Gulf Journal of Mathematics
Vol 6, Issue 1 (2018) 1-11

PROPERTY (m) UNDER PERTURBATIONS

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ABSTRACT. A Banach space operator is said to be obeys property (m) if the isolated points of the spectrum $\sigma(T)$ of T which are eigenvalues of finite multiplicity are exactly those points λ of the spectrum for which $T-\lambda$ is an upper semi-Browder. In this article, we study the stability of property (m), for a bounded operator acting on a Banach space, under perturbation by finite rank operators, by nilpotent operators, quasi-nilpotent operators, Riesz operator or algebraic operators commuting with T.

1. Introduction and preliminaries

Let $\mathscr{B}(\mathscr{X})$ denote the algebra of bounded operators acting on an infinite complex Banach space $\mathscr{X}$. We use I to denote the identity operator on $\mathscr{X}$, and $\mathscr{K}(\mathscr{X})$ to denote the ideal of all compact operators on $\mathscr{X}$ and $\mathscr{F}(\mathscr{X})$ to denote the ideal of all finite rank operators on $\mathscr{X}$. For an arbitrary operator $T \in \mathscr{B}(\mathscr{X})$, $\ker(T)$ denotes its kernel and $\mathcal{R}(T)$ denotes its range. We set $\alpha(T) = \dim\ker(T)$ and $\beta(T) = \dim \mathscr{X}/\mathcal{R}(T)$. Denote by

$$SF_+(\mathscr{X}) := \{T \in \mathscr{B}(\mathscr{X}) : \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed}\}$$

the class of all upper semi-Fredholm operators, and by

$$SF(\mathscr{X}) := \{T \in \mathscr{B}(\mathscr{X}) : \beta(T) < \infty\}$$

the class of all lower semi-Fredholm operators. The class of all semi-Fredholm operators is defined by $SF(\mathscr{X}) := SF_+(\mathscr{X}) \cup SF(\mathscr{X})$, while the class of all Fredholm operator is defined by $F(\mathscr{X}) := SF_+(\mathscr{X}) \cap SF(\mathscr{X})$. For a semi-Fredholm operator T we define the index, $ind(T)$, by $ind(T) = \alpha(T) - \beta(T)$. Let $a := a(T)$ be the ascent of an operator T ; i.e., the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, let $d := d(T)$ be the descent of an operator T ; i.e., the smallest nonnegative integer q such that $\mathcal{R}(T^q) = \mathcal{R}(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite then $a(T) = d(T)$ [12, Proposition 38.3]. Moreover, $0 < a(T-\lambda I) = d(T-\lambda I) < \infty$ precisely when λ is a pole of the resolvent of T, see Heuser [12, Proposition 50.2].

For a subset G of an arbitrary topological space, $\overline{G}$ denotes the closure of G.

Date: Received: Jan 14, 2018; Accepted: Feb 7, 2018.

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2010 *Mathematics Subject Classification.* 47A10; 47A11; 47A53.

Key words and phrases. Weyl's theorem, Weyl spectrum, Polaroid operators, Property (R), Property (m).

Let $\mathbb{C}$ denotes the complex plane. If K is a subset of $\mathbb{C}$, then $isoK$ denotes the set of all isolated points of K and $accK$ denotes the set of all points of accumulation of K. We use $\sigma(T)$ and $\sigma_a(T)$ to denote the spectrum and the approximate point spectrum of T respectively. We use T^* to denote the adjoint of $T \in \mathscr{B}(\mathscr{X})$.

Two important classes of operators are the class of all upper semi-Browder operators

$$B_+(\mathscr{X}) := \{T \in SF_+(\mathscr{X}) : a(T) < \infty\}$$

and the class of all lower semi-Browder operators

$$B_-(\mathscr{X}) := \{T \in SF_-(\mathscr{X}) : d(T) < \infty\}.$$

The class of all Browder operators is defined by $B(\mathscr{X}) := B_+(\mathscr{X}) \cap B_-(\mathscr{X})$. Recall that a bounded linear operator $T \in \mathscr{B}(\mathscr{X})$ is said be a Weyl operator, $T \in W(\mathscr{X})$ if $T \in \mathscr{F}(\mathscr{X})$ and has index 0. Obviously, if $T \in B(\mathscr{X})$ then $T \in W(\mathscr{X})$.

