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THE LAPLACIAN WITH ROBIN BOUNDARY CONDITIONS INVOLVING SIGNED MEASURES

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Abstract. In this work we propose to study the general Robin boundary value problem involving signed smooth measures on an arbitrary domain Ω of $\mathbb{R}^d$. A Kato class of measures is defined to make the associated form $(\mathcal{E}_\mu, \mathcal{F}^\mu)$ a closed one. Moreover, the associated operator Δ_μ is a realization of the Laplacian on $L^2(\Omega)$. In particular, when $|\mu|$ is locally infinite everywhere on $\partial\Omega$, Δ_μ is the laplacian with Dirichlet boundary conditions. On the other hand, we will prove that the semigroup $(e^{-t\Delta_\mu})_{t\geq 0}$ is sandwiched between $(e^{-t\Delta_{\mu^+}})_{t\geq 0}$ and $(e^{-t\Delta_{-\mu^-}})_{t\geq 0}$ and we will see that the converse is also true.

1. Introduction and preliminaries

In this paper we consider Robin boundary value problems involving signed smooth measures on arbitrary domains. Our approach follows the ideas developed in [1, 4] to deal with arbitrary domains and [2, 3] to deal with signed measures.

Let Ω be an open set on $\mathbb{R}^d$ and define the Dirichlet form $(\mathcal{E}, H^1(\Omega))$ on $L^2(\Omega)$ by:

$$\mathcal{E}(u, v) = \int_\Omega \nabla u \nabla v dx \quad , \forall u, v \in H^1(\Omega)$$

The dirichlet form $(\mathcal{E}, H^1(\Omega))$ need not to be regular for non smooth Ω but there is a way to "regularize" it. For this purpose we set ourselves in the context of [7] where a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ was considered(X is a locally compact separable metric space, and m a positive Radon measure on X with supp$[m] = X$). In our special case we take as in [5] $X = \overline{\Omega}$ and the measure m on the $\sigma-$algebra $\mathcal{B}(X)$ is given by $m(A) = \lambda(A \cap \Omega)$ for all $A \in \mathcal{B}(X)$ with λ the Lebesgue measure, it follows that $L^2(\Omega) = L^2(X, \mathcal{B}(X), m)$, and we define a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\Omega)$ by:

$$\mathcal{E}(u, v) = \int_\Omega \nabla u \nabla v dx \quad , \forall u, v \in \mathcal{F}$$

where $\mathcal{F} = \widetilde{H}^1(\Omega)$ is the closure of $H^1(\Omega) \cap C_c(\overline{\Omega})$ in $H^1(\Omega)$. In the special case where Ω is bounded with Lipschitz boundary, we have $\widetilde{H}^1(\Omega) = H^1(\Omega)$.

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The capacity associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called the relative capacity (denoted by $\mathrm{Cap}_{\overline{\Omega}}$) and it was introduced in [4] as a special case of the capacity as defined in [7]. It seems to be an efficient tool to analyse the phenomena occurring on the boundary $\partial\Omega$ of Ω.

The relative capacity has the properties of a capacity as described in [7]. In particular, $\mathrm{Cap}_{\overline{\Omega}}$ is also an outer measure (but not a Borel measure) and a Choquet Capacity.

A statement depending on $x \in A \subset \overline{\Omega}$ is said to hold relatively quasi-everywhere (r.q.e.) on A, if there exist a relatively polar set $N \subset A$ such that the statement is true for every $x \in A \setminus N$.

Now we may consider functions in $\widetilde{H}^1(\Omega)$ as defined on $\overline{\Omega}$, and we call a function $u : \overline{\Omega} \to \mathbb{R}$ relatively quasi-continuous (r.q.c.) if for every $\epsilon > 0$ there exists a relatively open set $G \subset \overline{\Omega}$ such that $\mathrm{Cap}_{\overline{\Omega}}(G) < \epsilon$ and $u|_{\overline{\Omega}\setminus G}$ is continuous. It follows [13] that for each $u \in \widetilde{H}^1(\Omega)$ there exists a relatively quasi-continuous function $\widetilde{u} : \overline{\Omega} \to \mathbb{R}$ such that $\widetilde{u}(x) = u(x)$ $m-$a.e. This function is unique relatively quasi-everywhere. We call $\widetilde{u}$ the relatively quasi-continuous representative of u.

For more details, we refer the reader to [4, 13], where the relative capacity is investigated, as well as its relation to the classical one.

