Gulf Journal of Mathematics
Vol 5, Issue 2 (2017) 71-86

ON THE WEAK HUB-INTEGRITY OF GRAPHS

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Abstract. The concept of weak hub-integrity is introduced as a new measure of the stability of a graph G and it is defined as $WHI(G) = \min\{|S| + m_e(G - S)\}$, where S is a hub set and $m_e(G - S)$ denotes the number of edges in the largest component of $G - S$. In this paper, the weak hub-integrity of some graphs is obtained. The relations between weak hub-integrity and other parameters are determined. Also, some results connecting the weak hub-integrity and binary graph operations are established.

1. Introduction and preliminaries

All the graphs considered here are finite, non-trivial, undirected and connected without loops or multiple edges. The symbols $\Delta(G)$, $\delta(G)$, $\alpha(G)$, $\kappa(G)$, $\lambda(G)$, and $\beta(G)$ denote the maximum degree, the minimum degree, the vertex cover number, the connectivity, the edge-connectivity, and the independence number of G, respectively. Further a $cut - set$ is any set of vertices whose removal leaves a disconnected graph. For a vertex $v \in V$, the open neighborhood of v in G, denoted by $N(v)$ is the set of all vertices that are adjacent to v. The distance $d(u, v)$ between two vertices u and v in a graph G is the length of the shortest path between u and v. The eccentricity $e(u)$ of a vertex u in G, with vertex set $V(G)$, is defined as $e(u) = max_{v\in V(G)} d(u, v)$. A vertex v is an eccentric vertex of a vertex u if $d(u, v) = e(u)$. For notation and terminology not defined here we refer to [8].

For a subset $S \subseteq V, \overline{S} = V - S$, $\lceil x \rceil$ denotes the smallest integer number that greater than or equals to x with $\lfloor x \rfloor$ to the greatest integer number that smaller than or equals to x. A friendship graph F_n is a graph which consists of n triangles with a common vertex. $V(F_n) = 2n + 1$. The double star graph $S_{n,m}$ is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1})$ and $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j \ : \ 1 \le i \le n - 1,\ 1 \le j \le m - 1\}$. A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G, denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G. The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G [8].

Date: Received: May 22, 2016; Accepted: Oct 15, 2016.
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2010 *Mathematics Subject Classification.* 05C38, 05C99, 05C76.
Key words and phrases. Integrity, Hub Integrity, Line graph, Middle graph, Total graph.

The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G [8]. The middle graph $M(G)$ of a graph G is the graph whose set of vertices is the union of the set of vertices and edges of G and in which two vertices are adjacent if they are adjacent edges of G or one is a vertex and other is an edge incident with it [13]. The total graph $T(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$, in which two vertices are adjacent if and only if they are adjacent or incident in G [8].
The advent of affordable but powerful workstations and the improved network cost-performance ratio have meant that much more powerful computing systems can now be constructed by interconnecting a large number of such units in a distributed working environment. The paramount importance in the design and maintenance of such system is the knowledge and the ability to maintain a certain level of sustainable computational power. Thus, the study of system reliability in general and network reliability, in particular, is critical to achieving performance goals. Previous work in this area has been mostly on a probabilistic basis. However, sometimes it is important to take subjective reliability estimates into consideration. Among the relevant issues of importance, we are particularly interested in one of the vulnerabilities. That is, in an unfriendly external environment, how vulnerable is such a distributed system to certain external destruction and how much computing power can be sustained in the face of destruction.

The concept of network vulnerability is motivated by the design and analysis of networks under a hostile environment. Several graph theoretic models under various assumptions have been proposed for the study and assessment of network vulnerability. Graph integrity, introduced by Barefoot et al. [3, 4], is one of these models that has received wide attention [1, 5, 6, 7]. Barefoot et al. studied two measures of network vulnerability, the integrity and the edge integrity of a graph. Bagga et al. introduced a similar measure called pure-edge integrity [2]. The integrity $I(G)$ of a graph G is defined as $I(G) = \min\{|S| + m(G-S) : S \subseteq V(G)\}$, where $m(G-S)$ denotes the order of a maximum component of $G-S$. The edge-integrity $I'(G)$ of a graph G is defined as $I'(G) = \min\{|S| + m(G-S) : S \subseteq E(G)\}$, where $m(G-S)$ denotes the order of a maximum component of $G-S$. Moreover the pure edge-integrity $I_p(G)$ of a graph G is defined as $I_p(G) = \min\{|S| + m_e(G-S) : S \subseteq E(G)\}$, where $m_e(G-S)$ denotes the number of edges in a largest component of $G-S$. Also the weak integrity was introduced by Kirlangic [10] and is defined as $I_w(G) = \min\{|S| + m_e(G-S) : S \subseteq V(G)\}$. Veena et al. [13] have investigated Vulnerability: vertex neighbor integrity of middle graph of some graphs.

In [14], Walsh defined a hub set as follows: Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - H$, there is an H-path in G between x and y. The smallest size of a hub set in G is called the hub number of G, and is denoted by $h(G)$.

Sultan et al. [12] have introduced the concept of hub-integrity of a graph as a new measure of vulnerability which is defined as follows.

Definition 1.1. [12] The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = \min\{|S| + m(G - S), S \text{ is a hub set of } G\}$, where $m(G - S)$ is the order of a maximum component of $G - S$.

We need the following to prove main results.

Proposition 1.2. [12] *For any graph G, $G \neq \overline{K}_p$, $\gamma(G)+1 \leq HI(G)$. The bound is sharp for $G = K_{1,n}$.*

Lemma 1.3. [9] *If $D \subseteq E(G)$, then $L(G - D) = L(G) - D$.*

Theorem 1.4. [10] *If a graph G of order n is isomorphic to a cycle graph or a tree, then $I_w(G) = I(G) - 1$.*

Theorem 1.5. [11] *Given the broom graph $G = B_{n,d}$ the eccentric graph G_e is*

- **(i):** $\overline{K_{\frac{d-1}{2}}} + K_1 + \overline{K_{n-d}} + \overline{K_{\frac{d-1}{2}}}$, *when d is odd*
- **(ii):** *a 3-partite graph, when d is even.*

Theorem 1.6. [8] *The total graph $T(G)$ is isomorphic to the square of the subdivision graph $S(G)$.*

Theorem 1.7. [8] *For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

2. Weak hub-integrity

We now introduce a new stability measure of a graph G and it is called weak hub-integrity. Formally, the weak hub-integrity $WHI(G)$ of a graph G is defined as

$$WHI(G) = \min\{|S| + m_e(G - S), S \text{ is a hub set of } G\},$$

where $m_e(G - S)$ denotes the number of edges in a largest component of $G - S$. Any set S with property that $|S| + m_e(G - S) = WHI(G)$ is called a $WHI-$set of G and it is obvious that $WHI(G) \geq HI(G) - 1$ for any graph G, and deal with the question "What is the size of the largest remaining group within which mutual communication can still occur?". The motivation was that, in some respects, connectivity is oversensitive to local weaknesses and does not reflect the overall vulnerability. For example, the stars $K_{1,p+1}$ and the graphs $K_1 + (K_1 \cup K_p)$ (where $+$ and $\cup$ denote the join and disjoint union) are all of connectivity one but differ vastly in how much damage is done to the corresponding communications network by the removal of a hub set vertex: in the former case all communications are destroyed, whereas in the latter all but two stations remain in mutual contact. We aim to lay the groundwork on weak hub-integrity as a specific graphical parameter, especially with regard to the fundamental properties of WHI and $WHI-$sets, we obtained the weak hub-integrity of some graphs, and some bounds is found. Weak hub-integrity of middle and total graphs is obtained.

