_1, ..., X_d] = \mathbb{Z}[t_1, ..., t_d]$. Then :

$$(\mathbb{Z}[X_1, ..., X_d])^{S_d} = (\mathbb{Z}[t_1, ..., t_d])^{S_d} = \mathbb{Z}[s_1, ..., s_d].$$

On the other hand :

$$\mathbb{Z}\left[X_1, ..., X_d, (X_1)^{-1}, ..., (X_d)^{-1}\right] = \mathbb{Z}[X_1, ..., X_d]\left[(X_1)^{-1}, ..., (X_d)^{-1}\right]$$

Which implies:

$$\begin{aligned} R_k(GL_d) &= \left(\mathbb{Z}\left[X_1, ..., X_d, X_1^{-1}, ..., X_d^{-1}\right]\right)^{S_d} \\ &= \left(\mathbb{Z}\left[X_1, ..., X_d][X_1^{-1}, ..., X_d^{-1}\right]\right)^{S_d} \\ &= (\mathbb{Z}[X_1, ..., X_d])^{S_d}\left[X_1^{-1}, ..., X_d^{-1}\right] \\ &= \mathbb{Z}[s_1, ..., s_d]\left[X_1^{-1}, ..., X_d^{-1}\right] \\ &= \mathbb{Z}\left[s_1, ..., s_d, (1 + s_1 + ... + s_d)^{-1}\right] \end{aligned}$$

□

The ring $R_k(GL_d)$ is augmented by $\epsilon_d : s \longmapsto dim(s)$, denote I_d its augmentation ideal and $\hat{R}_k(GL_d)$ its I_d-adic completed ring. Recall the general following result:

Lemma 2.4. *The augmentation ideal I_d is generated by $s_1, s_2, ..., s_d$.*

$$I_d = \langle s_1, s_{2,} ..., s_d \rangle$$

Proof. We know that I_d is generated by $X_i - 1$. Since $X_i - 1$ is an element of the ideal generated by s_i, then $I_d \subset \langle s_1, s_2, ..., s_d \rangle$.
Conversely, each symmetric polynomial s_i is expressed in terms of $X_i - 1$, so $\langle s_1, s_2, ..., s_d \rangle \subset I_d$. □

Lemma 2.5. *Let A be a commutative ring, I an ideal of A and $a \in A$ becomes invertible in A/I. Denote by $A_a = A[1/a]$ the local ring. Then for each $n \in \mathbb{N}$ the obvious map $A/I^n \to A_a/(A_a.I)^n$ is an isomorphism. Thus, the I-adic completion $\hat{A}$ of A and J-adic completion of $A[1/a]$, where J is the ideal generated by the image of I in A_a, are the same.*

$$\hat{A} = \widehat{A[1/a]} \tag{2.4}$$

Proof. See [5, Lemma III.91 on p.120]. □

Proposition 2.6. *The I_d-adic completion of the ring $R_k(GL_d)$ is the ring of formal power series $\mathbb{Z}\,[[s_1, ..., s_d]]$.*

$$\hat{R}_k(GL_d) \cong \mathbb{Z}\,[[s_1, s_2 ..., s_d]] \tag{2.5}$$

Proof. We deduce from the Proposition 2.3 that $R_k(GL_d)$ is isomorphic to the ring $\mathbb{Z}\,[s_1, ..., s_d, (1 + s_1 + ... + s_d)^{-1}]$. By I-adic completion where $I = \langle s_1, ..., s_d \rangle$ we obtain :

$$\hat{R}_k(GL_d) \cong \mathbb{Z}\,[s_1, ..., s_d, (\widehat{1 + s_1} + ... + s_d)^{-1}]$$

Since $1 + s_1 + ... + s_d$ is invertible through $R_k(GL_d) \to R_k(GL_d)/I_d$ we get:

$$\hat{R}_k(GL_d) \cong \mathbb{Z}\,[s_1, ..., s_d, (\widehat{1 + s_1} + ... + s_d)^{-1}] \cong \mathbb{Z}\,[\widehat{s_1, ..., s_d}] \cong \mathbb{Z}\,[[s_1, ..., s_d]]$$

