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SOLID AND $H-$SOLID TOPOLOGIES ON SEQUENCE GROUPS.

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Abstract. The present paper is concerned with the concept of solid topologies in non-archimedean and classical analysis. We introduce the notion of solid and $H-$solid topologies in sequence groups over a topological group; then we give characterizations of such topologies.

1. Introduction and Preliminary

In sequence spaces on a vector space, the notion of solid parts is defined using the external law of the vector space [1, p.2]. In the case of groups of sequences on a group, we will give adequate definitions for solids and $H-$solid parts taking into account the existence of only of internal law, and we define a solid topology and $H-$solid topology on a group of sequences. In [1, Theorem 6.3], Boos and Leiger extended wellknown inclusion theorems of Bennet and Kalton to a solid sequence spaces in the classical case. In non-archimedean analysis (n.a), De Grande-De Kimpe [2] gave a characterization of the natural topology Na over a perfect sequence space; in particular she demonstrated that this topology is solid; this result was generalized on a sequence space $E(X)$ over a topological vector space X by Ameziane Hassani, El amrani and Babahmed in [3]; we gave also some results concerning polar and solid topology on $E(X)$. In [4], we developed a theory of a duality in sequence spaces over a non-archimedean vector space, we introduce polar and solid topologies in such spaces, and we gave basic results characterizing compact, Ccompact, complete and AKcomplete subsets related to these topologies. We know that, in the case of sequence spaces over locally convex space, the topology is solid if, and only if it is defined by absolutely monotonic family of semi-norms [5, p.3] ; we will give similar results for sequences groups provided on topologies defined by a directed family of semi-norms.

Throughout this paper, K is a non-archimedean $(n.a)$ non trivially valued complete field with valuation $|.|$ and G denotes a commutative group additively denoted.

Definition 1.1. A map $p : G \longrightarrow \mathbb{R}$ checking out the following is called a semi-norm over G :

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i. $p(0) = 0$ and for all $x \in G$ $p(x) \geq 0$;
ii. for all $x \in G$; $p(-x) = p(x)$;
iii. for all $x, y \in G$; $p(x + y) \leq p(x) + p(y)$.
If of over p verifies:
iv. for all $x \in G$; $p(x) = 0 \Longrightarrow x = 0$;
p is called norm over G.

Definition 1.2. We called non-archimedean ($n.a.$) semi-norm (resp. norm) over G all semi-norm (resp. norm) p over G that verifies:

$$for\ all\ \ (x, y) \in G^2; \ \ p(x + y) \leq max\,(p(x), p(y))\,.$$

Let G a metrizable topological group, the topology of G is defined by an invariant metric d, or equivalently, by a norm p [6, p.44, 45]. We denote $\omega(G)$ the linear set of all sequences in G; equipped with the canonical addition $\omega(G)$ is a commutative group. We call sequence group on G all subgroup of $\omega(G)$. We define the following sequence groups over G :

$$c_0\,(G) = \{(x_k)_k \in \omega\,(G) : (x_k)_k \ \ converges\ to\ zero\ \}$$

$$c\,(G) = \{(x_k)_k \in \omega\,(G) : (x_k)_k \ \ converges\ in\ G\},$$

$$\varphi\,(G) = \{(x_k)_k \in \omega\,(G) : \ there\ exists\ k_0 \in \mathbb{N} : x_k = 0\ for\ all\ k \geq k_0\},$$

$$m\,(G) = \{(x_k)_k \in \omega\,(G) : (x_k)_k \ \ is\ bounded\ in\ G\}.$$

Let E a K vector space, a subset X of $\omega\,(E)$ is called solid if: $(x_k)_k \in X \Longrightarrow (c_k x_k)_k \in X$ for all $(c_k)_k \in \omega\,(K)$ such that $|c_k| \leq 1$ for all $k \geq 1$. In the case of groups, we introduce the following definition:

Definition 1.3. Let H a subset of $\omega\,(G)$; a subset X of $\omega\,(G)$ is Hsolid if: $(x_k)_k \in X \Longrightarrow (h_k + x_k)_k \in X$ for all $(h_k)_k \in H$. ($i.e.$ $H + X \subset X$). And we say that X is solid if X is Xsolid. ($i.e.$ $X + X \subset X$).