We denote by $S(\partial\Omega)$ the family of all smooth measures supported by $\partial\Omega$. The class $S(\partial\Omega)$ is quiet large and it contains all positive Radon measure on $\partial\Omega$ charging no set of zero relative capacity(see [2, 7] for more about smooth measure on locally compact topological spaces).

Consider now the diffusion process $M = (\Xi, X_t, \xi, P_x)$ on $\overline{\Omega}$ associated with the local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ and denote by $\mathcal{A}_c^+(\partial\Omega)$ the set of all positive continuous additive functionals associated with M and supported by $\partial\Omega$ in the sense of [7]. As the support of an additive functional is the quasi-support of its Revuz measure it is clear that the family of all equivalence classes of $\mathcal{A}_c^+(\partial\Omega)$ and the family $S(\partial\Omega)$ are in one to one correspondence under the Revuz correspondence. As a classical example of Revuz correspondence one can consider a bounded Lipschitz open domain and the surface measure σ. It is welle known [9] that $\frac{1}{2}\sigma$ is the Revuz measure of L_t, where L_t is the boundary local time of the reflecting Brownian motion M on $\overline{\Omega}$.

To deal with Robin boundary conditions we need to define Kato class of measures supported by $\partial\Omega$. Let $f \in \mathcal{B}(\partial\Omega)$ and set

$$\|f\|_{rq} = \inf_{\mathrm{Cap}_{\overline{\Omega}}(N)=0} \sup_{x\in\partial\Omega\setminus N} |f(x)|$$

where the index 'rq' reminds us "relatively quasi-everywhere". A smooth measure is said to be in Kato class of measures on $\partial\Omega$, and we denote $\mu \in S_K(\partial\Omega)$ if:

$$\lim_{t\searrow 0} \|E.A_t^\mu\|_{rq} = 0$$

where A_t^μ is the unique PCAF associated with μ. (Note that A^μ is also supported by $\partial\Omega$). $S_K(\partial\Omega)$ is defined in the spirit of [9, 10] and is a particular case of the Kato classe as defined in [3], and generalized in [12]. As a classical example, suppose that Ω is a bounded domain with Lipschitz boundary and let β be a Borel function on the $\partial\Omega$ and define the measure $\mu = \beta\sigma$, then the above definition is exactly the definition given in [9], which means that the situation in this paper is a generalization of the one in [9] and a particular case of the one in [2, 3].

In this paper we consider the form $(\mathcal{E}_\mu, \mathcal{F}^\mu)$ defined on $L^2(\Omega)$ for $\mu \in S(\partial\Omega) - S(\partial\Omega)$ by

$$\mathcal{E}_\mu(u,v) = \int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} uv d\mu, \ \forall u, v \in \mathcal{F}^\mu$$

where $\mathcal{F}^\mu = \widetilde{H}^1(\Omega) \cap L^2(\partial\Omega, |\mu|)$.

We will see that when $\mu \in S(\partial\Omega) - S_K(\partial\Omega)$, the Dirichlet form $(\mathcal{E}_\mu, \mathcal{F}^\mu)$ is closed and the associated selfadjoint operator Δ_μ is a realization of the Laplacian on $L^2(\partial\Omega)$. In the special case where $|\mu|$ is locally infinite on $\partial\Omega$, then Δ_μ is the Laplacian with Dirichlet boundary conditions. Moreover, $(\mathcal{E}_\mu, \mathcal{F}^\mu)$ is regular if and only if $|\mu|$ is a Radon measure. In section 3, we will prove a domination theorem. It says that the semigroup $(e^{-t\Delta_\mu})_{t\geq 0}$ is sandwiched between $(e^{-t\Delta_{\mu^+}})_{t\geq 0}$ and $(e^{-t\Delta_{-\mu^-}})_{t\geq 0}$. We will see that the converse is also true. That means that if one have a semigroup $(T(t))_{t\geq 0}$ sandwiched between $(e^{-t\Delta_{\mu^+}})_{t\geq 0}$ and $(e^{-t\Delta_{-\mu^-}})_{t\geq 0}$, then $T(t) = e^{-t\Delta_{\nu-\mu^-}}$, where ν is a Radon measure charging no set of zero relative capacity.

