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ON PATHOS BLOCK LINE GRAPH OF A TREE

HADONAHALLI MUDALAGIRAIAH NAGESH[1]

Abstract. Let T be a tree of order $n \geq 2$. A *pathos line graph of* T, written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T are adjacent or the edge lies on the corresponding path of the pathos. An extension is called a *pathos block line graph*, denoted by $PBL(T)$, and is similar to $PL(T)$, except that the adjacency of pathos vertices. For this class of graphs we discuss the planarity, outer planarity, maximal outer planarity, minimally nonouter planarity, and crossing number one properties of these graphs.

1. Introduction

Notations and definitions not introduced here can be found in [1]. There are many graph valued functions (or graph operators) with which one can construct a new graph from a given graph, such as the line graphs, the total graphs, and their generalizations. One such a graph operator is called a *pathos line graph.* This was introduced by M. H. Muddebihal et al., in [4].

Let T be a tree of order $n \geq 2$. A *pathos line graph of* T, written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T are adjacent or the edge lies on the corresponding path of the pathos. In this paper, we extend the definition of a pathos line graph to a pathos block line graph.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

The concept of *pathos* of a graph G was introduced by Harary [2] as a collection of minimum number of edge disjoint open paths whose union is G. The *path number* of a graph G is the number of paths in any pathos. The *path number* of a tree T equals k, where $2k$ is the number of odd degree vertices of T.

We need some concepts and notations on graphs. A graph G is *planar* if it has a drawing without crossings. For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane.

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[1] Corresponding author.
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If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be *outerplanar*, that is, $i(G) = 0$. An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing its outer planarity. A graph G is said to be *minimally nonouterplanar* if $i(G)$=1 [3]. The least number of edge crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by cr(G). A *wheel graph* W_n is a graph obtained by taking the join of a cycle and a single vertex. A *fan graph* $F_{m,n}$ is defined as the graph join $\overline{K_m} + P_n$, where $\overline{K_m}$ is the empty graph on m vertices and P_n is the path graph on n vertices.

2. Definition of $PBL(T)$

Definition 2.1. The degree of an edge pq of a tree T is the sum of degrees of p and q.

Definition 2.2. A *pathos vertex* is a vertex corresponding to a path of the pathos, and a *block vertex* is a vertex corresponding to a block (or an edge) of T.

Definition 2.3. A *pathos block line graph* of a tree T, written $PBL(T)$, is a graph whose vertices are the edges, paths of a pathos, and blocks of T, with two vertices of $PBL(T)$ adjacent whenever the corresponding edges of T are adjacent or the edge lies on the corresponding path of the pathos or the edge lies on the corresponding block; two distinct pathos vertices P_m and P_n of $PBL(T)$ are adjacent whenever the corresponding paths $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex.

Clearly, $L(T) \subseteq PL(T) \subseteq PBL(T)$. Here $\subseteq$ is the subgraph notation.

Since the pattern of pathos for a tree is not unique, the corresponding pathos block line graph is also not unique.

See Fig.1 and Fig.2 for an example of a tree T and its pathos block line graph $PBL(T)$.