Remark 1.4. It is clear that these two concepts are different, but there is some analogy of the fact that in the case of vector spaces, if we note $B_1 = \{(\lambda_k) \in \omega\,(K) / |\lambda_k| \leq 1\ for\ all\ k \geq 1\}$, a subset X is solid if $B_1.X \subset X$.

We have the followings properties:

Proof.
(1) $\omega\,(G)$ is Hsolid for all $H \subset \omega\,(G)$.
(2) If a subset X of $\omega\,(G)$ is Hsolid, then for all $K \subset H$, X is Ksolid.
(3) $c_0\,(G)$ is $\varphi\,(G)$ solid, $c\,(G)$ is $c_0\,(G)$ solid and $m\,(G)$ is $c\,(G)$ solid.
(4) $\omega\,(G)$ and all sub-group of $\omega\,(G)$ are solids.
(5) The sum of a finite family of H solids (resp. solids) subsets is Hsolid (resp. solid).
(6) The intersection of family of H solids (resp. solids) subsets is Hsolid (resp. solid).
(7) The union of family of H solids subsets is H solid. But the union of family of solid subsets is not necessarily solid.

□

Definition 1.5. Let $(X_\alpha)_{\alpha\in A}$ a family of subsets of $\omega(G)$; we say that this family is R saturated if: for all $x_\alpha \in X_\alpha$, for all $x_\beta \in X_\beta$, there exists $\gamma \in A$ such that $x_\alpha + x_\beta \in X_\gamma$.

All filtering family of solid subsets of $\omega(G)$ is Rsaturated.

Proposition 1.6. *Let $(X_\alpha)_{\alpha\in A}$ a family of subsets of $\omega(G)$; if $(X_\alpha)_{\alpha\in A}$ is R saturated then $\bigcup\{X_\alpha/\ \alpha\in A\}$ is solid.*

Let A a subset of commutative group S and n an integer ; we note by $n.A$ the set $A+A+....+A$ (n *times*) if $n\geq 1$ and $\{0\}$ if $n=0$.

Proposition 1.7. *Let H and X two subsets of $\omega(G)$.*

1. *if X is Hsolid, then $n.H+X\subset X$ for all $n\geq 1$;*
2. *If X is solid, then $n.X\subset X$ for all $n\geq 1$;*
3. *If X is solid, then X is Hsolid for all $H\subset X$.*

Let H and K two subsets of $\omega(G)$. We call Hsolid (resp. solid) envelope of X the small east Hsolid (resp. solid) subset containing X; we denote $EHS(X)$ (resp. $ES(X)$). If X and Y are two subsets of $\omega(G)$ such that $X\subset Y$, then $EHS(X)\subset EHS(Y)$ and $ES(X)\subset ES(Y)$.

Proposition 1.8. *Let H and X two subsets of $\omega(G)$.*

1. $EHS(X)=\cup\{n.H\ /\ n\geq 0\}+X$;
2. $ES(X)=\cup\{n.X\ /\ n\geq 1\}$;
3. *If H is solid, then $EHS(X)=(H\cup\{0\})+X$;*
4. *If H is a sub-group of $\omega(G)$, solid, then $EHS(X)=H+X$.*

Proposition 1.9. *Let H, X and Y three subsets of $\omega(G)$, and let $(X_\alpha)_{\alpha\in A}$ a family of subsets of $\omega(G)$.*