2. Signed measures case

Let $\mu = \mu^+ - \mu^-$ be a signed Broel measure on $\partial\Omega$. If $|\mu|$ is a smooth measure, then μ is called a signed smooth measure, and we shall write $\mu \in S(\partial\Omega) - S(\partial\Omega)$. It is evident that $\mu \in S(\partial\Omega) - S(\partial\Omega)$ if μ^+ and μ^- are both in $S(\partial\Omega)$. For $\mu \in S(\partial\Omega) - S(\partial\Omega)$ we shal set $A_t^\mu := A_t^{\mu^+} - A_t^{\mu^-}$. We shall call μ the Revuz measure of A_t^μ.

We define for $\mu \in S(\partial\Omega) - S(\partial\Omega)$

$$\mathcal{E}_\mu(u,v) = \int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} uv d\mu, \ \forall u, v \in \mathcal{F}^\mu$$

where $\mathcal{F}^\mu = \widetilde{H}^1(\Omega) \cap L^2(\partial\Omega, |\mu|)$

First, we begin with the following result

Theorem 2.1. *Let μ be a signed Borel measure on $\partial\Omega$ and assume that $|\mu|$ is locally infnite everywhere on $\partial\Omega$; i.e.,*

$$\forall x \in \partial\Omega \text{ and } r > 0 \quad |\mu|(B(x,r)) = \infty.$$

Then the form $\mathcal{E}_\mu$, which we denote by $\mathcal{E}_\infty$, is closed and is given by

$$\mathcal{E}_\infty(u,v) = \int_\Omega \nabla u \nabla v dx \quad , D(\mathcal{E}_\infty) = H_0^1(\Omega)$$

Proof. Let $u \in \mathcal{F}^\mu$ and $\widetilde{u}$ its relatively continous version, it follow from the fact that $\int_{\partial\Omega} |\widetilde{u}|^2 d|\mu| < \infty$ that $\widetilde{u} = 0$ r.q.e on $\partial\Omega$. Thus

$$\mathcal{F}^\infty := \mathcal{F}^\mu = \{u \in \widetilde{H}^1(\Omega) : \widetilde{u} = 0 \text{ r.q.e. on } \partial\Omega\}$$

One obtain that for all $u, v \in \mathcal{F}^\infty$,

$$\mathcal{E}_\infty(u,v) := \mathcal{E}_\mu(u,v) = \int_\Omega \nabla u \nabla v dx.$$

Following a characterization of $H_0^1(\Omega)$ with relative capacity [4, Theorem 2.3.], one conclude that $\mathcal{F}^\infty = H_0^1(\Omega)$. □

Exploiting the lines of the proof of Proposition 3.1. in [3] and by taking measures supported by the boundary of Ω, one can obtain the following result

Proposition 2.2. *Let $\mu \in S(\partial\Omega) - S_K(\partial\Omega)$. Then*

(1) *$(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is lower semibounded closed quadratic form,*
(2) *$(\mathcal{P}_t^\mu)_{t\geq 0}$ is the unique strongly continuous semigroup corresponding to $(\mathcal{E}^\mu, \mathcal{F}^\mu)$,*
(3) *$\mathcal{F}^\mu = \mathcal{F} \cap L^2(\partial\Omega, \mu^+)$.*

Now define the following subset of $\partial\Omega$,

$$\Gamma^\infty := \{x \in \partial\Omega : |\mu|\,(B(x,r)) = \infty, \forall r > 0\}$$

Note that Γ^∞ is a relatively closed subset of $\partial\Omega$. Since $\Gamma^{|\mu|} := \partial\Omega \setminus \Gamma^\infty$ is a locally compact metric space, it follows from [6, Theorem 2.18. p.48] that $|\mu|_{|\Gamma^{|\mu|}}$ is a regular Borel measure. Therefore $|\mu|$ is a Radon measure on $\Gamma^{|\mu|}$. As for Theorem 2.1, we have $\widetilde{u}_{|\Gamma_\infty} = 0$ r.q.e. for each function $u \in \mathcal{F}^{\mu^+}_{\Gamma^\infty}$, where

$$\mathcal{F}^{\mu^+}_{\Gamma^\infty} := \{u \in \mathcal{F}^{\mu^+} : \widetilde{u} = 0 \text{ r.q.e. on } \Gamma^\infty\}$$

It follows that Δ_μ, the operator associated with $(\mathcal{E}^\mu, \mathcal{F}^\mu)$,is the Laplacian with Dirichlet boundary conditions on Γ^∞, and with Robin boundary conditions on $\Gamma^{|\mu|}$.