These classes of operators motivate the definition of several spectra. The *upper semi-Browder spectrum* of $T \in \mathscr{B}(\mathscr{X})$ is defined by

$$\sigma_{ub}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B_+(\mathscr{X})\},$$

the *lower semi-Browder spectrum* of $T \in \mathscr{B}(\mathscr{X})$ is defined by

$$\sigma_{lb}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B_-(\mathscr{X})\},$$

while the *Browder spectrum* of $T \in bx$ is defined by

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin B(\mathscr{X})\},$$

The *Weyl spectrum* of $T \in \mathscr{B}(\mathscr{X})$ is defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin W(\mathscr{X})\},$$

We have that $\sigma_w(T) = \sigma_w(T^*)$, while $\sigma_{ub}(T) = \sigma_{lb}(T^*)$ and $\sigma_{ub}(T^*) = \sigma_{lb}(T)$. Evidently,

$$\sigma_w(T) \subseteq \sigma_b(T) = \sigma_w(T) \cup acc\sigma(T).$$

For $T \in \mathscr{B}(\mathscr{X})$, $SF_+^-(\mathscr{X}) := \{T \in SF_+(\mathscr{X}) : ind(T) \leq 0\}$ and $SF_-^+(\mathscr{X}) := \{T \in SF_-(\mathscr{X}) : ind(T) \geq 0\}$. The *Weyl (or essential) approximate point spectrum* is defined by $\sigma_{SF_+^-}(T) := \{T \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathscr{X})\}$. Note that $\sigma_{SF_+^-}(T)$ is the intersection of all approximate point spectra $\sigma_a(T+K)$ of compact perturbations K of T, see [14]. The *Weyl surjectivity spectrum* $\sigma_{SF_-^+}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SF_-^+(\mathscr{X})\}$. The spectrum $\sigma_{SF_-^+}(T)$ coincides with the intersection of all surjectivity spectra $\sigma_s(T+K)$ of compact perturbations K of T, see [14]. Clearly, the last two spectra are dual each other, i.e., $\sigma_{SF_+^-}(T) = \sigma_{SF_-^+}(T^*)$ and $\sigma_{SF_-^+}(T) = \sigma_{SF_+^-}(T^*)$. Moreover, $\sigma_w(T) = \sigma_{SF_+^-}(T) \cup \sigma_{SF_-^+}(T)$. Since $a(T) < \infty$ entails $ind(T) \leq 0$ and $d(T) < \infty$ entails $ind(T) \geq 0$, we have $\sigma_{SF_+^-}(T) \subseteq \sigma_{ub}(T)$ and $\sigma_{SF_-^+}(T) \subseteq \sigma_{lb}(T)$. Hence the relationship between these spectra are given by the following equalities:

$$\sigma_{ub}(T) = \sigma_{SF_+^-}(T) \cup acc\sigma_{\mathbf{a}}(T), \tag{1.1}$$

$$\sigma_{lb}(T) = \sigma_{SF_-^+}(T) \cup acc\sigma_{\mathbf{a}}(T), \tag{1.2}$$

see [16].

Following [10] we say that $T \in \mathscr{B}(\mathscr{X})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \to \mathscr{X}$ which satisfies the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$. An operator $T \in \mathscr{B}(\mathscr{X})$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

An operator $T \in \mathscr{B}(\mathscr{X})$ has the SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. The identity theorem for analytic functions ensures that for every $T \in \mathscr{B}(\mathscr{X})$, both T and T^* have the SVEP at the points of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, that both T and T^* have the SVEP at every isolated point of $\sigma(T) = \sigma(T^*)$. The SVEP is inherited by the restrictions to closed invariant subspaces, i.e., if $T \in \mathscr{B}(\mathscr{X})$ has the SVEP at λ_0 and M is closed T-invariant subspace then $T|_M$ has SVEP at λ_0. Let $S(T) := \{\lambda \in \mathbb{C} : T \text{ does not have the SVEP at } \lambda\}$. Observe that $T \in \mathscr{B}(\mathscr{X})$ has SVEP if and only if $S(T) = \emptyset$.