1. $EHS(X+Y)=EHS(X)+EHS(Y)$;
2. $ES(X+Y)\subset ES(X)+ES(Y)$; *if X and Y contain 0, then* $ES(X+Y)=ES(X)+ES(Y)$;
3. $EHS(\cap X_\alpha\ /\ \alpha\in A))\subset\cap EHS(X_\alpha)\ /\ \alpha\in A$;
4. $ES(\cap X_\alpha\ /\ \alpha\in A)\subset\cap ES(X_\alpha)\ /\ \alpha\in A$;
5. $EHS(\cup\{X_\alpha\ /\ \alpha\in A\})=\cup\{EHS(X_\alpha)\ /\ \alpha\in A\}$;
6. $\cap\{ES(X_\alpha)\ /\ \alpha\in A\}\subset ES(\cap\{X_\alpha)\ /\ \alpha\in A\})$;
7. *If $(X_\alpha)_{\alpha\in A}$ is Rsaturated, we have:*

$$\cap\{ES(X_\alpha)/\alpha\in A\}=ES(\{X_\alpha/\alpha\in A\})=\cup\{X_\alpha/\alpha\in A\};$$

Let E a sequence group over G and H a subset of $\omega(G)$; a subset X of E is called H solid in E if $EHS(X)\cap E=X$. $EHS(X)\cap E$ is called the Hsolid envelope of X in E.

Proposition 1.10. *Let H,X two subsets of $\omega(G)$ and E a sequence group over G. If X is Hsolid in E, then X is $(H\cap E)$ solid.*

Let E a sequence group over G and H a subset of $\omega(G)$; a topology of group over E is called Hsolid (resp. solid) if it have a Hsolid (resp. solid) fundamental system of neighborhood ($F.S.N.$) of 0 in E. We say that a semi-norm (resp. norm) over E is Hsolid (resp. solid) if the topology which define is Hsolid (resp. solid) topology , i.e. there exists a F.S.N. of 0 formed of Hsolid (resp. solid) balls in E.

Example 1.11. (1) Let G a topological group having a denombrable fundamental system of neighborhood of 0 formed of open sub-groups; then there exists d, an invariant non-archimedean ($n.a.$) metric or in an equivalent manner, a $n.a.$ norm $\|.\|$ defining its topology [6, Theorem 6.4, p.43]. For all $x = (x_k)_k \in m(G)$, we put $\|x\|_\infty = \sup_k \|x_k\|$; $\|.\|_\infty$ is solid over $m(G)$.

(2) Let G the same group; set ourselves $a \in G$ such that $\|a\| = r_0 > 0$. Let $(r_p)_{p \geq 0}$ a sequence of positive numbers such that $r_p \succ r_0$ for all $p \geq 1$. For all $i \geq 1$ we put: $\|x\|_i = \|x_i\|$ for all $x = (x_k) \in \omega(G)$; $B_i(0, \varepsilon) = \{x \in \omega(G) / \|x\|_i \prec \varepsilon\}$, for all $\varepsilon \succ 0$. Let τ the topology over $\omega(G)$ with a F.S.N. of 0 is formed by all finite intersections of balls $B_i(0, r_p)$, $i \geq 1$ and $p \geq 0$. τ is a topology of group over $\omega(G)$. Let $H = \cap \{B_i(0, r_0) / i \geq 1\}$; $x = (a, a, ...) \in H$. For all $i \geq 1$ and $p \geq 0$ we have $H + B_i(0, r_p) \subset B_i(0, r_p)$. Therefore τ is Hsolid.

Definition 1.12. Let E a sequence group over G, H a subset of $\omega(G)$ and p a semi-norm (resp. norm) over E.

(1) We say that p is H increasing if for all $x, y \in E$, we have $EHS(x) \subset EHS(y) \Longrightarrow p(x) \leq p(y)$;
(2) We say that p is Sincreasing if for all $x, y \in E$, we have $ES(x) \subset ES(y) \Longrightarrow p(x) \leq p(y)$.

Example 1.13. (1) All $n.a.$ semi-norm is $S-$ increasing.

