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TOEPLITZ AND WEIGHTED TOEPLITZ OPERATORS ON WEIGHTED SEQUENCE SPACES

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Abstract. Weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$ with $\phi \in L^\infty(\beta)$ is defined as $T_\phi^\beta = P^\beta M_\phi^\beta$, where P^β is the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. The paper describes some properties of weighted Toeplitz operators induced by some specific symbols.

1. Introduction

Multiplication operators play an important role in the theory of operators, with one of the reason that every normal operator is similar to a multiplication operator. These operators have been studied extensively over various spaces and in particular, on Hardy spaces of analytic functions. The class of multiplication operators becomes a never formidable notion with their tendency to provide the Toeplitz operators and Hankel operators. Toeplitz operators, introduced by O. Toeplitz [17] in the year 1911, arise in many applications, constitute one of the most important classes of non self-adjoint operators. The study of Toeplitz operators becomes more demanding with the inception of the notion of slant Toeplitz operators by Ho [13] in 1996, which has widely appeared in connection with the wavelet theory, having the property that their matrices with respect to the standard orthonormal basis could be obtained by eliminating every alternate row of the matrices of the corresponding Toeplitz operators. The study in this direction is enhanced with the introduction of various new classes of operators over various function spaces, like, k^{th}-order slant Toeplitz operators, essentially slant Toeplitz operators, λ-Toeplitz and essentially λ-Toeplitz operators, $\lambda \in \mathbb{C}$, the set of all complex numbers (see the references [[1]-[7],[10]-[12]] and the references therein). Around the year 1966, the notion of weighted sequence spaces was brought forth by R.L. Kelley [14]. Shields [16] focussed the attention of mathematician towards the study of multiplication operators and the weighted shift operators over weighted sequence spaces, which have the tendency to cover Hardy spaces, Bergman spaces and Dirichlet spaces. However, weighted Hardy spaces appeared in the work of Yousefi [18] and Zorboska [19, 20], where they discussed

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the notion of composition operators. In the year 2005, Lauric [15] discussed the notion of weighted Toeplitz operators on weighted sequence spaces, whereas, the notion of weighted Hankel operators on details was studied in [8, 9]. We begin with the following notational familiarity needed in the paper, for the details of which, we refer [8, 9, 16] and the references therein.

Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$, $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $r \leq \frac{\beta_n}{\beta_{n-1}} \leq 1$ for $n \leq 0$, for some $r > 0$ (this assumption is taken on β throughout the paper). Let $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be the formal Laurent series (whether or not the series converges for any values of z). Define $\|f\|_\beta$ as

$$\|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 {\beta_n}^2.$$

The space $L^2(\beta)$ consists of all $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ for which $\|f\|_\beta < \infty$. The space $L^2(\beta)$ is a Hilbert space with the norm $\|\cdot\|_\beta$ induced by the inner product

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n\, \bar{b}_n {\beta_n}^2,$$

for $f(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, $g(z) = \sum\limits_{n=-\infty}^{\infty} b_n z^n$. The collection $\{e_n(z) = z^n/\beta_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\beta)$.

The collection of all $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ (formal power series) for which $\|f\|_\beta^2 = \sum\limits_{n=0}^{\infty} |a_n|^2 {\beta_n}^2 < \infty$, is denoted by $H^2(\beta)$. $H^2(\beta)$ is a subspace of $L^2(\beta)$.

Let $L^\infty(\beta)$ denote the set of formal Laurent series $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$ such that $\phi L^2(\beta) \subseteq L^2(\beta)$ and there exists some $c > 0$ satisfying $\|\phi f\|_\beta \leq c\|f\|_\beta$ for each $f \in L^2(\beta)$. For $\phi \in L^\infty(\beta)$, define the norm $\|\phi\|_\infty$ as

$$\|\phi\|_\infty = \inf\{c > 0 : \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for each } f \in L^2(\beta)\}.$$

$L^\infty(\beta)$ is a Banach space with respect to $\|\cdot\|_\infty$. $H^\infty(\beta)$ denotes the set of formal Power series ϕ such that $\phi H^2(\beta) \subseteq H^2(\beta)$.

Let $P^\beta : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. Weighted Toeplitz operator on $H^2(\beta)$ induced by $\phi \in L^\infty(\beta)$ is denoted by T_ϕ^β and given by $T_\phi^\beta = P^\beta M_\phi^\beta$, where M_ϕ^β is the weighted Laurent operator on $L^2(\beta)$ induced by $\phi \in L^\infty(\beta)$. Along with some properties of weighted Toeplitz operators, the paper describes the notion of Toeplitz operators on $H^2(\beta)$ and relates it to the notion of weighted Toeplitz operators.