2. **Property** (m) **under Perturbation**

For a bounded operator $T \in \mathscr{B}(\mathscr{X})$, set

$$\pi^0(T) : \sigma(T) \setminus \sigma_b(T) = \{\lambda \in \sigma(T) : T - \lambda \in B(\mathscr{X})\}.$$

Note that every $\lambda \in \pi^0(T)$ is a pole of the resolvent and hence an isolated point of $\sigma(T)$, see [12, Proposition 50.2]. Moreover, $\pi^0(T) = \pi^0(T^*)$. Define

$$E^0(T) := \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

Obviously,

$$\pi^0(T) \subseteq E^0(T) \text{ for every } T \in \mathscr{B}(\mathscr{X}).$$

a bounded operator $T \in \mathscr{B}(\mathscr{X})$, let us define

$$E^0_a(T) := \{\lambda \in iso\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\},$$

and

$$\pi^0_a(T) := \sigma_a(T) \setminus \sigma_{ub}(T) = \{\lambda \in \sigma_a(T) : T - \lambda \in B_+(\mathscr{X})\}.$$

Lemma 2.1. *([3]) For every $T \in \mathscr{B}(\mathscr{X})$, we have*
(a) $\pi^0(T) \subseteq \pi^0_a(T) \subseteq E^0_a(T)$ *and*
(b) $E^0(T) \subseteq E^0_a(T)$.

Following Harte and W.Y. Lee [11], we shall say that T satisfies Browder's theorem if

$$\sigma_b(T) = \sigma_w(T),$$

while, $T \in \mathscr{B}(\mathscr{X})$ is said to satisfy a-Browder's theorem if

$$\sigma_{SF_+^-}(T) = \sigma_{ub}(T).$$

Obviously, a-Browder's theorem holds for T implies Browder's theorem holds for T and the converse is not true. Following Coburn [7], we say that Weyl's theorem holds for $T \in \mathscr{B}(\mathscr{X})$ if

$$\Delta(T) := \sigma(T) \setminus \sigma_w(T) = E^0(T).$$

An approximate point version of Weyl's theorem is a-Weyl's theorem: according Rakočevič [17] an operator $T \in \mathscr{B}(\mathscr{X})$ is said to satisfy a-Weyl's theorem if

$$\Delta_a(T) := \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T).$$

Note that

$$a\text{-Weyl's theorem holds for } T \Longrightarrow \text{Weyl's theorem holds for } T$$

while the converse in general does not hold.

Definition 2.2. A bounded operator $T \in \mathscr{B}(\mathscr{X})$ is said to satisfy
(i) property (w) if $\Delta_a(T) = E^0(T)$ [15].
(ii) property (t) if $\Delta_+(T) := \sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$ [19].

Definition 2.3. ([20]) A bounded operator $T \in \mathscr{B}(\mathscr{X})$ is said to satisfy property (m) if

$$\sigma(T) \setminus \sigma_{ub}(T) = E^0(T).$$

Weyl's theorem corresponds to the half of property (m), in the following sense:

Theorem 2.4. *([20]) If $T \in \mathscr{B}(\mathscr{X})$ then the following assertions are equivalent:*
(i) *property (m) holds for T;*
(ii) *T satisfies Weyl's theorem and $\sigma_{ub}(T) = \sigma_w(T)$.*

Theorem 2.5. *Let $T \in \mathscr{B}(\mathscr{X})$. Then the following assertions are equivalent:*
(i) *Property (t) holds for T;*
(ii) *T satisfies property (m) and $\sigma_{ub}(T) = \sigma_{SF_+^-}(T)$.*

Proof. (i) $\Longrightarrow$ (ii) Assume that T obeys property (t) then $\Delta_+(T) = E^0(T)$. If $\lambda \in \sigma(T) \setminus \sigma_{ub}(T)$, then $\lambda \in \Delta_+(T) = E^0(T)$ and so $\sigma(T) \setminus \sigma_{ub}(T) \subseteq E^0(T)$. To prove the other inclusion. Let $\lambda \in E^0(T)$ be an arbitrary given. Then λ is an isolated in $\sigma(T)$ and hence T and T^* has SVEP at λ. As T has property (t), we have $T - \lambda \in SF_+^-(\mathscr{X})$ and hence $\lambda \in B_+(\mathscr{X})$. The SVEP of T and T^* at λ implies by [3, Remark 1.2] that $a(T-\lambda) = d(T-\lambda) < \infty$. As $\alpha(T-\lambda) < \infty$ then it follows by [1, Theorem 3.4] that $\alpha(T-\lambda) = \beta(T-\lambda) < \infty$ and so $\lambda \in \pi^0(T)$. Hence $\lambda \in \sigma(T) \setminus \sigma_{ub}(T)$. Therefore, $E^0(T) \subseteq \sigma(T) \setminus \sigma_{ub}(T)$ and so T obeys property (m).
(ii) $\Longrightarrow$ (i) Suppose that T obeys property (m) and $\sigma_{ub}(T) = \sigma_{SF_+^-}(T)$. Then

$$E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T).$$

That is, T obeys property (t). ∎

Let $H_{nc}(\sigma(T))$ denotes the set of all complex-valued functions f, dened and regular in some neighborhood of $\sigma(T)$, such that f is not constant on the connected components of its domain of denition.