(2) Let G the group from the previous example, on $\omega(G)$ we define the following semi-norm:

$$\|x\|_1 = \|x_1\| \ \forall x = (x_k)_k \in \omega(G).$$

We pose $H = \{(x_k)_k \in \omega(G) / \ x_1 = 0\}$. Let $x, y \in \omega(G)$ such that $EHS(x) \subset EHS(y)$, there exist $h_1, h_2, ..., h_n \in H$ such that $x = h_1 + ... + h_n + y$; $\|x\|_1 = \|y_1\| = \|y\|_1$, then $\| \ . \ \|_1$ is $H-$ increasing.

Let S a commutative group and $\mathcal{P} = (p_i)_{i \in I}$ a filtering family of semi-norms on S. For all $i \in I$ and all $\varepsilon > 0$, we pose: $B_i(0, \varepsilon) = \{x \in X / \ p_i(x) \leq \varepsilon\}$. We show that the set of finite intersections of balls $B_i(0, \varepsilon)$ form a $F.S.N.$ of 0 of a topology of group $\tau(\mathcal{P})$ on S called $\mathcal{P}-$topology.

Definition 1.14. Let E a sequence group on G, H a subset of $\omega(G)$ and $\mathcal{P}$ a filtering family of semi-norms over E; the $\mathcal{P}-$ topology on E is called $H-$increasing (resp.) Sincreasing if for all $p \in \mathcal{P}$, p is a Hincreasing (resp.) S increasing semi-norm.

2. Main results

Proposition 2.1. *Let E a sequence group on G and H a subset of $\omega(G)$. If τ is a $H-$ solid topology on E, then any point of E admits an $H-$solid F.S.N. in E.*

Proof. Let $\{U_\alpha/\alpha \in A\}$ an $H-$solid $F.S.N.$ of 0 in E and let x an element of E. $\{(x+U_\alpha)/\alpha \in A\}$ is an $F.S.N.$ of x. For all $\alpha \in A$, $EHS(U_\alpha) \cap E = U_\alpha$. Let $z \in EHS(x+U_\alpha) \cap E$, there exists $h_1, ..., h_n \in H$ and $y \in U_\alpha$ such that $z = h_1 + ... + h_n + x + y \in x + U_\alpha$, then $x + U_\alpha$ is $H-$solid in E. This result is not generally true for solid topology. □

Proposition 2.2. *Let H a subset of $\omega(G)$ and τ a topology on $\omega(G)$. If τ is $H-$solid (resp. solid), then the $H-$solid (resp. solid) closure of the envelope of an subset X of $\omega(G)$ is $H-$solid (resp. solid).*

Proposition 2.3. *Let E a sequence group on G, H a subset of $\omega(G)$ and τ a Hsolid topology on E. If the Hsolid envelope of an subset X of E is contained in E, then its closure in E is Hsolid.*

Proof. Let $\{U_\alpha/\alpha \in A\}$ a $H-$ solid $F.S.N.$ of 0 in E, and let $z \in EHS(\overline{EHS(X)} \cap E)$ there exists $h_1, ..., h_m \in H$, and $y \in \overline{EHS(X)}$ such that $z = h_1 + .. + h_m + y$. For all $\alpha \in A, y \in EHS(X) + U_\alpha$, there exists $k_1, .., k_n \in H$, $x \in X$, and $x_\alpha \in U_\alpha$ such that $y = k_1 + ... + k_n + x + x_\alpha$; then $z \in EHS(X) + U_\alpha$, $\forall \alpha \in A$, and so $z \in \overline{EHS(X)}$. □

If τ is a solid topology on E, the solid envelope of all subset X of E is contained in E and its closure in E is solid.