Proposition 2.1. (1) *For any complete graph K_p, $WHI(K_p) = p - 1$.*
(2) *For any path P_p with $p \geq 3$, $WHI(P_p) = p - 2$.*

(3) *For any cycle C_p, $p \geq 4$,*

$$WHI(C_p) = \begin{cases} p-2, & \text{if } p = 4, 5\ ; \\ p-3, & \text{if } p \geq 6. \end{cases}$$

(4) *For the star $K_{1,p-1}$, $WHI(K_{1,p-1}) = 1$.*
(5) *For the double star $S_{n,m}$, $WHI(S_{n,m}) = 2$.*
(6) *For the complete bipartite graph $K_{n,m}, WHI(K_{n,m}) = \min\{n, m\}$.*
(7) *For the wheel graph $W_{1,p-1}, p \geq 5$, $WHI(W_{1,p-1}) = \lceil 2\sqrt{p-1} \rceil - 1$.*

Remark 2.2. In general, the inequality $WHI(G') \leq WHI(G)$ is not true for a subgraph G' of G, For example, for the graph G and a subgraph G' of G shown in Figure 1, we have $WHI(G) = 5$, while $WHI(G') = 6$.

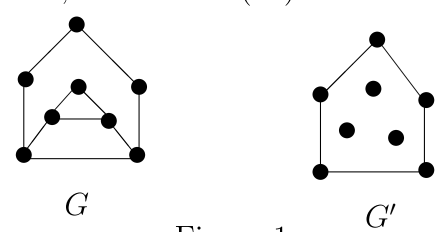


Figure 1

Theorem 2.3. *For any subset X of vertices in a graph G,*

$$WHI(G - X) \geq WHI(G) - |X|.$$

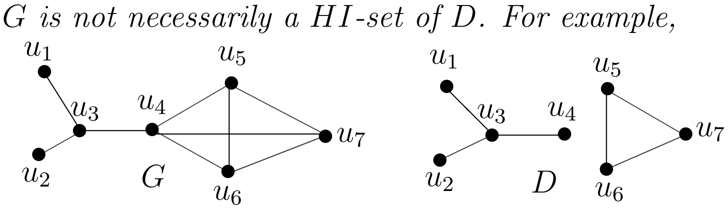
Proof. Let S be a WHI-set of $G - X$, then S is a hub set of $G - X$ and $WHI(G - X) = |S| + m_e((G - X) - S)$. Let $S^\star = S \cup X$, then $S^\star$ is a hub set of G and $m_e(G - S^\star) = m_e((G - X) - S)$. Therefore,

$$\begin{aligned} WHI(G) &\leq |S^\star| + m_e(G - S^\star) \\ &= |S| + |X| + m_e((G - X) - S) \\ &= WHI(G - X) + |X|. \end{aligned}$$

□

Lemma 2.4. *Let G be a graph ,*

(1) *If G is non-complete, then every $WHI-$set of G is a cut-set of G and hence has cardinality at least $\kappa(G)$.*
(2) *For any graph $H \leq G$, $WHI(L(H)) \leq WHI(L(G))$.*
(3) *If D is a spanning subgraph of G and $WHI(G) = WHI(D)$, then a HI-set of G is not necessarily a HI-set of D. For example,*



$WHI(G) = WHI(D) = 4$, but HI-set of $G = \{u_3, u_4, u_7\}$, while HI-set of $D = \{u_3, u_5, u_6, u_7\}$.

Proposition 2.5. *For any graph G , $1 \leq WHI(G) \leq p - 1$. The lower bound attains for $K_{1,p-1}$ and the upper bound attains for a complete graph K_p, $p \geq 2$.*

Theorem 2.6. *For any graph G, $WHI(G) \geq \delta(G)$.*

Proof. Let S be a WHI-set of G such that $WHI(G) = |S| + m_e(G-S)$. Therefore $m_e(G-S) \geq \delta(G-S) \geq \delta(G) - |S|$, So, $WHI(G) = |S| + m_e(G-S) \geq |S| + \delta(G) - |S| = \delta(G)$. □

Theorem 2.7. *For any graph G, $WHI(G) \geq \lambda(G)$.*

Proof. Proof follows from Theorem (2.6) and Theorem (1.7). □

Theorem 2.8. *For any graph G, $WHI(G) \geq \kappa(G)$.*

Proof. Proof follows from Theorem (2.7) and Theorem (1.7). □

Theorem 2.9. *For any tree T, $WHI(T) \geq \alpha(T)$.*

Proof. Let S be a WHI-set of T, and $S^\star$ be a minimum covering set of T. Then

$$\begin{aligned} WHI(T) &= |S| + m_e(T-S) \\ &\geq |S^\star| + m_e(T-S^\star) \\ &\geq |S^\star| \\ &= \alpha(T). \end{aligned}$$

□

As described above, $\alpha(G), \delta(G), \lambda(G)$ and $\kappa(G)$ are lower bounds of $WHI(G)$. However, the independence number β, has no such relationship with WHI. For example :

- $WHI(K_{1,p-1}) < \beta(K_{1,p-1})$.
- $WHI(K_p) > \beta(K_p)$.
- $$WHI(K_{n,m}) = \begin{cases} n = m = \beta(K_{n,m}), & \text{if } n = m \text{ ;} \\ min\{n,m\} < \beta(K_{n,m}), & \text{if } n \neq m. \end{cases}$$

  where $\beta(K_{n,m}) = max\{n,m\}$.