Recall that $T \in \mathscr{B}(\mathscr{X})$ is isolated, provided that all isolated points of $\sigma(T)$ are eigenvalues of T. $T \in \mathscr{B}(\mathscr{X})$ is a-isolated provided that all isolated points of $\sigma_a(T)$ are eigenvalues of T. It is well-known that $\partial\sigma(T) \subseteq \sigma_a(T)$, so all isolated points of $\sigma(T)$ are also isolated points of $\sigma_a(T)$. Now it is obvious that if T is a-isolated, then it is also isolated.

We shall consider nilpotent perturbations of operators satisfying property (m). It easy to check that if N is a nilpotent operator commuting with T, then

$$\sigma(T)=\sigma(T+N) \text{ and } \sigma_a(T)=\sigma_a(T+N) \text{and } \sigma_{ub}(T)=\sigma_{ub}(T+N). \tag{2.1}$$

Hence it follows from Equation (2.1)

$$E^0(T)=E^0(T+N) \text{ and } E_a^0(T)=E_a^0(T+N), \tag{2.2}$$

from [9, Theorem 2.13], we have

$$\sigma_{SF_+^-}(T)=\sigma_{SF_+^-}(T+N). \tag{2.3}$$

Theorem 2.6. *Let $T\in\mathscr{B}(\mathscr{X})$ and let N be a nilpotent operator commuting with T. If property (m) holds for T then it also holds for $T+N$.*

Proof. Firstly we prove that $E^0(T)=E^0(T+N)$. It is enough to prove that if $0\in E^0(T)$, then $0\in E^0(T+N)$. Suppose that $0\in E^0(T)$, so $0<\dim\ker(T-\lambda)<\infty$.

We prove that $\dim\ker(T+N)<\infty$. If $(T+N)x=0$ for some $x\neq 0$, then $Tx=Nx$. Since N commutes with T, it follows that for every positive integer m: $T^m x=(1)^m N^m x$. Let n be the smallest positive integer such that $N^n=0$. We get that there is some positive integer r, $r\leq n$, such that $T^r x=0$. Thus $\ker(T+N)\subseteq\ker(T^r)$ and $\ker(T+N)$ is finite dimensional.

We prove that $\dim\ker(T+N)>0$. There is some $x\neq 0$ such that $Tx=0$. Then $(T+N)^n x=0$, $0\in\sigma_p(T+N)\subseteq\sigma(T+N)$ and $\dim\ker(T+N)>0$. By Eq. (2.1) we know that $\sigma(T)=\sigma(T+N)$, so it follows that $0\in E^0(T+N)$. Thus, using Eq. (2.1) we get

$$\sigma_{ub}(T+N)=\sigma_{ub}(T)=\sigma(T)\setminus E^0(T)=\sigma(T+N)\setminus E^0(T+N).$$

Thus property (m) holds for $T+N$. ∎

The following example shows that the result of Theorem 2.6 does not hold if we do not assume that the nilpotent operator commutes with T.

Example 2.7. Let$\mathscr{X}:=\ell^2(\mathbb{N})$ and T and N be defined by

$$T(x_1,x_2,\cdots):=\left(0,\frac{x_1}{2},\frac{x_2}{3},\cdots\right),\quad (x_n)\in\mathscr{X}$$

and

$$N(x_1,x_2,\cdots):=\left(0,-\frac{x_1}{2},0,0,\cdots\right),\quad (x_n)\in\mathscr{X}.$$

Clearly, N is a nilpotent operator and T is a quasinilpotent operator satisfying property (m). On the other hand, it easily seen that $0\in E^0(T+N)$ and $0\notin\sigma(T+N)\setminus\sigma_{ub}(T+N)$, so that $T+N$ does not satisfies property (m).

The next result from [8] is very useful.

Lemma 2.8. *If $\alpha(T)=n$ and $\dim\mathcal{R}(T)=m$, then*

$$\alpha(T+N)\leq n+m,$$

where m and n are non-negative integers.