2. Structural Properties

A weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$, defined as $T_\phi^\beta = P^\beta M_\phi^\beta$, induced by $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ satisfies that for $k \geq 0$,

$$T_\phi^\beta e_k = \frac{1}{\beta_k} \sum_{n=0}^{\infty} a_{n-k} \beta_n e_n \text{ and } T_\phi^{\beta *} e_k = \beta_k \sum_{n=0}^{\infty} \overline{a}_{k-n} \frac{e_n}{\beta_n}.$$

Once we put $\beta_n = 1$ for each n, then the notion of weighted Toeplitz operators on $H^2(\beta)$ coincides with the notion of Toeplitz operators on the Hardy space. It is known that the adjoint of a Toeplitz operator is a Toeplitz operator and the operator equation $U^*TU = T$ characterizes a Toeplitz operator, where $U(= M_z,$ the multiplication operator$)$ denotes the unilateral shift. Now, one can expect a similar characterization for weighted Toeplitz operators, i.e. a characterization of weighted Toeplitz operator by the operator equation $U^{\beta *} T U^\beta = T$, where $U^\beta (= M_z^\beta)$ is the operator satisfying $U^\beta z^n = z^{n+1}$ or given by $U^\beta e_n = \frac{\beta_{n+1}}{\beta_n} e_{n+1}$ for each $n \geq 0$. We notice that weighted Toeplitz operators need not satisfy the operator equation $U^{\beta *} T U^\beta = T$. For, consider the following example.

Example 2.1. Consider the sequence $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ defined as

$$\beta_n = \begin{cases} 1 & \text{if } n \leq 0 \\ 2^n & \text{if } n \geq 1 \end{cases}.$$

Let $\phi(z) = z^2$. Then, $\phi \in L^\infty(\beta)$ as $\|\phi f\|_\beta \leq 4\|f\|_\beta$ for each $f \in L^2(\beta)$. Consider the weighted Toeplitz operator T_ϕ^β induced by ϕ. Then, the structure of T_ϕ^β provides $T_\phi^\beta e_0 = \frac{1}{\beta_0} \sum_{n=0}^{\infty} a_n \beta_n e_n = 4e_2$ and $U^{\beta *} T_\phi^\beta U^\beta (e_0) = U^{\beta *} T_\phi^\beta (\frac{\beta_1}{\beta_0} e_1) = \frac{\beta_1}{\beta_0} U^{\beta *} (\frac{\beta_3}{\beta_1} e_3) = 16 e_2$.

Now consider the operator $V^\beta : H^2(\beta) \mapsto H^2(\beta)$ defined as $V^\beta e_n = \frac{\beta_n}{\beta_{n+1}} e_{n+1}$ for each $n \geq 0$. Then $\|V^\beta\| \leq 1$. It is easy to verify that a weighted Toeplitz operator T on $H^2(\beta)$ always satisfies $V^{\beta *} T U^\beta = T$. For, if $[\lambda_{i,j}]_{i,j \geq 0}$ is matrix representation of a weighted Toeplitz operator T_ϕ^β, $\phi = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$, on $H^2(\beta)$ with respect to the basis $\{e_n | n \geq 0\}$ then $\lambda_{i,j} = a_{i-j} \frac{\beta_i}{\beta_j}$ so that

$$\frac{\beta_{j+1}}{\beta_j} \lambda_{i+1,j+1} = \frac{\beta_{i+1}}{\beta_i} \lambda_{i,j}$$

for each $i, j \geq 0$. This observation helps to conclude that $V^{\beta *} T U^\beta = T$. Further, the adjoint of a weighted Toeplitz operator need not be a weighted Toeplitz operator, which can be justified through the following weighted Toeplitz operator.