Now define the subset $X_0 = \overline{\Omega} \setminus \Gamma^\infty$, then by Theorem 5.8. in [2] X_0 is a relatively open set satisfying $\Omega \subset X_0 \subset \overline{\Omega}$ such that the form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is regular on X_0. In particular, $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ is regular on $\overline{\Omega}$ if and only if $|\mu|$ is a Radon measure.

3. Domination results

In this section, we will prove a domination theorem. It says that the semigroup $(e^{-t\Delta_\mu})_{t\geq 0}$ is sandwiched between $(e^{-t\Delta_{\mu^+}})_{t\geq 0}$ and $(e^{-t\Delta_{-\mu^-}})_{t\geq 0}$. A very natural question arise: Is the converse also true? The answer is yes under a locality assymption.

The following theorem is proved using a result characterizing domination of positive semigroups due to Ouhabaz and contained in [8, Théorème 3.1.7.]

Theorem 3.1. *Let $\mu \in S(\partial\Omega) - S_K(\partial\Omega)$, and Δ_μ the closed operator on the $L^2(\Omega)$ associated with the closed form $(\mathcal{E}^\mu, \mathcal{F}^\mu)$. Then*

$$0 \leq e^{-t\Delta_{\mu^+}} \leq e^{-t\Delta_\mu} \leq e^{-t\Delta_{-\mu^-}}$$

for all $t \geq 0$ in the sens of positive operators.

Proof. Recall that the forms associated with $e^{-t\Delta_{\mu^+}}$, $e^{-t\Delta_{-\mu^-}}$ and $e^{-t\Delta_\mu}$ are given repectively by

$$\mathcal{E}_{\mu^+}(u,v) = \int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} uv d\mu^+,\ \forall u,v \in \mathcal{F}^{\mu^+} = \widetilde{H}^1(\Omega) \cap L^2(\partial\Omega, \mu^+)$$

$$\mathcal{E}_{\mu}(u,v) = \int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} uv d\mu,\ \forall u,v \in \mathcal{F}^{\mu} = \mathcal{F}^{\mu^+}$$

and

$$\mathcal{E}_{-\mu^-}(u,v) = \int_\Omega \nabla u \nabla v dx - \int_{\partial\Omega} uv d\mu^-,\ \forall u,v \in \mathcal{F}^{\mu^-} = \widetilde{H}^1(\Omega)$$

It is clear that $\mathcal{E}_\mu(u,v) \leq \mathcal{E}_{\mu^+}(u,v)$ for all $u,v \in \mathcal{F}^\mu_+ = \mathcal{F}^{\mu^+}_+$. Then by Theorem **??** we have $e^{-t\Delta_{\mu^+}} \leq e^{-t\Delta_\mu}$ for all $t \geq 0$ in the sense of positive operator. On the other hand, one have $\mathcal{E}_{-\mu^-}(u,v) \leq \mathcal{E}_\mu(u,v)$ for all $u,v \in \mathcal{F}^{\mu^+}_+$. It still to prove that $\mathcal{F}^{\mu^+}$ is an ideal of $\widetilde{H}^1(\Omega)$. Let $u \in \mathcal{F}^{\mu^+}$ and $v \in \widetilde{H}^1(\Omega)$ such that $0 \leq |v| \leq |u|$.We may assume that u and v are r.q.c., it follows that $0 \leq |v| \leq |u|$ r.q.e. and therefore μ^+-a.e.(since μ charges no set of zero relative capacity) . It follows that

$$\int_{\partial\Omega} |v|^2 d\mu^+ \leq \int_{\partial\Omega} |u|^2 d\mu^+ < \infty$$

and then $v \in L^2(\partial\Omega, \mu^+)$, which implies that $v \in \widetilde{H}^1(\Omega) \cap L^2(\partial\Omega, \mu^+)$.