Theorem 2.9. *Suppose that F is an arbitrary nite rank operator and $TF=FT$. If T is isoloid and property (m) holds for T, then property (m) holds for $T+F$.*

Proof. It is enough to prove that $0 \in \sigma(T+F) \setminus \sigma_{ub}(T+F)$ if and only if $0 \in E^0(T+F)$. Firstly we prove that if $0 \in \sigma(T+F) \setminus \sigma_{ub}(T+F)$, then $T+F \in B_+(\mathscr{X})$ and $0 < \alpha(T+F) < \infty$. We need to prove that $0 \in iso\sigma(T+F)$. It follows that $T \in B_+(\mathscr{X})$, so $0 \notin \sigma_{ub}(T)$. It is possible that $0 \notin \sigma(T)$. In this case we get $0 \notin acc\sigma(T)$ and hence $0 \notin acc\sigma_a(T+F)$, so $0 \in E^0(T+F)$. The second possibility is that $0 \in \sigma(T)$. Since property (m) holds for T, we get that $0 \notin acc\sigma(T)$ and again $0 \in E^0(T+F)$.

To prove the opposite implication, suppose that $0 \in E^0(T+F)$. Then $0 \in iso\sigma(T+F)$ and $0 < \alpha(T+F) < \infty$. Hence $0 \notin acc\,\sigma(T)$ and by Lemma 2.8 it follows that $0 \leq \alpha(T) < \infty$. Again we distinguish two cases. Firstly, if $0 \notin \sigma(T)$, then $T \in B_+(\mathscr{X})$ and $T+F \in B_+(\mathscr{X})$, $0 \in \sigma(T+F) \setminus \sigma_{ub}(T+F)$. On the other hand, if $0 \in \sigma(T)$ then $0 \in iso\sigma(T)$. Since T is isoloid, we get that $0 < \alpha(T) < \infty$ and $0 \notin \sigma_{ub}(T)$. Now, we have $T \in B_+(\mathscr{X})$, $T+F \in B_+(\mathscr{X})$ and $0 \in \sigma(T+F) \setminus \sigma_{ub}(T+F)$. ∎

Note that the operator N in Example 2.7 is also a nite rank operator not commuting with T. In general, property (m) is also not transmitted under commuting nite rank perturbation.

Example 2.10. Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be an injective quasinilpotent operator, and let $U : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be dened:

$$U(x_1, x_2, \cdots) := (x_1, 0, 0, \cdots) \quad (x_n) \in \ell^2(\mathbb{N}).$$

Dene on $\mathscr{X} := \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ the operators T and F by

$$T := I \oplus S \text{ and } F := U \oplus 0.$$

Clearly, F is a nite rank operator and $TF = FT$. It is easy to check that

$$\sigma(T) = \sigma_a(T) = \sigma_w(T) = \sigma_{ub}(T) = \{0, 1\}.$$

Now, both T and T^* have SVEP, since $\sigma(T) = \sigma(T^*)$ is nite. Moreover, $E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \emptyset$, so T satises property (m). On other hand, $\sigma(T+F) = \sigma_{ub}(T+F) = \{0, 1\}$, and $E^0(T+F) = \{0\}$, so that property (m) does not hold for $T+F$.

Recall that $T \in \mathscr{B}(\mathscr{X})$ is said to be a Riesz operator if $T - \lambda \in \mathscr{F}(\mathscr{X})$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators. The proof of the following result may be found in Rakočević [18]:

Lemma 2.11. *Let $T \in \mathscr{B}(\mathscr{X})$ and R be a Riesz operator commuting with T. Then*

(i) $T \in B_+(\mathscr{X}) \Longleftrightarrow T + R \in B_+(\mathscr{X})$.
(ii) $T \in B_-(\mathscr{X}) \Longleftrightarrow T + R \in B_-(\mathscr{X})$.
(iii) $T \in B(\mathscr{X}) \Longleftrightarrow T + R \in B(\mathscr{X})$.

Define

$$E^{0f} := \{\lambda \in iso\,\sigma(T) : \alpha(T-\lambda) < \infty\}.$$

Evidently, $E^0(T) \subseteq E^{0f}(T)$ for every operator $T \in \mathscr{B}(\mathscr{X})$.

Lemma 2.12. *Let $T \in \mathscr{B}(\mathscr{X})$. If R is a Riesz operator that commutes with T, then*

$$E^0(T+R) \cap \sigma(T) \subseteq iso\,\sigma(T). \tag{2.4}$$

Proof. By [13, Lemma 2.3], we have

$$E^0(T+R) \cap \sigma(T) \subseteq E^{0f}(T+R) \cap \sigma(T) \subseteq iso\,\sigma(T).$$

■

Lemma 2.13. *Let $T \in \mathscr{B}(\mathscr{X})$ be an isoloid operator satisfying property (m). If F is an operator that commutes with T and for which there exists a positive integer n such that F^n is nite rank, then $E^0(T+F) = \pi_a^0(T+F)$.*