Theorem 2.10. *Let G be a connected graph of order $p > 1$. Then $WHI(G) = 1$ if and only if $\alpha(G) = 1$.*

Proof. Let S be a WHI-set of G. Since $WHI(G) = |S| + m_e(G-S) = 1$ and $m_e(G-S) \geq 0$ it follows that $|S| = 1$ and $m_e(G-S) = 0$. Thus, $|S| = \alpha(G) = 1$. Converesly, consider $\alpha(G) = 1$. Then $G \cong K_{1,p-1}$. Thus, $WHI(G) = 1$. □

Theorem 2.11. *For any connected graph G, $WHI(G) = \kappa(G)$ if and only if $\kappa(G) = \alpha(G)$.*

Proof. Suppose that $WHI(G) = \kappa(G)$. Let S be a WHI-set of a graph G such that $WHI(G) = |S| + m_e(G-S)$. If G is complete, then by Proposition (2.1), $WHI(G) = \kappa(G)$. Thus we may assume that G is non-complete. Since $WHI(G) = |S| + m_e(G-S) = \kappa(G)$, it follows by Lemma (2.4), that $|S| \geq \kappa(G)$. Thus $\kappa(G) + m_e(G-S) \leq \kappa(G)$. Therefore, $m_e(G-S) \leq 0$. Since $m_e(G-S) \geq 0$, we have $m_e(G-S) = 0$, and $m(G-S) = 1$. So, we have the following cases:
Case 1: $m(G-S) = 1$. S is a cover set and we have $|S| = \alpha(G)$.

Case 2: $WHI(G) = |S| + m_e(G-S) = \kappa(G)$ and $m_e(G-S) = 0$. Then $|S| = \kappa(G)$. Consequently, $\kappa(G) = \alpha(G)$.
Conversely, let S be a hub set of a graph G. Then we have the following cases:
Case 1: $|S| \leq \kappa(G)$. By lemma (2.4), $G \cong K_p$. Therefore,

$$WHI(K_p) = p - 1. \tag{2.1}$$

Case 2: $|S| = \kappa(G)$. Since $\kappa(G) = \alpha(G)$, it follows that $m_e(G-S) = 0$. So,

$$WHI(G) = \kappa(G). \tag{2.2}$$

Case 3: $|S| \geq \kappa(G)$. Since $\kappa(G) = \alpha(G)$, it follows that $m_e(G-S) \geq 0$ and so,

$$WHI(G) \geq \kappa(G). \tag{2.3}$$

Since $\kappa(G) \leq p-1$ for every graph G, from (2.1), (2.2) and (2.3), we have $WHI(G) = \kappa(G)$. □

Theorem 2.12. *For any subset E' of edges in a graph G,*

$$WHI(G - E') \geq WHI(G) - |E'|.$$

Proof. Let S' be a WHI-set of $G - E'$, and $S'' = S' \cup E'$, then $|S''| = |S'| + |E'|$, and $G - S'' = G - (S' \cup E') = (G - E') - S'$. Therefore,

$$\begin{aligned} WHI(G) &= |S| + m_e(G-S), \textit{where } S \textit{ is a } WHI - \textit{set of } G \\ &\leq |S''| + m_e(G - S'') \\ &= |S'| + |E'| + m_e[(G-E') - S'] \\ &= WHI(G - E') + |E'|. \end{aligned}$$

□

Proposition 2.13. *If $G \cong \overline{G}$, then $WHI(G) = WHI(\overline{G})$.*

Remark 2.14. The converse of Proposition (2.13) is not true. For example, if $G \cong K_p$, we have $WHI(K_p) = WHI(\overline{K_p})$, but K_p and $\overline{K_p}$ are not isomorphic.

Theorem 2.15. *For any graph $G \neq \overline{K_p}, WHI(G) \geq \gamma(G)$.*

Proof. The proof follows by Proposition (1.2). □

Lemma 2.16. $WHI(\overline{K_n} + \overline{K_m}) = min\{n, m\}$.

Proof. Since $\overline{K_n} + \overline{K_m} = K_{n,m}$, the proof follows by Proposition (2.1). □

Theorem 2.17. *Let T be a tree with p vertices and m terminal vertices. Then $WHI(G) = p - m$.*

Proof. Let S be a hub set of T. The set $p - m$ of all internal vertices in T forms a hub set, since the unique path between any two terminals never passes through another terminal. Therefore any proper subset of $p - m$ cannot be a hub set. So $|S| = p - m$, since every internal vertex is a cut-vertex. If we delete all $p - m$ internal vertices, each component is of size 0. So, $WHI(T) = |S| + m_e(T - S) = p - m$. □

Definition 2.18. [11] A broom graph $B_{p,d}$ consists of a path P_d with d vertices, together with $(p - d)$ pendant vertices all adjacent to the same end vertex of P_d.

Lemma 2.19. $h(B_{p,d}) = d - 1$.

Proof. Let $B_{p,d} = (V, E)$ be a broom graph. Let $V(B_{p,d}) = \{u_1, u_2, ..., u_d, v_1, v_2, ..., v_{p-d}\}$ such that $u_1, u_2, ..., u_d$ is a path on d vertices and $v_1, v_2, ..., v_{p-d}$ are pendant vertices that are adjacent to u_d. We note that in $B_{p,d}$,

(1) $d(u_1) = d(v_i) = 1, 1 \leq i \leq p - d$.
(2) $d(u_j) = 2, 2 \leq j \leq d - 1$.
(3) $d(u_d) = p - d + 1$.

Consider $S = \{u_2, u_3, ..., u_d\}$ a hub set for $B_{p,d}$, $|S| = d - 1$. As u_2 is adjacent to u_1 and $N(u_d) = \{v_1, v_2, v_3, ..., v_{p-d}\}$, for any $v_i, v_j \in V(B_{p,d} - S), 1 \leq i, j \leq p - d$, then there exists $S-$path between them, and between u_1 with $v_i, 1 \leq i \leq p - d$. If some $u_i, 2 \leq i \leq d$ is removed from set S, then there does not exist path between u_1 and $v_i, 1 \leq i \leq p - d$. Thus S is a minimum hub set. Hence $h(B_{p,d}) = d - 1$. □

Theorem 2.20. $WHI(B_{p,d}) = d - 1$.

Proof. Let $V(B_{p,d}) = \{u_1, u_2, ..., u_d, v_1, v_2, ..., v_{p-d}\}$ such that $u_1, u_2, ..., u_d$ is a path on d vertices and $v_1, v_2, ..., v_{p-d}$ are pendant vertices that are adjacent to u_d. From Lemma (2.19) , we have $h(B_{p,d}) = d - 1$, let $H = \{u_2, u_3, ..., u_d\}$, be a hub set of graph $B_{p,d}$. Then $m_e(B_{p,d} - H) = 0$. Therefore,

$$WHI(B_{p,d}) \leq h(B_{p,d}) + m_e(B_{p,d} - H) = d - 1. \tag{2.4}$$

If S is any hub set other than H, if $m_e(B_{p,d} - H) \geq 1$, then trivially

$$|H| + m_e(B_{p,d} - H) > d - 1. \tag{2.5}$$

From (2.4) and (2.5), $WHI(B_{p,d}) = d - 1$. □

Corollary 2.21. *If $p-d = 2$ in a broom graph, then $WHI(B_{p,d}) = WHI(L(B_{p,d}))$.*

Definition 2.22. [11] The eccentric graph G_e of a graph G is defined as a graph having the same set of vertices as G with two vertices u and v adjacent in G_e if and only if either u is an eccentric vertex of v or v is an eccentric vertex of u.