Example 2.2. Consider the space $L^2(\beta)$, where the sequence $\beta = \{\beta_n\}$ is given by $\beta_n = 2^{|n|}$ for each $n \in \mathbb{Z}$. Define a formal Laurent series $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, where

$$a_n = \begin{cases} 0 & \text{if } n \leq 0 \\ \frac{1}{n^2 2^n} & \text{if } n \geq 1 \end{cases}.$$

Then, $\phi \in L^2(\beta)$ [[16], Theorem 10'(vii)]. Consider T_ϕ^β, the weighted Toeplitz operator induced by the above defined ϕ. Let, if possible, $T_\phi^{\beta *} = T_\psi^\beta$ for some $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n \in L^\infty(\beta)$. Now, on equating the entries of the matrix of the complex conjugate of T_ϕ^β with the matrix of T_ψ^β, we find that $b_n = \frac{\overline{a}_{-n}}{{\beta_n}^2}$ and $b_{-n} = \overline{a}_n {\beta_n}^2$ for $n \geq 0$. Now,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |b_n|^2 {\beta_n}^2 &= \sum_{n=-\infty}^{-1} |b_n|^2 {\beta_n}^2 + \sum_{n=0}^{\infty} |b_n|^2 {\beta_n}^2 \\ &= \sum_{n=1}^{\infty} |a_n|^2 {\beta_n}^4 {\beta_{-n}}^2 \\ &= \sum_{n=1}^{\infty} \frac{2^{4n}}{n^4}, \end{aligned}$$

which is divergent. Hence $\psi e_0 \notin L^2(\beta)$ and as a consequence $\psi \notin L^\infty(\beta)$. Thus, we can conclude that $T_\phi^{\beta *}$ is not a weighted Toeplitz operator.

The matrix structure of a weighted Toeplitz operator provides the following, which can be obtained without any extra efforts.

Theorem 2.3. *Let* $\phi = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$.

(1) *If the adjoint of a weighted Toeplitz operator* T_ϕ^β *is a weighted Toeplitz operator, then the inducing symbol of the adjoint is given by* $\psi = \sum_{n=-\infty}^{\infty} b_n z^n \in L^\infty(\beta)$, *where*

$$b_{i-j} = (\frac{\beta_j}{\beta_i})^2 \overline{a}_{j-i}$$

for $i, j \geq 0$.

(2) *A necessary condition for the adjoint of a weighted Toeplitz operator to be a weighted Toeplitz operator is that* $\{\frac{\beta_{n+1}}{\beta_n}\}_{n \geq 0}$ *is a constant sequence.*

(3) *A necessary and sufficient condition for the weighted Toeplitz operator* T_ϕ^β *to be self adjoint is that* $a_{i-j} = (\frac{\beta_j}{\beta_i})^2 \overline{a}_{j-i}$.

Proof. We just prove (3). For, let $T_\phi^\beta = T_\phi^{\beta *}$. Then $< \frac{1}{\beta_j} \sum_{n=0}^{\infty} a_{n-j} \beta_n e_n,\ e_i >=< \beta_j \sum_{n=0}^{\infty} \overline{a}_{j-n} \frac{e_n}{\beta_n}, e_i >$. Hence, $\frac{1}{\beta_j} a_{i-j} \beta_i = \beta_j \overline{a}_{j-i} \frac{1}{\beta_i}$, which gives $a_{i-j} = (\frac{\beta_j}{\beta_i})^2 \overline{a}_{j-i}$ for $i, j \geq 0$. The converse follows on retracing the above steps in backward direction. □

The condition obtained in Theorem 2.3(2) is not sufficient for the adjoint of a weighted Toeplitz operator to be weighted Toeplitz operator and this can be justified through Example 2.2. A straight forward computation shows that the operators commuting with T_z^β are lower triangular matrices given in the following form.

Proposition 2.4. *An operator A on $H^2(\beta)$ having matrix representation $[\alpha_{i,j}]_{i,j\geq 0}$ commutes with T_z^β if and only if*

$$\alpha_{i,j} = \begin{cases} 0 & \text{if } i < j \\ C & \text{if } i = j \\ \frac{\beta_i}{\beta_{i-j}\beta_j}\alpha_{i-j,0} & \text{if } i > j \end{cases},$$

where $C \in \mathbb{C}$ is any constant.

Proof. Proof can be obtained using the equality $< AT_z^\beta e_j, e_i >=< T_z^\beta A e_j, e_i >$ for different values of $i, j \geq 0$. □

3. Isometric, Hyponormal and Hilbert-Schmidt Operators

The aim of this section is to investigate the isometric weighted Toeplitz operators. Existence of some non isometric operators can be viewed through Example 2.1, where we find that $||T_\phi^\beta e_0||^2 = \frac{1}{{\beta_0}^2}{a_2}^2{\beta_2}^2 = 16$. It is interesting to find that $\phi(z) = z^n, n > 0$ generates the isometric weighted Toeplitz operator only in the case of Hardy spaces, but not in the case of other spaces with specifically defined sequences $\beta = \{\beta_n\}$, like Fischer space and Dirichlet space.