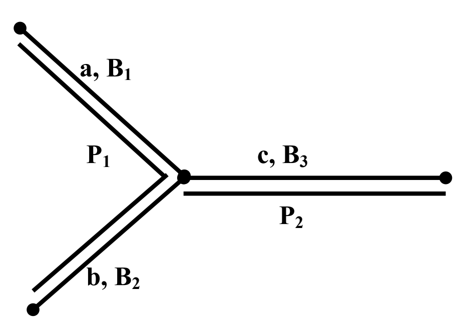

Fig.1. A tree T

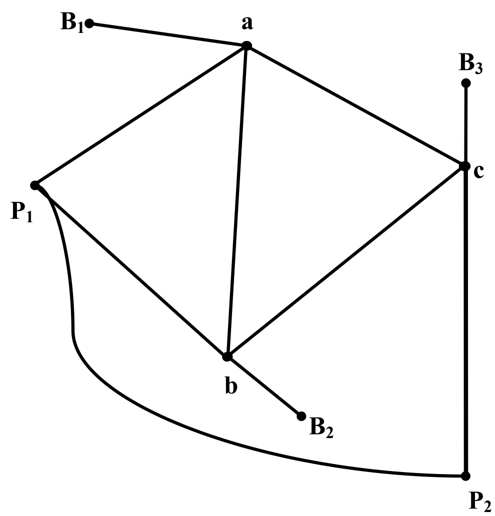

Fig.2. $PBL(T)$

3. Basic properties of $PBL(T)$

Remark 3.1. If T is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the number of edges whose end vertices are the pathos vertices in $PBL(T)$ is $\left[\frac{k(k-1)}{2}\right]$, where k is the path number of T.

Theorem 3.2. (***F. Harary*** [1]) *If G is a graph on p vertices and q edges, then $L(G)$ has q vertices and $-q + \frac{1}{2}\sum_{i=1}^{p} d_i^2$ edges, where d_i is the degree of vertices of G.*

Proposition 3.3. *Let T be a star graph $K_{1,n}$ on $n \geq 2$ vertices. Then the order and size of $PBL(T)$ are $2m + k$ and $m + \frac{1}{2}\sum_{i=1}^{n} d_i^2 + \frac{k(k-1)}{2}$, respectively, where m is the size, k is the path number, and d_i is the degree of vertices of T.*

Proof. Let T be a star graph $K_{1,n}$ on $n \geq 2$ vertices and m edges. By definition, $PBL(T)$ contains $2m + k$ vertices. The size of $PBL(T)$ equals the sum of size of $L(T)$, twice the number of edges of T, and the number of edges whose end vertices are the pathos vertices. By Remark 3.1 and Theorem 3.2, the size of $PBL(T)$ is $-m + \frac{1}{2}\sum_{i=1}^{n} d_i^2 + 2m + \frac{k(k-1)}{2}$

$$\Rightarrow m + \frac{1}{2}\sum_{i=1}^{n} d_i^2 + \frac{k(k-1)}{2}$$ □

Proposition 3.4. *The degree of an edge pq of a tree T equals the degree of the corresponding vertex pq of $PBL(T)$.*

Proof. Let n be the degree of an edge pq in T. Then the edge pq is adjacent to $n-2$ edges. Clearly, $n-2$ degrees are contributed for the corresponding vertex pq in $L(T)$. Since every edge of T lie on exactly one path of the pathos and every edge of T is a block, by definition, the degree of the corresponding vertex in $PBL(T)$ will be incremented by two. Hence $d(pq) = n-2+2 = n$. This completes the proof. □

4. Characterization of $PBL(T)$

Theorem 4.1. *A pathos block line graph $PBL(T)$ of a tree T is planar if and only if $\Delta(T) \leq 3$, for every vertex $v \in T$.*

Proof. Suppose $PBL(T)$ is planar. Assume that $\Delta(T) \geq 4$, for every vertex $v \in T$. If there exists a vertex v of degree four in T, that is, $T = K_{1,4}$. Let $E(T) = \{e_1, e_2, e_3, e_4\}$ be the edge set of T. By definition, $L(T)$ is K_4. Let $P(T) = \{P_1,\, P_2\}$ be a pathos set of T such that P_1 lies on the edges e_1 and e_2; P_2 lies on e_3 and e_4. Then $PL(T)$ is the graph obtained from $L(T)$ by joining the vertices P_1 and e_1; P_1 and e_2; P_2 and e_3; P_2 and e_4 such that $\mathrm{cr}(PL(T)) = 0$. Furthermore, every edge of T is a block. Then $PBL(T)$ is obtained from $L(T)$ by adjoining a pendant edge at each vertex of $L(T)$ such that $\mathrm{cr}(PBL(T)) = 0$. Since the pathos vertices P_1 and P_2 are adjacent in $PBL(T)$, the crossing number of $PBL(T)$ becomes one, i.e., $\mathrm{cr}(PBL(T)) = 1$, a contradiction.