□

Proposition 3.2. *Let $\mu \in S(\partial\Omega) - S_K(\partial\Omega)$. Then $(\mathcal{E}_\mu, \mathcal{F}^{\mu^+})$ is local.*

Proof. The proof is similar to Proposition 3.4.20. in [13]. □

The main result of this paper is the converse of Theorem 3.1. More precisely, if $(T(t))_{t\geq 0}$ is a C_0-semigroup on $L^2(\Omega)$ satisfaying

$$e^{-t\Delta_{\mu^+}} \leq T(t) \leq e^{-t\Delta_{-\mu^-}}$$

for all $t \geq 0$ in the sense of positive operators, under which conditions $T(t)$ is given by a signed measure ν on $\partial\Omega$? We suppose that $\Gamma^\mu = \partial\Omega$, we have then the following theorem:

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^d$ be an open set and $T = (T(t))_{t\geq 0}$ be a symmetric C_0-semigroup on $L^2(\Omega)$ satisfaying*

$$e^{-t\Delta_{\mu^+}} \leq T(t) \leq e^{-t\Delta_{-\mu^-}}$$

for all $t \geq 0$ in the sense of positive operators, where $\mu^+ \in S(\partial\Omega)$ and $\mu^- \in S_K(\partial\Omega)$. Let $(\mathcal{E}, D(\mathcal{E}))$ be the closed form on $L^2(\Omega)$ associated with T. Suppose in addition that $(\mathcal{E}, D(\mathcal{E}))$ is regular. Then the following assertions are equivalent to each other:

1. *$T(t) = e^{-t\Delta_{\nu-\mu^-}}$ for a unique positive Radon measure ν charging no set of zero relative capacity on $\partial\Omega$.*
2. *$(\mathcal{E}, D(\mathcal{E}))$ is local.*

Proof. (1)⇒(2) This part follows from Proposition 3.2.

(2)⇒(1) We have $D(\mathcal{E})$ is an ideal of $\widetilde{H}^1(\Omega)$ and for all $u, v \in D(\mathcal{E})_+$ we have,

$$\int_\Omega \nabla u \nabla v dx - \int_{\partial\Omega} uv d\mu^- \leq \mathcal{E}(u,v)$$

For $u, v \in D(\mathcal{E}) \cap C_c(\overline{\Omega})$ we let

$$b(u,v) = \mathcal{E}(u,v) + \int_{\partial\Omega} uv d\mu^- - \int_\Omega \nabla u \nabla v dx$$

Let $\{G^{\mathcal{E}}_\beta : \beta > 0\}$ be the resolvent of the operator associated with the closed form $(\mathcal{E}, D(\mathcal{E}))$ and $\{G^{-\mu^-}_\beta : \beta > 0\}$ be the resolvent of $\Delta_{-\mu^-}$. Let $\mathcal{E}^{(\beta)}$ and $\mathcal{E}^{(\beta)}_{-\mu^-}$ be the approximation forms of $\mathcal{E}$ and $\mathcal{E}_{-\mu^-}$ and let

$$\begin{aligned} b^{(\beta)}(u,v) &:= \mathcal{E}^{(\beta)}(u,v) - \mathcal{E}^{(\beta)}_{-\mu^-}(u,v) \\ &= \beta\left(u - \beta G^{\mathcal{E}}_\beta u, v\right) - \beta\left(u - \beta G^{-\mu^-}_\beta u, v\right) \\ &= \beta(\beta(G^{-\mu^-}_\beta - G^{\mathcal{E}}_\beta)u, v) \end{aligned} \tag{3.1}$$

Since by domination criterion, $b^{(\beta)}(u,v) \geq 0$ for all positive $u, v \in D(\mathcal{E}) \cap C_c(\overline{\Omega})$, we have that $\beta(G^{-\mu^-}_\beta - G^{\mathcal{E}}_\beta)$ is a positive symmetric operator on $L^2(\omega)$ and it then follows from [7](Lemma 1.4.1.) that there exists a unique positive Radon measure ν_β on $\overline{\Omega} \times \overline{\Omega}$ such that for all $u, v \in D(\mathcal{E}) \cap C_c(\overline{\Omega})$ we have

$$b^{(\beta)}(u,v) = \beta(\beta(G^{-\mu^-}_\beta - G^{\mathcal{E}}_\beta)u, v) = \beta \int_{\overline{\Omega}} u(x)v(y) d\nu_\beta$$