Lemma 2.23. $h(G_e) = 2$*, where G_e is the eccentric graph of $B_{p,d}$.*

Proof. Let G_e be the eccentric graph of a graph $B_{p,d}$ and from Theorem (1.5), G_e is

- $(\overline{K_{\frac{d-1}{2}}} + K_1) \cup (K_1 + \overline{K_{p-d}}) \cup (\overline{K_{p-d}} + \overline{K_{\frac{d-1}{2}}})$, when d is odd,
- a 3 -partite graph, when d is even.

Consider $S = \{u_1, v_1\}$, a hub set of G_e, and $|S| = 2$. We have two cases:
Case 1: d is odd. In G_e, $N(u_1) = \{u_i, i > \frac{d+1}{2}\} \cup \{v_j, 1 \leq j \leq p - d\}$, also $N(v_1) = \{u_i, 1 \leq i \leq \frac{d+1}{2}\}$, so for any $u_i, v_j \in V(G_e - S), 1 < i < d, 1 < j \leq p - d$, there exists S- path between them. Now we claim that set S is a minimum hub set. Removal of u_1 from set S leads to nonexistence of S-path between vertices of $\{u_i, i > \frac{d+1}{2}\}$ and $\{v_j, 1 < j \leq p - d\}$, it follows that S is a minimum hub set. Hence $h(G_e) = 2$.
Case 2: d is even. In G_e, every vertex $u_i, i \geq \frac{d}{2} + 1$ is adjacent to u_1 and $N(u_1) = \{u_i, i > \frac{d}{2} + 1\} \cup \{v_j, 1 \leq j \leq p - d\}$, also $N(v_1) = \{u_i, 1 \leq i \leq \frac{d}{2} + 1\}$, so for any $u_i, v_j \in V(G_e - S), 1 < i \leq d, 1 < j \leq p - d$, there exists S- path between

them. Now we claim that set S is a minimum hub set of G_e. Removal of u_1 from set S leads to nonexistence of S-path between any vertices of $\{u_i, i > \frac{d}{2}+1\}$ with $\{v_j, 1 < j \leq p-d\}$, it follows that S is a minimum hub set. Hence $h(G_e) = 2$. □

Theorem 2.24.

$$WHI(G_e) = \begin{cases} \lceil \frac{d}{2} \rceil, & \textit{if } d \textit{ is odd;} \\ \frac{d}{2}+1, & \textit{if } d \textit{ is even.} \end{cases}$$

Proof. Let $V(G_e) = \{u_1, u_2, ..., u_d, v_1, v_2, ..., v_{p-d}\}$, now depending upon the number of vertices in a path P_d, the following subsets are considered:

- For $0 \leq i < \frac{d}{2}+1 : S = \{u_{1+i}\}, |S| = \frac{d}{2}+1, \ for \ d \equiv 0(mod\ 2), d \geq 2,$
- For $0 \leq i < \lceil \frac{d}{2} \rceil : S = \{u_{1+i}\}, |S| = \lceil \frac{d}{2} \rceil, \ for \ d \equiv 1(mod\ 2), d > 2.$

In all the above cases, S is a hub set of G_e and $m_e(G_e - S) = 0$. Now we discuss the minimality of $|S| + m_e(G_e - S)$. In case 1, if S_1 is a hub set other than S and $m_e(G_e - S_1) = 0$, then

$$WHI(G_e) \geq \frac{d}{2} + 2. \tag{2.6}$$

If S_2 is a hub set other than S such that $m_e(G_e - S_2) = 1$, then $|S_2| \geq d$, so

$$WHI(G_e) \geq 1 + d. \tag{2.7}$$

In case 2, if S_1 is a hub set of G_e other than S and $m_e(G_e - S_1) = 0$, then

$$WHI(G_e) > \lceil \frac{d}{2} \rceil. \tag{2.8}$$

Let S_2 be any a hub set of G_e other than S such that $m_e(G_e - S_2) = 1$, then $|S_2| \geq d$, so

$$WHI(G_e) \geq 1 + d. \tag{2.9}$$

Then from (2.6), (2.7), (2.8) and (2.9), we have

$$WHI(G_e) = \begin{cases} \lceil \frac{d}{2} \rceil, & \text{if } d \text{ is odd;} \\ \frac{d}{2}+1, & \text{if } d \text{ is even.} \end{cases}$$

□

Definition 2.25. [8] For a simple connected graph G the square of G denoted by G^2, is defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 in G .

Theorem 2.26.

$$WHI(C_p^2) = \begin{cases} 5, & p = 8; \\ 7, & p = 11, 12; \\ 8, & p = 13; \\ 9, & p = 14. \end{cases}$$

Let $V(C_p) = \{v_1, v_2, ..., v_p\}$. Then, $|V(C_p^2)| = p$ and $|E(C_p^2)| = 3p - p, p \geq 5$. We consider the following cases:

Case 1: For $p = 8$, consider $S = \{v_1, v_2, v_5, v_6\}$, a hub set of C_8^2, then $m_e(C_8^2 - S) = 1$. This implies that $WHI(C_8^2) \leq |S| + m_e(C_8^2 - S) = 5$.

Clearly there does not exist any hub set S_1 of C_8^2 such that $|S_1| + m_e(C_8^2 - S_1) <$