Theorem 2.4. *Let H a subset of $\omega(G)$ and p a semi-norm on $\omega(G)$. If p is Hincreasing, then p is Hsolid.*

Proof. Suppose that p is not H solid; there exists a ball $B(0, r)$ which is not H solid. Let $x \in B(0, r)$ and $h \in H$ such that $x + h \notin B(0, r)$. We have $p(x) \leq r < p(x + h)$, then $EHS(x + h) \not\subset EHS(x)$, which is absurd. □

Theorem 2.5. *Let H a subgroup of $\omega(G)$ and p a semi-norm on $\omega(G)$. If p is H solid, then there exist a Hincreasing semi-norm defining the same topology on $\omega(G)$.*

Proof. Let:

$$\begin{cases} q(x) = Sup\{p(x+h)/h \in H\} & if\ x \neq 0; \\ q(0) = 0. \end{cases}$$

q is well defined: Let $x \in \omega(G)$, there exists $r > 0$ such that $B_p(x, r)$ is Hsolid ; then $Sup\{p(x+h)/h \in H\} \leq p(x) + r < \infty$. q is a semi-norm on $\omega(G)$:

$q(-x) = Sup\{p(-x+h)/h \in H\} = Sup\{p(x+k)/k \in H\} = q(x)$;

$q(x+y) = Sup\{p(x+y+h)/h \in H\} = Sup\{p(x+y+h+k)/h, k \in H\} \leq q(x)+q(y)$

q is $H-$ increasing: Let $x, y \in \omega(G)$ such that $EHS(x) \subset EHS(y)$, then $x + H \subset y + H$, and so $q(x) \leq q(y)$. q defines the same topology as p :

$$\forall x \in \omega(G),\ p(x) \leq q(x).$$

Furthermore, let $B_q(0, r)$ a neighborhood of 0 for $q; p$ being $H-$ solid, then there exists $\rho \leq r$ such that $B_p(0, \rho)$ is $H-$ solid, then we have $B_p(0, \rho) \subset B_q(0, \rho) \subset B_q(0, r)$. □

We can combine the two previous theorems assuming H a sub-group of $\omega(G)$; we have the following theorem:

Theorem 2.6. *Let H a subgroup of $\omega(G)$, p a seminorm on $\omega(G)$ and τ the topology defined by p. In order that τ is $H-$ solid it is necessary and sufficient that there exists an $H-$ increasing semi-norm on $\omega(G)$ equivalently to p.*

To show the theorem analogous to Theorem 2.5 in the case of solid semi-norms we need enter the following definition:

Definition 2.7. Let E a sequence group on G; a semi-norm p on E is said to be everywhere solid if it is solid and each point of E has a solid neighborhood.

A $\mathcal{P}-$ topology is said to be everywhere solid if for all $p \in \mathcal{P}$, p is everywhere solid.

Proposition 2.8. *Let E a sequence group on G and p a n.a. semi-norm on E; the topology defined by p is an everywhere solid topology.*

Proof. It is clear that this topology is solid. Let $x \in E$ and $r = p(x)$. If $r = 0$, we have $B_p(x,1) + B_p(x,1) \subset B_p(x,1)$. If $r > 0$, we have $B_p(x,r) + B_p(x,r) \subset B_p(x,r)$. □

Theorem 2.9. *Let p a semi-norm on $\omega(G)$. If p is everywhere solid, then there exist an S increasing semi-norm defining the same topology on $\omega(G)$.*

Proof. Let:

$$\begin{cases} q(x) = Sup\,\{p(n.x)/n \geq 1\} & if\ x \neq 0; \\ q(0) = 0. \end{cases}$$

q is well defined: Let $x \in \omega(G)$, there exist $r > 0$ such that $B_p(x,r)$ is solid; then for all $n \geq 1,\ n.x \in B_p(x,r)$, and consequently Sup $Sup\,\{p(n.x)/n \geq 1\} \leq r < \infty$.
q is a semi-norm on $\omega(G)$:

$$\begin{aligned} q(-x) &= Sup\,\{p(n.(-x))\ /\ n \geq 1\} \\ &= Sup\,\{p(n.x)/n \geq 1\} \\ &= q(x) \end{aligned}$$