Conversely, suppose $\Delta(T) \leq 3$, for every vertex $v \in T$. We consider the following two cases.
Case 1: Suppose that T is a path of order n ($n \geq 2$). If T is the path P_2, then $PBL(T)$ is the path P_3, which is planar. If T is a path of order n ($n \geq 3$), then $PL(T)$ is the fan graph $F_{1,n-1}$. Since every edge of T is a block, $PBL(T)$ is obtained from $L(T)$ by adjoining a pendant edge at each vertex of $L(T)$. Clearly $\mathrm{cr}(PBL(T)) = 0$.
Case 2: Suppose that the degree of every vertex of T is at most 3. Then $L(T)$ is a connected graph in which every block is either K_2 or K_3. The path number of T is at least one, and the edges joining these blocks from the corresponding pathos vertices are incident with at most two vertices of each block of $L(T)$ such that $\mathrm{cr}(PL(T)) = 0$. Since every edge of T is a block, $PBL(T)$ is obtained from $L(T)$ by adjoining a pendant edge at each vertex of $L(T)$ such that $\mathrm{cr}(PBL(T)) = 0$. Finally, the edges joining pathos vertices does not increase the crossing number of $PBL(T)$. Hence $PBL(T)$ is planar. □

Theorem 4.2. *A pathos block line graph $PBL(T)$ of a tree T is outerplanar if and only if T is a path P_n of order n ($n \geq 2$).*

Proof. Suppose $PBL(T)$ is outerplanar. Assume that there exists a vertex v of degree three in T, that is, $T = K_{1,3}$. Let $E(T) = \{e_1, e_2, e_3\}$ be the edge set and $P(T) = \{P_1,\, P_2\}$ be a pathos set of T such that P_1 lies on the edges e_1, e_2; and P_2 lies on the edge e_3. Then $PL(T)$ is the kite graph.

Since every edge of T is a block, $PBL(T)$ is obtained from $L(T)$ by adjoining a pendant edge at each vertex of $L(T)$. Since the pathos vertices P_1 and P_2 are adjacent in $PBL(T)$, $i(PBL(T)) = 2$, a contradiction.

Conversely, suppose that T is a path P_n of order n ($n \geq 2$). By Case 1 of sufficiency of Theorem 4.1, $PBL(T)$ is obtained from $L(T)$ by adjoining a pendant edge at each vertex of $L(T)$. Clearly $i(PBL(T)) = 0$. Hence $PBL(T)$ is outerplanar. □

Theorem 4.3. (***F.Harary*** [1]) *Every maximal outerplanar graph G with n vertices has $2n-3$ edges.*

Theorem 4.4. *For any tree T, $PBL(T)$ is not maximal outerplanar.*

Proof. We use contradiction. Suppose $PBL(T)$ is maximal outerplanar. We consider the following four cases.
Case 1: Suppose that $\Delta(T) \geq 4$, for every vertex $v \in T$. By Theorem 4.1, $PBL(T)$ is nonplanar, a contradiction.
Case 2: Suppose there exists a vertex of degree three in T. By Theorem 4.2, $PBL(T)$ is nonouterplanar, a contradiction.
Case 3: Suppose that T is the path P_2. Then $PBL(T)$ is P_3, which is not maximal outerplanar, a contradiction.
Case 4: Suppose that T is a path P_n of order n ($n \geq 3$). Then the order and size of $PBL(T)$ are $2\alpha+3$ and $3\alpha+2$, respectively, where $\alpha = (n-2), n \geq 3$. But $3\alpha + 2 < 4\alpha + 3 = 2(2\alpha + 3) - 3$. By Theorem 4.3, $PBL(T)$ is not maximal outerplanar, again a contradiction. Hence by all the above cases, $PBL(T)$ is not maximal outerplanar. □