Proof. Let $\lambda \in E^0(T+F)$ be an arbitrary given. We distinguish two cases. Firstly, if $\lambda \notin \sigma(T)$, then $T+F-\lambda \in B_+(\mathscr{X})$, and hence $\lambda \in \sigma_{ub}(T)$. Suppose that $\lambda \in \sigma(T)$, it follows, by Lemma 2.12, that $\lambda \in iso\,\sigma(T)$. Furthermore, since the operator $(T+F-\lambda)^n|_{\ker(T-\lambda)} = F^n|_{\ker(T-\lambda)}$ is both of finite-dimensional range and kernel, kernel, we obtain easily that also $\ker(T-\lambda)$ is finite-dimensional, and therefore that $\lambda \in E^0(T)$, because T is isoloid. On the other hand, if T obeys property (m), then $E^0(T) \cap \sigma_{ub}(T) = \emptyset$. Consequently, $T-\lambda \in B_+(\mathscr{X})$ and hence $T+F-\lambda \in B_+(\mathscr{X})$, which implies that $\lambda \in \pi_a^0(T+F)$.
To prove the other inclusion, let $\lambda \in \pi_a^0(T+F)$ be arbitrary given. Then $\lambda \in iso\,\sigma_a(T+F)$ and $T+F-\lambda \in B_+(\mathscr{X})$, so $\alpha(T+F-\lambda) < \infty$. Since $T+F-\lambda$ has closed range, the condition $\lambda \in \sigma_a(T+F)$ entails that $\alpha(T+F-\lambda) > 0$. Therefore, in order to show that $\lambda \in E^0(T+F)$, we need only to prove that λ is an isolated point of $\sigma(T+F)$. We know that $\lambda \in iso\,\sigma_a(T)$. We have from Lemma 2.11 that $(T+F)-\lambda-F = T\lambda \in B_+(\mathscr{X})$ so that $\lambda \in \sigma_a(T) \setminus \sigma_{ub}(T) = \pi_a^0(T)$. Now, by assumption T obeys property (m) so, by [20, Theorem 2.2], $\pi_a^0(T) = E^0(T)$. Moreover, T satises Weyl's theorem and hence

$$E^0(T) = \pi^0(T) = \sigma(T) \setminus \sigma_b(T).$$

Therefore, $T-\lambda$ is Browder and hence $T+F-\lambda$ is Browder, so

$$0 < a(T+F-\lambda) = d(T+F-\lambda) < \infty$$

and hence λ is a pole of the resolvent of $T+F$. Consequently, λ is an isolated point of $\sigma(T+F)$. ■

Theorem 2.14. *If $T \in \mathscr{B}(\mathscr{X})$ has property (m) and R is a Riesz operator for which $TR = RT$, then $E^0(T) \subseteq E^0(T+R)$.*

Proof. Suppose that T has property (m). Since $\sigma(T) = \sigma(T+R)$ holds for every Riesz operator commuting with T, we have

$$E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \sigma(T+R) \setminus \sigma_{ub}(T+R). \tag{2.5}$$

Let $\lambda \in E^0(T)$ be arbitrary given. Taking into account that $S := T+R$ commutes with R, by Lemma 2.12 we then have

$$\begin{aligned} \lambda \in E^0(T) \cap \sigma(T+R) &= E^0(S-R) \cap \sigma(S) \\ &\subseteq iso\,\sigma(S) = iso\,\sigma(T+R). \end{aligned}$$

Moreover, from (2.5) we know that $T + R - \lambda \in B_+(\mathscr{X})$ and hence has closed range. Since $\lambda \in \sigma(T + R)$ it then follows that λ is an eigenvalue, so $0 < \alpha(T + R - \lambda) < \infty$, i.e., $\lambda \in E^0(T + R)$. ∎

Lemma 2.15. *Let $T \in \mathscr{B}(\mathscr{X})$ be an isoloid operator satisfying property (m). If F is an operator that commutes with T and for which there exists a positive integer n such that F^n is nite rank, then $E^0(T) = E^0(T + F)$.*