It is clear that $b^{(\beta)}(u,v) \to b(u,v)$ as $\beta \nearrow \infty$ for all $u, v \in D(\mathcal{E}) \cap C_c(\overline{\Omega})$. Since for each $\beta > 0$ and $u \in D(\mathcal{E}) \cap C_c(\overline{\Omega})$

$$b^{(\beta)}(u,v) \leq \mathcal{E}(u,v)$$

it follows that the sequence $(\beta\nu_\beta)$ is uniformly bounded on each compact subsets of $\overline{\Omega} \times \overline{\Omega}$ and hence a subsequence converges as $\beta_n \to \infty$ vaguely on $\overline{\Omega} \times \overline{\Omega}$ to a positive Radon measure ν. The form $(\mathcal{E}, D(\mathcal{E}))$ is regular and then ν is unique and therefore for all $u, v \in D(\mathcal{E}) \cap C_c(\overline{\Omega})$

$$b(u,v) = \int_{\overline{\Omega}} u(x)v(y) d\nu$$

Since $(\mathcal{E}, D(\mathcal{E}))$ and $(\mathcal{E}_{-\mu^-}, \widetilde{H}^1(\Omega))$ are local, it follows that $b(u,v) = 0$ for all $u, v \in D(\mathcal{E}) \cap C_c(\overline{\Omega})$ with $\operatorname{supp}[u] \cap \operatorname{supp}[v] = \emptyset$. This implies that $\operatorname{supp}[\nu] \subset \{(x,x) : x \in \overline{\Omega}\}$, and therefore

$$b(u,v) = \int_{\overline{\Omega}} u(x)v(x) d\nu$$

Since $b(u,v) = 0$ for all $u, v \in \mathcal{D}(\Omega) \subset D(\mathcal{E})$, we have $\operatorname{supp}[\nu] \subset \overline{\Omega} \setminus \Omega = \partial\Omega$ and thus

$$b(u,v) = \int_{\partial\Omega} u(x)v(x) d\nu$$

Consequently, for all $u, v \in D(\mathcal{E}) \cap C_c(\overline{\Omega})$ we have

$$\mathcal{E}(u,v) = \int_\Omega \nabla u \nabla v dx + \int_{\partial\Omega} uvd\nu - \int_{\partial\Omega} uvd\mu^-$$

The positive Radon measure ν charges no set of zero relative capacity. In fact, we have $\mathcal{E}(u,u) \leq \mathcal{E}_{\mu^+}(u,u)$ for all $u \in \mathcal{F}^{\mu^+} \subset D(\mathcal{E})$, which implies

$$\int_{\partial\Omega} |u|^2 d\nu \leq \int_{\partial\Omega} |u|^2 d|\mu|$$

With a particular choice of the function u, we have for all Borel subsets $\mathcal{O} \subset \partial\Omega$

$$\nu(\mathcal{O}) \leq |\mu|(\mathcal{O})$$

If $\mathcal{O}$ is of zero relative capacity then $\nu(\mathcal{O}) = 0$, thus ν also charges no set of zero relative capacity.

To finish , it still to prove that $(\mathcal{E}, D(\mathcal{E})) = (\mathcal{E}_{\nu-\mu^-}, \mathcal{F}^\nu)$.

It is clear that $\mathcal{F}^\nu$is a closed subspace of $D(\mathcal{E})$. We show that $D(\mathcal{E})$ is a subspace of $\mathcal{F}^\nu$. Let $u \in D(\mathcal{E})$. For $n \in \mathbb{N}$ we let $u_n = u \wedge n$. Then $u_n \in \widetilde{H}^1(\Omega)$ is relatively quasi-continous. Since $0 \leq u_n \leq n$ and $\nu(\partial\Omega) < \infty$, it follows that $u_n \in L^2(\partial\Omega, \nu)$ and therefore $u_n \in \mathcal{F}^\nu$. It is also clear that $u_n \to u$ $\widetilde{H}^1(\Omega)$ and thus after taking a subsequence if necessary, we may assume that $u_n \to u$ r.q.e. (see proposition 2.1. [5]). since ν charges no set of zero relative capacity, it follows that $u_n \to u$ $\nu-$a.e. on $\partial\Omega$. Finally, since $0 \leq u_n \leq k$, the Lebesgue Dominated Convergence Theorem implies that $u_n \to u$ in $L^2(\partial\Omega, \nu)$ and thus $u_n \to u$ in $\mathcal{F}^\nu$ and therefore $u \in \mathcal{F}^\nu$.

□

We can drop out the condition that $(\mathcal{E}_\mu, \mathcal{F}^\mu)$ is regular, but in this case we shoud add with the locality assymption the fact that $D(\mathcal{E}) \cap C_c(\overline{\Omega})$ is dense in $D(\mathcal{E})$. One can then follow the proof of Theorem 4.1 in [5] and the technics in Theorem 3.3 to prove the existence of such measure ν. The inconvenient in this case is that ν is not necessary unique.

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