□

Proposition 2.7. *Let i_d^* be the restriction morphisms: $R_k(GL_{d+1}) \longrightarrow R_k(GL_d)$. We have $i_d^*(s_{d+1}) = 0$ and $i_d^*(s_i) = s_i$ for each $i \leq d$.*

Proof. Recall that the torus $\mathfrak{G}_m^d$ is the group of the diagonal matrices and X_i its characters. From each inclusion :

$$\begin{array}{rcl} i_d : \mathfrak{G}_m^d & \longrightarrow & \mathfrak{G}_m^{d+1} \\ M & \longmapsto & i_d(M) \end{array}$$

arises a morphism

$$\begin{array}{rcl} i_d^* : R(\mathfrak{G}_m^{d+1}) & \longrightarrow & R(\mathfrak{G}_m^d) \\ X_i & \longmapsto & i_d^*(X_i) \end{array}$$

- For $i = d + 1$:
  We have $i_d^*(X_{d+1}) = X_{d+1} \circ i_d : M \longrightarrow 1$.
  Then $i_d^*(t_{d+1}) = i_d^*(X_{d+1} - 1) = 0$.
  Which implies $i_d^*(s_{d+1}) = 0$.
- For $i \leq d$:
  Let $M = (a_{ii})_{i \geq 1}$ be a diagonal matrix in $\mathfrak{G}_m^d$.
  We have $i_d^*(X_i) = X_i \circ i_d : M \longrightarrow a_{ii}$ for each $i \leq d$.
  Then $i_d^*(X_i) = X_i$ for each $i \leq d$. We obtain $i_d^*(t_i) = t_i$ for each $i \leq d$.
  Thus $i_d^*(s_i) = s_i$ for each $i \leq d$.

Finally $i_d^*(s_{d+1}) = 0$ and $i_d^*(s_i) = s_i$. □

Corollary 2.8. *The transition morphisms* $\hat{i}_d^* : \widehat{R}_k(GL_{d+1}) \longrightarrow \widehat{R}_k(GL_d)$ *are surjective for each* d.

3. Representation ring of the special linear group

Definition 3.1. Let d be an integer.

(1) The special group SL_d is the group of matrices with a determinant of 1.
(2) Its maximal torus is denoted by $T_d = G_m^d \cap SL_d$.

Proposition 3.2. *The representation ring of* T_d *is given by:*

$$R_k(T_d) = \mathbb{Z}\left[X_1, ..., X_d, X_1^{-1}, ..., X_d^{-1}\right] / X_1 \times X_2 ... \times X_d = 1 \tag{3.1}$$

Proof. We have $T_d = \{A = (a_{ii})_{i=1,..,d}$ with $a_{ii} \in k^*$ such that $a_{11} \times a_{22} \times ... \times a_{nn} = 1\}$, then T_d is isomorphic to $(k^*)^d / a_{11} \times a_{22} \times ... \times a_{nn} = 1$. The irreductible representations of T_d are the restrictions of the irreductible representations GL_1^d to T_d. Since $a_1 \times a_2 \times ... \times a_n = 1$ in T_d then $X_1 \times X_2 ... \times X_d = 1$ in $R_k(T_d)$. As a result $R_k(T_d) = \mathbb{Z}\left[X_1, ..., X_d, X_1^{-1}, ..., X_d^{-1}\right] / X_1 \times X_2 ... \times X_d = 1$. ☐

Proposition 3.3. *Let* $t_i = X_i - 1$. *Denote* s_i *the* i^{th} *symmetric polynomial on* t_i. *Hence,*

$$R_k(SL_d) = \mathbb{Z}\left[s_1, ..., s_d\right] / s_1 + ... + s_d = 0 \tag{3.2}$$

Proof. According to [6] we have $R_k(T_d)^{S_d} = R_k(SL_d)$ and it follows from the Proposition 3.2 that

$$R_k(T_d) = \mathbb{Z}\left[X_1, ..., X_d, X_1^{-1}, ..., X_d^{-1}\right] / X_1 \times X_2 ... \times X_d = 1$$

Then

$$\begin{aligned} R_k(SL_d) &= \left(\mathbb{Z}\left[X_1, ..., X_d, X_1^{-1}, ..., X_d^{-1}\right] / X_1 \times X_2 ... \times X_d = 1\right)^{S_d} \\ &= \mathbb{Z}\left[X_1, ..., X_d, X_1^{-1}, ..., X_d^{-1}\right]^{S_d} / X_1 \times X_2 ... \times X_d = 1 \end{aligned}$$