Proof. Observe rst that F is a Riesz operator, so, by Theorem 2.14, we need only to prove the inclusion $E^0(T + F) \subseteq E^0(T)$. Let $\lambda \in E^0(T + F)$. Then λ is an isolated point of $\sigma(T + F)$, and since $\alpha(T + F - \lambda) > 0$ we then have $\lambda \in \sigma(T + F) = \sigma(T)$. Therefore, by Lemma 2.12, $\lambda \in E^0(T + F) \cap \sigma(T) \subseteq iso\,\sigma(T)$. Since T is isoloid then $\alpha(T\lambda) > 0$. We show now that $\alpha(T - \lambda) < \infty$. Let U denote the restriction of $(T + F - \lambda)^n$ to $\ker(T - \lambda)$. Clearly, if $x \in \ker(T - \lambda)$ then $Ux = (1)^n F^n x \in \mathscr{F}(\mathscr{X})$, thus U is a nite rank operator. Moreover, since $\lambda \in E^0(T+F)$ we have $\alpha(T+F-\lambda) < \infty$ and hence $\alpha(U) \leq \alpha(T+F-\lambda)^n < \infty$. By [5, Remark 2.5] it then follows that $\ker(T-\lambda)$ is nite-dimensional. Therefore, $\lambda \in E^0(T)$ and consequently $E^0(T + F) \subseteq E^0(T)$. ∎

Theorem 2.16. *Let $T \in \mathscr{B}(\mathscr{X})$ be an isoloid operator satisfying property (m). If F is an operator that commutes with T and for which there exists a positive integer n such that F^n is nite rank, then $T + F$ satises property (m).*

Proof. Since F is a Riesz operator we have, by [18], $\sigma_{ub}(T) = \sigma_{ub}(T + F)$, thus

$$E^0(T + F) = E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \sigma(T + F) \setminus \sigma_{ub}(T + F),$$

hence $T + F$ satises property (m). ∎

Example 2.17. Generally, property (m) is not transmitted from T to a quasi-nilpotent perturbation $T + Q$. Let $Q : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ dened by

$$Q(x_1, x_2, \cdots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \cdots\right) \text{ for all } (x_n) \in \ell^2(\mathbb{N}).$$

Then Q is quasi-nilpotent, so $\sigma(T) = \sigma_{ub}(T) = \{0\}$ and hence $\{0\} = E^0(Q) \neq \sigma(Q) \setminus \sigma_{ub}(Q) = \emptyset$. Take $T = 0$. Clearly, T satises property (m) but $T + Q = Q$ fails this property.

Recall that an operator $T \in \mathscr{B}(\mathscr{X})$ is nite-isoloid if isolated points of $\sigma(T)$ are eigenvalues of T of nite multiplicity.

Theorem 2.18. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ is a nite-isoloid operator which obeys property (m). If R is a Riesz operator which commutes with T, then $T + R$ obeys property (m).*

Proof. We show rst that $E^0(T) = E^0(T + R)$. By Theorem 2.14 it suces to prove that $E^0(T + R) \subseteq E^0(T)$. Let $\lambda \in E^0(T + R)$ be arbitrary given. Then λ is an isolated of $\sigma(T + R)$ and $0 < \alpha(T + R - \lambda) < \infty$. Since $\sigma(T) = \sigma(T + R)$ holds for every Riesz operator commuting with T, we have by Lemma 2.12 that $\lambda \in E^0(T + R) \cap \sigma(T) \subseteq iso\,\sigma(T)$. Since T is nite-isoloid then $0 < \alpha(T - \lambda) < \infty$

and so $\lambda \in E^0(T)$. Therefore, $E^0(T) = E^0(T+R)$.
As T obeys property (m) and $E^0(T) = E^0(T+R)$, we have

$$E^0(T+R) = E^0(T) = \sigma(T) \setminus \sigma_{ub}(T) = \sigma(T+R) \setminus \sigma_{ub}(T+R).$$

That is, property (m) holds for $T+R$. ■

Since every compact operator is a Riesz operator we have:

Corollary 2.19. *Let $T \in \mathscr{B}(\mathscr{X})$ be a nite-isoloid operator that obeys property (m). If K is a compact operator commuting with T, then $T+K$ obeys property (m).*

Since every quasi-nilpotent operator is a Riesz operator we have:

Corollary 2.20. *Let $T \in \mathscr{B}(\mathscr{X})$ be a nite-isoloid operator that obeys property (m). If Q is a quasi-nilpotent operator commuting with T, then $T+Q$ obeys property (m).*

Theorem 2.21. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ and Q an injective quasi-nilpotent operator commuting with T. If T obeys property (m), then $T+Q$ obeys property (m).*

Proof. As T obeys property (m), we have

$$\sigma(T+Q) \setminus \sigma_{ub}(T+Q) = \sigma(T) \setminus \sigma_{ub}(T) = E^0(T). \tag{2.6}$$

To show property (m) for $T+Q$ it suces to prove that

$$E^0(T+Q) = E^0(T) = \emptyset.$$

Suppose that $E^0(T) \neq \emptyset$ and let $\lambda \in E^0(T)$. From (2.6) we know that $T - \lambda \in B_+(\mathscr{X})$ and hence from [2, Lemma 2.11] it then follows that $\alpha(T-\lambda) = 0$, a contradiction.