$|S| + m_e(C_8^2 - S)$. Hence, $WHI(C_8^2) = 5$.
Case 2: For $p = 11$, consider $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}\}$, a hub set of C_{11}^2, then $m_e(C_{11}^2 - S) = 1$. Therefore $WHI(C_{11}^2) \leq |S| + m_e(C_{11}^2 - S) = 7$.
Clearly there does not exist any hub set S_1 of C_{11}^2 such that $|S_1| + m_e(C_{11}^2 - S_1) < |S| + m_e(C_{11}^2 - S)$. Hence, $WHI(C_{11}^2) = 7$.
Case 3: For $p = 12$, consider $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}\}$, a hub set of C_{12}^2, then $m_e(C_{12}^2 - S) = 1$. This implies that $WHI(C_{12}^2) \leq |S| + m_e(C_{12}^2 - S) = 7$.
Clearly there does not exist any hub set S_1 of C_{12}^2 such that $|S_1| + m_e(C_{12}^2 - S_1) < |S| + m_e(C_{12}^2 - S)$. Hence, $WHI(C_{12}^2) = 7$.
Case 4: For $p = 13$, consider $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, v_{12}\}$, a hub set of C_{13}^2, then $m_e(C_{13}^2 - S) = 1$. This implies that $WHI(C_{13}^2) \leq |S| + m_e(C_{13}^2 - S) = 8$.
Clearly there does not exist any hub set S_1 of C_{13}^2 such that $|S_1| + m_e(C_{13}^2 - S_1) < |S| + m_e(C_{13}^2 - S)$. Hence, $WHI(C_{13}^2) = 8$.
Case 5: For $p = 14$, consider $S = \{v_1, v_2, v_5, v_6, v_9, v_{10}, v_{13}, v_{14}\}$, a hub set of C_{14}^2, then $m_e(C_{14}^2 - S) = 1$. This implies that $WHI(C_{14}^2) \leq |S| + m_e(C_{14}^2 - S) = 9$.
Clearly there does not exist any hub set S_1 of C_{14}^2 such that $|S_1| + m_e(C_{14}^2 - S_1) < |S| + m_e(C_{14}^2 - S)$. Hence, $WHI(C_{14}^2) = 9$.

Theorem 2.27.

$$WHI(C_p^2) = \begin{cases} \frac{2p}{3}, & p \equiv 0 (mod\ 3), p \neq 12; \\ \frac{2(p-1)}{3} + 1, & p \equiv 1 (mod\ 3), p \neq 13; \\ \frac{2(p+1)}{3}, & p \equiv 2 (mod\ 3), p \neq 8, 11, 14. \end{cases}$$

Proof. Let $V(C_p) = \{v_1, v_2, ..., v_p\}$. Then, $|V(C_p^2)| = p$ and $|E(C_p^2)| = 3p - p, p \geq 5$. We consider the following cases:
Case 1: If $p \equiv 0(mod\ 3), p \neq 12$, then $p = 3k$ for some integer $k \geq 1$.
Consider $S = \{v_{1+3i}, v_{2+3i}/0 \leq i \leq k-1\}$ and $|S| = 2k$, we have $|S| = \frac{2p}{3}$, so $m_e(C_p^2 - S) = 0$.
Case 2: If $p \equiv 1(mod\ 3), p \neq 13$, then $p = 3k + 1$ for some integer $k \geq 1$.
Consider $S = \{v_{1+3i}, v_{2+3i}/0 \leq i \leq k-1\}$ and $|S| = 2k$, we have $|S| = \frac{2(p-1)}{3}$, so $m_e(C_p^2 - S) = 1$.
Case 3: If $p \equiv 2(mod\ 3), p \neq 8, 11, 14$, then $p = 3k - 1$ for some integer $k \geq 2$.
Consider $S = \{v_{1+3i}, v_{2+3i}/0 \leq i \leq k-1\}$ and $|S| = 2k - 1$, we have $|S| = \frac{2(p+1)}{3}$, so $m_e(C_p^2 - S) = 0$. We discuss the minimality of $|S| + m_e(C_p^2 - S)$. If we consider any hub set S_1 of C_p^2 such that $|S_1| \leq |S|$, then

$$|S| + m_e(C_p^2 - S) \leq |S_1| + m_e(C_p^2 - S_1). \quad (2.10)$$

(In case 1), if S_2 is any hub set of C_p^2 such that $m_e(C_p^2 - S_2) \geq 1$, then

$$|S| + m_e(C_p^2 - S) \leq |S_2| + m_e(C_p^2 - S_2). \quad (2.11)$$

(In case 3), if S_3 is any hub set of C_p^2 such that $m_e(C_p^2 - S_3) \geq 1$, then

$$|S| + m_e(C_p^2 - S) \leq |S_3| + m_e(C_p^2 - S_3). \quad (2.12)$$

(In case 2), if S_4 is any hub set of C_p^2 such that $m_e(C_p^2 - S_4) \geq 1$, then

$$|S| + m_e(C_p^2 - S) \leq |S_4| + m_e(C_p^2 - S_4). \quad (2.13)$$

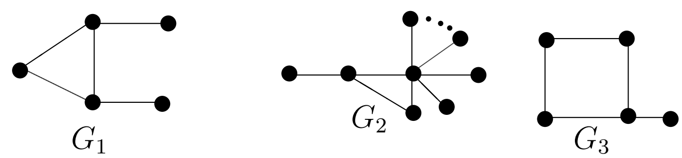
Now, if $m_e(C_p^2 - S) \geq 1$, then $|S| + m_e(C_p^2 - S) \geq \frac{2(p+1)}{3}$. Hence, from (2.10), (2.11), (2.12) and (2.13), we have

$$WHI(C_p^2) = \begin{cases} \frac{2p}{3}, & p \equiv 0(mod\ 3), p \neq 12; \\ \frac{2(p-1)}{3} + 1, & p \equiv 1(mod\ 3), p \neq 13; \\ \frac{2(p+1)}{3}, & p \equiv 2(mod\ 3), p \neq 8, 11, 14. \end{cases}$$

□

Remark 2.28. Let G be a graph of order p,

- $WHI(G) = 1$ if and only if G is a star.
- $WHI(G) = 2$ if and only if G contains any one of the following graphs: $S_{n,m}, C_3, C_4, P_4, 3K_1, K_4 - e, K_{2,n}, n \geq 3, K_1 \cup K_{1,n}, n \geq 1, F_n, K_1 + (P_3 \cup rK_1), K_1 + (K_2 \cup rK_1), K_1 + (rK_2 \cup sK_1)$ and three graphs G_1, G_2 and G_3 below:



Theorem 2.29. *For any graph G, $WHI(G) \geq HI(G) - 1$.*

Proof. Let S be a WHI-set of G. Since $HI(G) \leq |S| + m(G - S)$ and $m(G - S) \leq m_e(G - S) + 1$, for every $S \subset V(G)$, hence the result. □

Theorem 2.30. *If a graph G of order p is isomorphic to a cycle graph or a tree, then $WHI(G) = HI(G) - 1$.*

Proof. Let S be a hub set of G such that $|S| + m(G - S) = HI(G)$. If we remove the vertices in S, then each of the components of $G - S$ has two vertices or one vertex and $m(G - S) = m_e(G - S) + 1$. Hence $WHI(G) \leq |S| + m_e(G - S) = |S| + m(G - S) - 1 = HI(G) - 1$. □

Theorem 2.31. *For any graph G, $WHI(T(G)) = WHI(S(G))^2$.*

Proof. The proof follows by Theorem (1.6). □

3. Weak hub-integrity of middle and total graphs

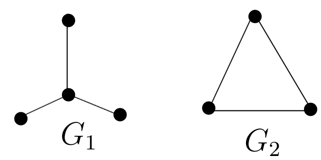
Theorem 3.1. *Let G be a connected graph and $\alpha(G) = 1$, then*

(1) $WHI(L(G)) = p - 2$,
(2) $WHI(M(G)) = p - 1$,
(3) $WHI(T(G)) = p$.