Using the Lemma 2.2 and the Proposition 2.3 we get

$$\mathbb{Z}\left[X_1, ..., X_d, X_1^{-1}, ..., X_d^{-1}\right]^{S_d} = \mathbb{Z}\left[s_1, s_2, ..., s_d, (1 + s_1 + ... + s_d)^{-1}\right]$$

We have $X_1 \times X_2 \times ... \times X_d = (t_1+1)(t_2+1)...(t_d+1) = 1 + s_1 + ... + s_d$ which implies $s_1 + ... + s_d = 0$. Then $R_k(SL_d) = \mathbb{Z}\left[s_1, ..., s_d, (1 + s_1 + .. + s_d)^{-1}\right] / s_1 + ... + s_d = 0$. Finally $R_k(SL_d) = \mathbb{Z}\left[s_1, ..., s_d\right] / s_1 + .. + s_d = 0$. ☐

Proposition 3.4. *The completed ring of* $R_k(SL_d)$ *is*

$$\widehat{R}_k(SL_d) = \mathbb{Z}\left[\left[s_1, ..., s_d\right]\right] / s_1 + ... + s_d = 0 \tag{3.3}$$

4. Links between the previous representation rings

Consider the following embeddings:

- $i_d : GL_d \hookrightarrow GL_{d+1}$
- $j_d : SL_d \hookrightarrow SL_{d+1}$
- $k_d : SL_d \hookrightarrow GL_d$

We denote by s_d the group morphism bellow:

$$\begin{array}{ccc} GL_d & \longrightarrow & SL_{d+1} \\ A & \longmapsto & \begin{bmatrix} A & 0 \\ 0 & det(A)^{-1} \end{bmatrix} \end{array} \qquad (4.1)$$

We have $s_d \circ k_d = j_d$. Each of them induces a morphism between the k-representation rings.

Proposition 4.1. *The morphism*

$$\begin{array}{ccc} s_d^* : R(SL_{d+1}) & \longrightarrow & R(GL_d) \\ \sigma & \longrightarrow & s_d^*(\sigma) \end{array}$$

is an isomorphism.

Proof. Let $\mathfrak{G}_m^{d+1}$ be the maximal torus of GL_{d+1} and S_{d+1} the symmetric group. The following diagram is commutative:

$$\begin{array}{ccc} R(GL_d) & \cong & (R(\mathfrak{G}_m^{d+1}))^{S_d} \\ \uparrow s_d^* & & \uparrow t_d^* \\ R(SL_{d+1}) & \cong & R(\mathfrak{G}_m^{d+1} \cap SL_{d+1})^{S_{d+1}} \end{array}$$

Where t_d is the restriction of s_d to $\mathfrak{G}_m^d$

$$t_d^* : \frac{\mathbb{Z}\left[X_1, X_2, ..., X_{d+1}, X_1^{-1}, X_2^{-1}, ..., X_{d+1}^{-1}\right]}{X_1 X_2 .... X_{d+1} = 1} \longrightarrow \mathbb{Z}\left[X_1, X_2, ..., X_d, X_1^{-1}, X_2^{-1}, ..., X_d^{-1}\right]$$

$$t_d^*\left(X_i\right)\left(diag(a_{jj})\right) = X_i \begin{bmatrix} diag(a_{jj}) & \\ & \pi(a_{jj})^{-1} \end{bmatrix}$$

We have

- For $1 \leq i \leq d$ $\qquad t_d^*(X_i) = X_i$
- For $i = d+1$ $\qquad t_d^*(X_{d+1}) = X_1^{-1} X_2^{-1} .... X_d^{-1}$

So :

$$t_d^* : \frac{\mathbb{Z}\left[X_1, X_2, ..., X_d, X_{d+1}, X_1^{-1}, X_2^{-1}, ..., X_d^{-1}, X_{d+1}^{-1}\right]}{X_1 X_2 ... X_d X_{d+1} = 1} \longrightarrow \mathbb{Z}\left[X_1, X_2, ..., X_d, X_1^{-1}, X_2^{-1}, ..., X_d^{-1}\right]$$

From the diagram that s_d^* is an isomorphism. □

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