Theorem 3.1. *The weighted Toeplitz operator T_ϕ^β, induced by $\phi(z) = z^{n_0}, n_0 > 0$ is an isometry if and only if $\beta_n = 1$ for all $n \geq 0$.*

Proof. Let T_ϕ^β be an isometry. Then for all $j \geq 0$, $||T_\phi^\beta e_j|| = 1$. For j=0, this gives that $\beta_{n_0} = 1$, which provides that $\beta_k = 1$ for $1 \leq k \leq n_0$. Further, on putting $j = m(\geq 1)$, $||T_\phi^\beta e_m|| = 1$ implies $\beta_{n_0+m} = \beta_m$. Hence, we have $\beta_n = 1$ for each $n \geq 0$. Converse follows evidently. □

It is straight forward from here to state the following.

Corollary 3.2. *Each Toeplitz operator T_{z^n}, $n > 0$ on Hardy space is an isometry.*

Proof. Proof follows as weighted Toeplitz operator on weighted sequence space $H^2(\beta)$ becomes Toeplitz operator on the Hardy space when $\beta_n = 1$ for each n. □

It is worth mentioning that the weighted Toeplitz operator $T_{z^n}^\beta$, $n < 0$ can not be an isometry as $||T_{z^n} e_j|| = 0$ for $0 \leq j < -n$. Further, we observe the following.

Theorem 3.3. *The weighted Toeplitz operator T_ϕ^β induced by the polynomial symbols of the type $\phi(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, $a_n \neq 0, n \geq 1$, can't be an isometry.*

Proof. If we assume that a symbol $\phi(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0, n > 0$, is such that T^β_ϕ is an isometry then $||T^\beta_\phi e_0||^2 = {\beta_0}^2 + |a_1|^2 {\beta_1}^2 + \cdots + |a_n|^2 {\beta_n}^2 = 1$. As a consequence, we have $\beta_n = 0$, which is a contradiction. This completes the proof. □

Working along similar lines, we see that the weighted Toeplitz operator T^β_ϕ, with ϕ in $L^\infty(\beta)$ of the form $\sum\limits_{m=-\infty}^{n} a_m z^m$, such that $|a_0| = 1, n \geq 1$ and $a_n \neq 0$, fails to become an isometric operator.

Now we check some symbols for the hyponormal weighted Toeplitz operators and obtain the following.

Theorem 3.4. *Weighted Toeplitz operator T^β_ϕ, induced by $\phi(z) = a_n z^n$, $n < 0$, is hyponormal if and only if $a_n = 0$ (equivalently $\phi = 0$).*

Proof. Suppose $\phi(z) = a_n z^n, n < 0$ is such that T^β_ϕ is hyponormal. Then $||T^\beta_\phi e_0|| \geq ||T^{\beta^*}_\phi e_0||$, which yields that $\sum\limits_{m=0}^{\infty} |a_m|^2 {\beta_m}^2 \geq \sum\limits_{m=0}^{\infty} |a_{-m}|^2 \ \frac{1}{{\beta_m}^2}$. This gives that $|a_n|^2 \leq 0$. Thus $a_n = 0$ or $\phi = 0$. Converse follows trivially. □

With the above theorem, we can say that $T^\beta_{z^n}, n < 0$ can not be hyponormal. Every normal operator is hyponormal so the above theorem also provides the following.

Corollary 3.5. *Weighted Toeplitz operator T^β_ϕ, induced by $\phi(z) = a_n z^n$, $n < 0$, is normal if and only if $\phi = 0$.*

For $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$, we have

$$\sum_{n=0}^{\infty} ||T^\beta_\phi e_n||^2 = \sum_{n=0}^{\infty} \frac{1}{{\beta_n}^2} \sum_{k=0}^{\infty} |a_{k-n}|^2 \beta_k^2 = A + B,$$

where $A = \sum\limits_{i=0}^{\infty} |a_i|^2 \big(\sum\limits_{k=0}^{\infty} \frac{\beta^2_{i+k}}{\beta^2_k} \big)$ and $B = \sum\limits_{i=1}^{\infty} |a_{-i}|^2 \big(\sum\limits_{k=0}^{\infty} \frac{\beta^2_k}{\beta^2_{i+k}} \big)$. Now it is easy to attain the following.