Proof. (1) Suppose $\alpha(G) = 1$, then $G \cong K_{1,p-1}$. Then $L(G) = K_{p-1}$, it follows from Proposition (2.1), $WHI(L(G)) = p - 2$.
(2) Since $G \cong K_{1,p-1}$, $M(G)$ contains a complete graph K_{p-1} as its subgraph, if we choose the set S as all vertices of K_{p-1}, then there exist p components each containing only one vertex, so $m_e(M(G) - S) = 0$. Then $WHI(M(G)) = p - 1$.
(3)Since $G \cong K_{1,p-1}$, $T(G)$ contains a wheel graph $W_{1,p-1}$ as its subgraph, if we choose the set S as all vertices of $W_{1,p-1}$, then there exist $p - 1$ components each containing only one vertex, so $m_e(T(G) - S) = 0$. Then $WHI(T(G)) = p$. □

Proposition 3.2. **(a):** *For any path P_p with $p \geq 3$, $WHI(M(P_p)) = p-1$.*
(b): *For any cycle C_p, $WHI(M(C_p)) = p$.*
(c): *For the star $K_{1,p-1}$, $WHI(M(K_{1,p-1})) = p-1$.*
(d): *For the double star $S_{n,n}$, $WHI(M(S_{n,n})) = 2n+1$.*

Remark 3.3. If $WHI(M(G_1)) = WHI(M(G_2))$, then it is not necessary that $WHI(G_1) = WHI(G_2)$, for example



Lemma 3.4. *For any graph G_1, G_2, if $G_1 \cong G_2$, then $WHI(M(G_1)) = WHI(M(G_2))$.*

Lemma 3.5. *If $D \subseteq E(G)$, then $WHI(L(G-D)) = WHI(L(G)-D)$.*

Proof. Proof follows by Lemma (1.3). □

Theorem 3.6. *Let G be a connected graph with $\Delta(G) \leq 2$, then $WHI(M(G)) = |E(G)|$ if and only if $G \cong P_p$ or $G \cong C_p$.*

Proof. Suppose that G is a connected graph with $\Delta(G) \leq 2$. Then G is path or cycle, and by Proposition (3.2), we have $WHI(M(G)) = |E(G)|$. The converse is obvious. □

Proposition 3.7. (1) *For any path P_p with $p \geq 3$, $WHI(T(P_p)) = p + \lceil 2\sqrt{p+1}\rceil - 4$.*
(2) *For any cycle C_p, $WHI(T(C_p)) = p + \lceil 2\sqrt{p}\rceil - 2$.*
(3) *For the star $K_{1,p-1}$, $WHI(T(K_{1,p-1})) = p$.*
(4) *For the double star $S_{n,n}$, $WHI(T(S_{n,n})) = 2n+3$.*

Lemma 3.8. *Let $G \cong K_p - e, e \in E(G)$. Then $WHI(\overline{G}) = p-1$.*

Proof. Let $G \cong K_p - e$, then $\overline{G} \cong K_2 \cup (p-2)K_1$. Then

$$\begin{aligned} WHI(\overline{G}) &= p-2+WHI(K_2) \\ &= p-2+1 = p-1. \end{aligned}$$

□

Corollary 3.9. *Let $G \cong K_p - e, e \in E(G)$. Then $WHI(\overline{G}) = WHI(G)+1$.*

Theorem 3.10. *Let $G \cong K_p - e, e \in E(G)$. Then*
(1) *$WHI(M(\overline{G})) = p-1$.*
(2) *$WHI(T(\overline{G})) = p$.*

Proof. Since $G \cong K_p - e$, then $\overline{G} \cong K_2 \cup (p-2)K_1$ and $M(\overline{G}) \cong P_3 \cup (p-2)K_1$. Thus, $WHI(M(\overline{G})) = 1+p-2 = p-1$, and $T(\overline{G}) \cong K_3 \cup (p-2)K_1$. Thus, $WHI(T(\overline{G})) = p$. □

In general, if $G \cong K_p - F$, where F is a set of independent edges in G, then $\overline{G} \cong |F|K_2 \cup (p-2|F|)K_1$.

Theorem 3.11. *Let $G \cong K_p - F$, where F is a set of maximum number of independent edges in G, then*

$$WHI(M(\overline{G})) = \begin{cases} 3|F| - 2, & p \text{ is is even;} \\ 3|F| - 1, & p \text{ is is odd.} \end{cases}$$

Proof. Since $G \cong K_p - F$, then $\overline{G} \cong |F|K_2 \cup (p - 2|F|)K_1$. Therefore, $M(\overline{G}) = |F|P_3 \cup (p - 2|F|)K_1$, thus,

$$\begin{aligned} WHI(M(\overline{G})) &= \underbrace{3 + 3 + 3 + \ldots + 3}_{|F|-1 \ times} + WHI(P_3) \\ &= 3|F| - 3 + 1 \\ &= 3|F| - 2, p \ is \ even. \end{aligned}$$

If p is odd, then

$$\begin{aligned} WHI(M(\overline{G})) &= 1 + \underbrace{3 + 3 + 3 + \ldots + 3}_{|F|-1 \ times} + WHI(P_3) \\ &= 3|F| - 3 + 2 \\ &= 3|F| - 1. \end{aligned}$$

□

Theorem 3.12. *Let $G \cong K_p - F$, where F is a set of maximum number of independent edges in G, then*

$$WHI(T(\overline{G})) = \begin{cases} 3|F| - 1, & p \text{ is is even;} \\ 3|F|, & p \text{ is is odd.} \end{cases}$$

Proof. Since $G \cong K_p - F$, then $\overline{G} \cong |F|K_2 \cup (p - 2|F|)K_1$. Therefore, $T(\overline{G}) = |F|K_3 \cup (p - 2|F|)K_1$, p is odd and $T(\overline{G}) = |F|K_3$, p is even. Thus,

$$\begin{aligned} WHI(T(\overline{G})) &= \underbrace{3 + 3 + 3 + \ldots + 3}_{|F|-1 \ times} + WHI(K_3) \\ &= 3|F| - 3 + 2 \\ &= 3|F| - 1, hence \ the \ result. \end{aligned}$$

□

Remark 3.13. $WHI(\overline{G}) + WHI(L(\overline{G})) = WHI(T(\overline{G}))$.

Lemma 3.14. $WHI(L(K_4)) = WHI(T(K_3)) = 4$.

Proof. Since $L(K_4)$ and $T(K_3)$ is isomorphic, hence the result. □

Theorem 3.15. $WHI(G) = WHI(T(G))$ *if and only if* $G \cong \overline{K_p}$.