$$\begin{aligned} q(x+y) &= Sup\,\{p(n.(y+h))/n \geq 1\} \\ &= Sup\,\{p(n.x)/n \geq 1\} + Sup\,\{p(n.y)/n \geq 1\} \\ &= q(x) + q(y) \end{aligned}$$

q is $S-$ increasing :
Let $x, y \in \omega(G)$ such that $ES(x) \subset ES(y)$, therefore $\{n.x/n \geq 1\} \subset \{n.y/n \geq 1\}$, and consequently $q(x) \leq q(y)$. q define the same topology as p :

$$\forall x \in \omega(G),\ p(x) \leq q(x).$$

On the other hand, let $B_q(0,r)$ a neighborhood of 0 for q; p is solid, there exists $\rho \leq r$ such that $B_p(0,\rho)$ is solid and then we have $B_p(0,\rho) \subset B_q(0,\rho) \subset B_q(0,r)$. □

Theorem 2.10. *Let E a sequence group on G, H a sub-group of $\omega(G)$ and τ a $\mathcal{P}$ topology on E. In order that τ is Hsolid it is necessary and sufficient that there exists a Hincreasing Qtopology defining the same topology on E.*

Proof. Suppose that τ is a Hsolid topology. For all $p \in \mathcal{P}$, let:

$$\begin{cases} q_p(x) = Sup\{p(x+h)/h \in H\} & if\ x \neq 0; \\ q_p(0) = 0. \end{cases}$$

Let $Q = \{q_p/\ p \in \mathcal{P}\}$. Q is a Hincreasing directed family of semi-norms which define the topology τ.

Conversely, let τ a Hincreasing $\mathcal{P}$ topology on E. Suppose that τ is not Hsolid, then there exists $\mathcal{J} \in \mathcal{F}(\mathcal{I})$ such that $\cap\{B_i(0,\varepsilon_i)\ /\ \varepsilon_i > 0,\ i \in \mathcal{J}\}$ is not Hsolid in E. Let $z \in ([H + \cap\{B_i(0,\varepsilon_i)/\ \varepsilon_i > 0,\ i \in \mathcal{J}\}] \cap E) \backslash \cap \{B_i(0,\varepsilon_i)/\ \varepsilon_i > 0,\ i \in \mathcal{J}\}$. Let $h \in H$ and $x \in \cap\{B_i(0,\varepsilon_i)/\ \varepsilon_i > 0,\ i \in \mathcal{J}\}$ such that $z = x + h$, there exists $i \in \mathcal{J}$ such that $p_i(x) \leq \varepsilon_i < p_i(x+h)$; p_i is Hincreasing , therefore $EHS(x+h) \subset EHS(s)$, which is absurd. □

Proposition 2.11. *Let E a sequence group on G and $\mathcal{P}$ a directed family of n.a. semi-norms on E; then the $\mathcal{P}-$topology on E is everywhere solid.*

Theorem 2.12. *Let E a sequence group on G and $\mathcal{P}$ a directed family of semi-norms on E. If the $\mathcal{P}-$ topology is everywhere solid, then there exists an $S-$increasing $Q-$topology defining the same topology on E.*

Proof. For all $p \in \mathcal{P}$, we pose:

$$\begin{cases} q_p(x) = Sup\{p(n.x)/n \geq 1\} & if\ x \neq 0; \\ q_p(0) = 0. \end{cases}$$

For all $p \in \mathcal{P}$, q_p is an Sincreasing semi-norm on E. Let $Q = \{q_p/p \in \mathcal{P}\}$; Q is a directed family. For all $p \in \mathcal{P}$, we have $p(x) \leq q_p(x) \forall x \in E$. Let $p \in \mathcal{P}$, $B_{q_p}(0,r)$ an neighborhood of 0 for the Qtopology; p is solid, there exits $\rho \leq r$ such that $B_p(0,\rho)$ is solid, and then we have $B_p(0,\rho) \subset B_{q_p}(0,\rho) \subset N_{q_p}(0,r)$, therefore $\mathcal{P}$ and Q define the same topology on E. □

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