Proposition 3.6. *Let $\phi(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$. If the weighted Toeplitz operator T^β_ϕ on $H^2(\beta)$ is Hilbert-Schmidt then $\phi = 0$.*

Proof. If T^β_ϕ is a Hilbert-Schmidt operator then $\sum\limits_{n=0}^{\infty} ||T^\beta_\phi e_n||^2$ is finite and so are A and B. But with the assumption, $\frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$, on the sequence $\beta = \{\beta_n\}$, each series enclosed in parenthesis in A is a divergent series, which gives that $a_i = 0$ for each $i \geq 0$. Further it is seen that if the series $\sum\limits_{k=0}^{\infty} \frac{\beta^2_k}{\beta^2_{i+k}}$ is divergent for some $i (\geq 1)$ then $a_{-k} = 0$ for all $1 \leq k \leq i$. Now the proof follows as we see the following.

(1) The series $\sum_{k=0}^{\infty} \frac{\beta_k^2}{\beta_{i+k}^2}$ is divergent for all $i(\geq 1)$. In which case, $\phi = 0$.

(2) $\sum_{k=0}^{\infty} \frac{\beta_k^2}{\beta_{i+k}^2}$ is convergent for some $i(\geq 1)$. This situation doesn't arise because with the assumption that $r \leq \frac{\beta_n}{\beta_{n+1}}$ for $n \geq 0$, for some $r > 0$, we find that the series $\sum_{k=0}^{\infty} \frac{\beta_k^2}{\beta_{i+k}^2}$ can not converge.

This completes the proof. □

The following observation is now immediate.

Theorem 3.7. *The weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$ induced by $\phi = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$ is Hilbert-Schmidt if and only if $\phi = 0$.*

Corollary 3.8. *No non-zero Toeplitz operator T_ϕ, $\phi = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty$, on the Hardy space H^2 is Hilbert-Schmidt.*

Proof. If we take the particular case of $\beta_n = 1$ for each $n \in \mathbb{Z}$ in Theorem 3.7, then the weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$ becomes the Toeplitz operator T_ϕ on H^2. This gives the result. □

Towards the end, we discuss the notion of a Toeplitz operator T_ϕ, $\phi \in L^\infty(\beta)$ on $H^2(\beta)$ and see its relation with weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$.

An operator T on the Hilbert space $H^2(\beta)$ is called Toeplitz operator if its matrix representation $[\alpha_{i,j}]_{i,j\geq 0}$ w.r.t orthonormal basis $\{e_n(z) = \frac{z^n}{\beta_n}\}_{n\geq 0}$ satisfies $\alpha_{i,j} = \alpha_{i+1,j+1}$ for each $i, j \geq 0$.

Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta)$. If there exists an operator on $H^2(\beta)$ whose matrix representation $[\alpha_{i,j}]_{i,j\geq 0}$ w.r.t orthonormal basis $\{e_n(z) = \frac{z^n}{\beta_n}\}_{n\geq 0}$ satisfies

$$\alpha_{i,j} = a_{i-j}$$

for $i, j \geq 0$, then this operator is a Toeplitz operator and is denoted by T_ϕ.

The symbol $\mathbb{T}$ denotes the unit circle. We recall a result of [[9], Lemma 2.7] which states the following.

Proposition 3.9. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be bounded. Then for each operator H on $H^2(\beta)$, there is an operator on $H^2(\mathbb{T})$ with the same matrix representation as of H and conversely. Moreover, the norm of both the operators are same.*

Now, we investigate the following when the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is bounded.

Theorem 3.10. *If $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ is bounded, then each weighted Toeplitz operator T_ϕ^β on $H^2(\beta)$, where $\phi \in L^\infty(\beta)$, is similar to the Toeplitz operator T_ϕ on $H^2(\beta)$.*

Proof. Let the sequence $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be bounded. Consider the diagonal operator D_β given by $D_\beta e_n = \beta_n e_n$ for $n \geq 0$. It is an invertible operator on $H^2(\beta)$. Now, if $\phi \in L^\infty(\beta)$ then it is easy to verify that $T_\phi^\beta = D_\beta T_\phi D_\beta^{-1}$. Hence the result follows. □

Theorem 3.10 along with Proposition 3.9 helps us to provide following information regarding Toeplitz operators on $H^2(\beta)$ and $H^2(\mathbb{T})$.

Theorem 3.11. *Let $\beta = \{\beta_n\}_{n\in\mathbb{Z}}$ be bounded. Then there is an isometry between the classes of Toeplitz operators on $H^2(\beta)$ and $H^2(\mathbb{T})$.*

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