Definition 3.16. [8] The (Cartesian)product $G \times H$ of graphs G and H has $V(G) \times V(H)$ as its vertex set and (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 or $u_2 = v_2$ and u_1 is adjacent to v_1 .

Lemma 3.17. $h(M(K_2 \times P_p)) = 2p - 1$.

Proof. The number of vertices in middle graph $M(K_2 \times P_p)$ is $5p-2$. Let $V(M(K_2 \times P_p)) = \{w_1, w_2, ..., w_{3p-2}\} \cup \{v_1, v_2, ..., v_{2p}\}$. Two vertices w_p and w_{2p} in $M(K_2 \times P_p)$ are adjacent to four vertices, and the vertices $w_1, w_{p-1}, w_{p+1}, w_{2p-1}$ are adjacent to five vertices, while the remaining vertices, with respect to w, are adjacent to six vertices. The graph $M(K_2 \times P_6)$ is shown in Figure 2 for better understanding of the notations and arrangement of vertices. We have two cases:

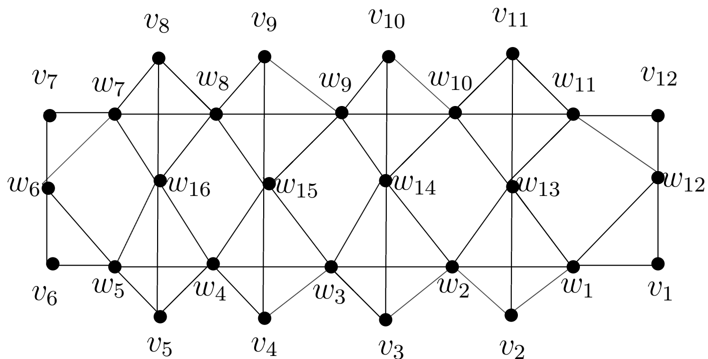


Figure 2 : $M(K_2 \times P_6)$

Case 1: p is even. Consider $H = \{w_1, w_3, ..., w_{p-1}\} \cup \{w_{p+2}, w_{p+4}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\} \cup \{w_p, w_{2p}\}$, a hub set of $M(K_2 \times P_p)$ and $|H| = 2p-1$. We claim that H is minimum hub set. If w_p or w_{2p} are removed from set H, then there does not exist H-path between v_1 with v_{p+1} and v_{2p}. Hence H is minimum hub set, therefore $h(M(K_2 \times P_p)) = 2p-1$.
Case 2: p is odd. Consider $H = \{w_1, w_3, ..., w_{p-2}\} \cup \{w_{p+1}, w_{p+3}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\} \cup \{w_p, w_{2p}\}$, a hub set of $M(K_2 \times P_p)$ and $|H| = 2p-1$. Proof is similar to case 1. Also we can consider $S = \{w_1, w_2, w_3, ..., w_{2p-1}\}$, (In two cases) a hub set of $M(K_2 \times P_p)$ and $|S| = 2p-1$. Clearly that S is a minimum hub set. Hence $h(M(K_2 \times P_p)) = 2p-1$. □

Theorem 3.18.

$$WHI(M(K_2 \times P_p)) = \begin{cases} 4, & p = 2 \ ; \\ 2p+1, & p \geq 3 \ . \end{cases}$$

Proof. The number of vertices in middle graph $M(K_2 \times P_p)$ is $5p-2$. Let $V(M(K_2 \times P_p)) = \{w_1, w_2, ..., w_{3p-2}\} \cup \{v_1, v_2, ..., v_{2p}\}$. We have two cases:
Case 1: $p = 2$, consider $S = \{w_1, w_2, w_3, w_4\}$, a hub set of $M(K_2 \times P_2)$ and $|S| = 4$, then $m_e(M(K_2 \times P_2) - S) = 0$. Therefore $WHI(M(K_2 \times P_2) \leq |S| + m_e(M(K_2 \times P_2) - S) = 4$. Clearly, there does not exist any hub set S_1 of $M(K_2 \times P_2)$ such that $|S_1| + m_e(M(K_2 \times P_2) - S_1) < |S| + m_e(M(K_2 \times P_2) - S)$. Hence, $WHI(M(K_2 \times P_2)) = 4$.
Case 2: $p \geq 3$. From Lemma (3.17) , $h(M(K_2 \times P_p)) = 2p-1$, and $H = \{w_1, w_3, ..., w_{p-1}\} \cup \{w_{p+2}, w_{p+4}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\} \cup \{w_p, w_{2p}\}$ if p is even, and if p is odd, $H = \{w_1, w_3, ..., w_{p-2}\} \cup \{w_{p+1}, w_{p+3}, ..., w_{2p-2}\} \cup \{w_{2p+1}, w_{2p+2}, ..., w_{3p-2}\} \cup \{w_p, w_{2p}\}$, then $m_e(M(K_2 \times P_p) - H) = 2$. Therefore

$$WHI(M(K_2 \times P_p)) \leq h(M(K_2 \times P_p)) + m_e(M(K_2 \times P_p) - H)) = 2p+1. \quad (3.1)$$

If S is any hub set other than H and $m_e(M(K_2 \times P_p) - S) = 1$, then $|S| \geq 2p+1$, so

$$WHI(M(K_2 \times P_p)) \geq 2p + 2. \tag{3.2}$$

Now, if S_1 is any hub set other than H and S with $m_e(M(K_2 \times P_p) - S_1) \geq 2$, then trivially

$$|S_1| + m_e(M(K_2 \times P_p) - S_1) \geq 2p + 1. \tag{3.3}$$

From (3.1), (3.2) and (3.3), $WHI(M(K_2 \times P_p)) = 2p + 1$. □

Corollary 3.19. $WHI(M(K_2 \times P_p)) \geq 2WHI(M(P_p)) + 3$.

Theorem 3.20. $WHI(M(K_2 \times C_p)) = 2p + 2, p \geq 3$.

Proof. The number of vertices in middle graph $M(K_2 \times C_p)$ is $5p$. Let $V(M(K_2 \times C_p)) = \{e_1, e_2, ..., e_{3p-2}\} \cup \{v_1, v_2, ..., v_{2p}\}$. The graph $M(K_2 \times C_4)$ is shown in Figure 3.