Theorem 4.5. *For any tree T, $PBL(T)$ is not minimally nonouterplanar.*

Proof. We use contradiction. Suppose $PBL(T)$ is minimally nonouterplanar. We consider the following three cases.
Case 1: Suppose that $\Delta(T) \geq 4$, for every vertex $v \in T$. By Theorem 4.1, $PBL(T)$ is nonplanar, a contradiction.
Case 2: Suppose that $\Delta(T) \leq 2$, for every vertex $v \in T$. By Theorem 4.2, $PBL(T)$ is outerplanar, a contradiction.
Case 3: Suppose there exists a vertex of degree three in T. By necessity of Theorem 4.2, $i(PBL(T)) = 2$, again a contradiction. Hence by all the above cases, $PBL(T)$ is not minimally nonouterplanar. □

Theorem 4.6. *A pathos block line graph $PBL(T)$ of a tree T has crossing number one if and only if T is either $K_{1,4}$ or T contains exactly one vertex of degree four such that the degree of remaining vertices is at most 3.*

Proof. Suppose that $PBL(T)$ has crossing number one. We consider the following two cases.
Case 1: Assume that T is a star graph $K_{1,n}$ on ≥ 5 vertices. Suppose $T = K_{1,5}$. Then $L(T)$ is K_5. Since $L(T) \subseteq PL(T) \subseteq PBL(T)$, $\mathrm{cr}(PBL(T)) > 1$, a contradiction.

Case 2: Assume that T contains two vertices of degree four such that the degree of the remaining vertices is at most 3. Then $L(T)$ is a connected graph in which every block is either K_2 or K_3 of K_4 (two copies of K_4). The path number of T is at least three, and the edges joining these blocks from the corresponding pathos vertices are incident with at most two vertices of each block of $L(T)$ such that $\text{cr}(PL(T)) = 0$. Since every edge of T is a block, $PBL(T)$ is obtained by $L(T)$ by adjoining a pendant edge at each vertex of $L(T)$ such that $\text{cr}(PBL(T)) = 0$. Finally, the edges joining pathos vertices will increase the crossing number of $PBL(T)$ by two, i.e., $\text{cr}(PBL(T)) = 2$, a contradiction.

Conversely, suppose T is $K_{1,4}$. By necessity of Theorem 4.1, $\text{cr}(PBL(T)) = 1$. Suppose now that T contains one vertex of degree four and the degree of remaining vertices is at most 3. Then $L(T)$ is a connected graph in which every block is either K_2 or K_3 of K_4 (one copy of K_4). The path number of T is at least three, and the edges joining these blocks from the corresponding pathos vertices are incident with at most two vertices of each block of $L(T)$ such that $\text{cr}(PL(T)) = 0$. Since every edge of T is a block, $PBL(T)$ is obtained by $L(T)$ by adjoining a pendant edge at each vertex of $L(T)$ such that $\text{cr}(PBL(T)) = 0$. Finally, the edges joining pathos vertices will increase the crossing number of $PBL(T)$ by one, i.e., $\text{cr}(PBL(T)) = 1$. This completes the proof. □

We finally conclude with the following observation. It is known that since the pattern of pathos for a tree T is not unique, the corresponding pathos block line graph $PBL(T)$ is also not unique. Since the path number of a path of order n ($n \geq 2$) is exactly one, the pathos block line graph of a path is unique. One can easily observe that for different pattern of pathos for a star graph $K_{1,n}$ on $n \geq 2$ vertices, the corresponding pathos block line graphs are isomorphic. Therefore, the necessity of Theorem 4.1 holds for any pattern of pathos of $K_{1,4}$. Also, for a tree T with $\Delta(T) \leq 3$, for different pattern of pathos, we find the same conclusion as described in Case 2 of sufficiency of Theorem 4.1.

5. Open question:

(1) One can naturally extend these concepts to the directed graph version. What can one say about the properties of the directed version?

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[1] Department of Science and Humanities, PES Institute of Technology - South Campus, Bengaluru, Karnataka, India.
E-mail address: nageshhm@pes.edu