To show that $E^0(T+Q) = \emptyset$. Suppose that $\lambda \in E^0(T+Q)$. Then $0 < \alpha(T+Q-\lambda) < \infty$, so there exists $x \neq 0$ such that $(T+Q-\lambda)x = 0$. Since Q commutes with $T+Q-\lambda$, a similar argument of proof of [2, Lemma 2.11] shows that $\alpha(T+Q-\lambda) = \infty$, a contradiction. ■

Theorem 2.22. *Let T be an operator on $\mathscr{X}$ that obeys property (m) and such that $\sigma_p(T) \cap iso\sigma(T) \subseteq E^0(T)$. If Q is a quasi-nilpotent operator that commutes with T, then $T+Q$ obeys property (m)*

Proof. As T obeys property (m), we have by [20, Theorem 2.10] that T satises Weyl's theorem and $\sigma_w(T) = \sigma_{ub}(T)$. Hence by [13, Proposition 2.9], we have $T+Q$ satises Weyl's theorem. Since $\sigma_{ub}(T+Q) = \sigma_{ub}(T)$ and $\sigma_w(T) = \sigma_w(T+Q)$ we have $\sigma_{ub}(T+Q) = \sigma_w(T+Q)$ and so $T+Q$ obeys property (m). ■

Definition 2.23. A bounded linear operator T is said to be algebraic if there exists a non-trivial polynomial h such that $h(T) = 0$.

From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a nite set. A nilpotent operator is a trivial example of an algebraic operator. Also nite rank operators K are algebraic; more generally, if K^n is a nite rank operator for some $n \in \mathbb{N}$ then K is algebraic. Clearly, if T is algebraic then its dual T^* is algebraic, as well as T' in the case of Hilbert space operators.

A bounded operator $T \in \mathscr{B}(\mathscr{X})$ is said to be polaroid (respectively, a-polaroid) if $iso\,\sigma(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent of T (respectively, if $iso\,\sigma_a(T) = \emptyset$ or every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T).

Theorem 2.24. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ and $K \in \mathscr{B}(\mathscr{X})$ is an algebraic operator which commutes with T.*

(i) *If T^* is hereditarily polaroid and has SVEP, then $T + K$ obeys property (m).*
(ii) *If T is hereditarily polaroid and has SVEP, then $T^* + K^*$ obeys property (m).*

Proof. (i) Obviously, K^* is algebraic and commutes with T^*. Moreover, by [5, Theorem 2.15], we have $T^* + K^*$ is polaroid, or equivalently, $T + K$ is polaroid. Since T^* has SVEP then by [4, Theorem 2.14], we have $T^* + K^*$ has SVEP . Therefore, $T + K$ obeys property (m) by [20, Theorem 3.3 (i)].
(ii) It follows from the proof of Theorem 2.15 of [5] that $T + K$ is polaroid and hence by duality $T^* + K^*$ is polaroid. Since T has SVEP then it follows from [4, Theorem 2.14] that $T + K$ has SVEP. Therefore, $T^* + K^*$ obeys property (m) by [20, Theorem 3.3 (ii)]. ■

Theorem 2.25. *Suppose that $T \in \mathscr{B}(\mathscr{X})$ and $K \in \mathscr{B}(\mathscr{X})$ is an algebraic operator which commutes with T.*

(i) *If T^* is hereditarily polaroid and has SVEP, then $f(T+K)$ obeys property (m) for all $f \in H_{nc}(\sigma(T))$.*
(ii) *If T is hereditarily polaroid and has SVEP, then $f(T^*+K^*)$ obeys property (m) for all $f \in H_{nc}(\sigma(T))$.*

Proof. (i) We conclude from [5, Theorem 2.15] that $T + K$ is polaroid and hence by [6, Lemma 3.11], we have $f(T + K)$ is polaroid and from [4, Theorem 2.14] that $T^{+}K^*$ has SVEP. The SVEP of $T^* + K^*$ entails the SVEP for $f(T^* + K^*)$ by [1, Theorem 2.40]. So, $f(T+K)$ obeys property (m) by [20, Theorem 3.3 (i)].
(ii) The proof of part (ii) is analogous. ■

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