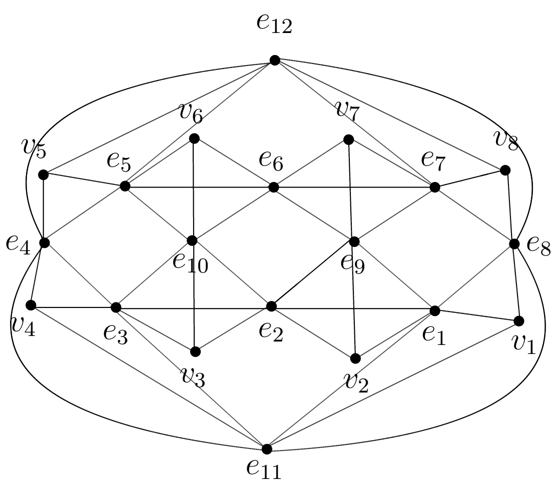


Figure 3: $M(K_2 \times C_4)$

Consider $S = \{e_1, e_2, e_3, ..., e_{2p}\}$, a hub set of $M(K_2 \times C_p)$, $|S| = 2p$, then $m_e(M(K_2 \times C_p) - S) = 2$. Therefore,

$$WHI(M(K_2 \times C_p)) \leq |S| + m_e(M(K_2 \times C_p) - S) = 2p + 2. \tag{3.4}$$

If S_1 is any hub set S with $m_e(M(K_2 \times C_p) - S_1) = 1$, then $|S_1| \geq 3p$. This implies that

$$|S_1| + m_e(M(K_2 \times C_p) - S_1) > |S| + m_e(M(K_2 \times C_p) - S). \tag{3.5}$$

If S_2 is any hub set other than S and $m_e(M(K_2 \times C_p) - S_2) \geq 2$, then trivially

$$|S_2| + m_e(M(K_2 \times C_p) - S_2) \geq 2p + 2. \tag{3.6}$$

Therefore, from (3.4), (3.5) and (3.6), we have $WHI(M(K_2 \times C_p)) = 2p + 2$. □

Corollary 3.21. $WHI(M(K_2 \times C_p)) \geq 2WHI(M(C_p)) + 2$.

Lemma 3.22. $h(M(K_2 \times K_{1,p-1})) = 2p - 1$.

Proof. The graph $M(K_2 \times K_{1,p-1})$ consists of two complete graphs each with p vertices and $\{v, v_1, v_2, ..., v_{p-1}, w_1, w_2, ..., w_{p-1}, w'_1, w'_2, ..., w'_{p-1}\}$ vertices as shown in Figure 4. The number of the vertices $M(K_2 \times K_{1,p-1})$ is $5p - 2$.
A vertex v in $M(K_2 \times K_{1,p-1})$ is adjacent to all vertices in both K_p, and the vertices $\{v_1, v_2, ..., v_{p-1}\}$ are adjacent to one vertex of both complete graph K_{p-1}, the vertices $\{w_1, w_2, ..., w_{p-1}\}$ are adjacent to one vertex of one complete K_{p-1}, and the vertices $\{w'_1, w'_2, ..., w'_{p-1}\}$ are adjacent to one vertex of another complete K_{p-1}. Consider $H = \{v, u_1, u_2, ..., u_{p-1}, u'_1, u'_2, ..., u'_{p-1}\}$, a hub set of graph $M(K_2 \times K_{1,p-1})$. We claim that H is a minimum hub set. If the vertex v is removed from set H, then there does not exist H-path between vertices of both complete graph K_p. Thus $h(M(K_2 \times K_{1,p-1}) = 2p - 1$. □

Theorem 3.23. $WHI(M(K_2 \times K_{1,p-1}) = 2p + 1$.

Proof. The graph $M(K_2 \times K_{1,p-1})$ consists of two complete graphs each with p vertices and $\{v, v_1, v_2, ..., v_n, w_1, w_2, ..., w_{p-1}, w'_1, w'_2, ..., w'_{p-1}\}$ vertices as shown in Figure 4.

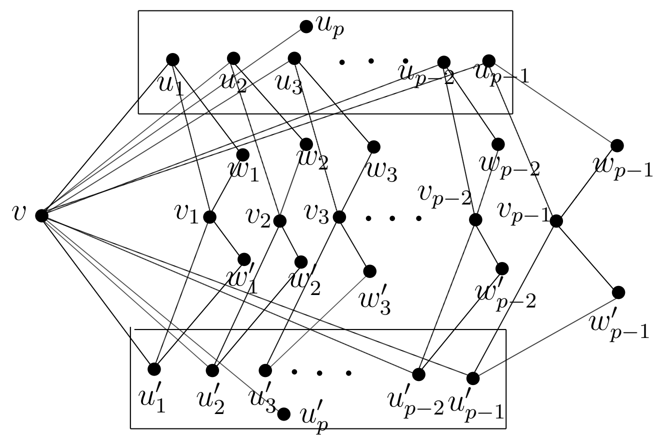


Figure 4: $M(K_2 \times K_{1,p-1})$

The number of the vertices in $M(K_2 \times K_{1,p-1})$ is $5p - 2$. From Lemma (3.22), $h(M(K_2 \times K_{1,p-1}) = 2p - 1$, and $H = \{v, u_1, u_2, ..., u_{p-1}, u'_1, u'_2, ..., u'_{p-1}\}$, a hub set of graph $M(K_2 \times K_{1,p-1})$. Then $m_e(M(K_2 \times K_{1,p-1}) - H) = 2$. This implies that

$$\begin{aligned} WHI(M(K_2 \times K_{1,p-1})) &\leq h(M(K_2 \times K_{1,p-1})) + m_e(M(K_2 \times K_{1,p-1}) - H) \\ &= 2p - 1 + 2 = 2p + 1. \end{aligned} \tag{3.7}$$

If S is any hub set other than S and $m_e(M(K_2 \times K_{1,p-1}) - H) = 1$, then $|S| \geq 3p - 2$, so

$$\begin{aligned} WHI(M(K_2 \times K_{1,p-1})) &\geq |S| + m_e(M(K_2 \times K_{1,p-1}) - S) \\ &= 3p - 2 + 1 = 3p - 1. \end{aligned} \tag{3.8}$$

If S_1 is any hub set other than H and S such that $m_e(M(K_2 \times K_{1,p-1}) - S) = 0$, then $|S| \geq 3p-2$, so

$$\begin{aligned} WHI(M(K_2 \times K_{1,p-1})) &\geq |S_1| + m_e(M(K_2 \times K_{1,p-1}) - S_1) \\ &= 3p - 2 + 0 = 3p - 2. \end{aligned} \tag{3.9}$$

If S_2 is any hub set such that $m_e(M(K_2 \times K_{1,p-1})) \geq 2$, then

$$|S_2| + m_e(M(K_2 \times K_{1,p-1}) - S_2) \geq 2p + 1. \tag{3.10}$$

From (3.7), (3.8), (3.9) and (3.10), $WHI(M(K_2 \times K_{1,p-1})) = 2p + 1$.

□

Corollary 3.24. *For $p \geq 5$, $WHI(M(K_2 \times K_{1,p-1})) \geq 2WHI(M(K_{1,p-1})) + 2$.*

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