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A GENERIC UNIQUENESS RESULT FOR AN INTERPOLATION PROBLEM FOR THE JOIN OF A TANGENTIAL VARIETY $\tau(X)$ AND SEVERAL COPIES OF THE VARIETY $X \subset \mathbb{P}^r$

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Abstract. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety non-singular in codimension 1. Let $\tau(X,2) = \tau(X) \subset \mathbb{P}^r$ be the tangential variety of X. Set $n := \dim(X)$. For any integer $b > 2$ let $\tau(X,b)$ be the join of $\tau(X)$ and $b-2$ copies of X. Mimicking the notion of weakly k-degenerate varieties introduced by Chiantini and Ciliberto we give a conditions on X assuring that a general $q \in \tau(X,b)$ is in the linear span of a unique scheme $Z \subset X$ with $Z = v \cup \{p_1, \dots, p_{b-2}\}$ with v connected of degree 2 and $v \subset X_{\mathrm{reg}}$.

1. Introduction and preliminaries

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate projective variety defined over an algebraically closed field with characteristic 0. Let $\tau(X) \subseteq \mathbb{P}^r$ be the tangential variety of X, i.e. the closure in $\mathbb{P}^r$ of the union of all linear spaces T_pX, $p \in X_{\mathrm{reg}}$. Set $\tau(X,2) := \tau(X)$. For any integer $b > 2$ let $\tau(X,b)$ be the join of $\tau(X)$ and $b-2$ copies of X. Set $n := \dim X$. Each $\tau(X,b)$ is an integral and non-degenerate variety of dimension at most $b(n+1)-2$. From now on we assume that $b \ge 2$ and that $b(n+1)-2 < r$. For a general $q \in \tau(X)$ there is $p \in X_{\mathrm{reg}}$ such that $q \in T_pX$ and hence there is a line L tangent to X at p and with $q \in L$. There is a degree 2 connected zero-dimensional scheme $v \subset X$ with $v_{\mathrm{red}} = \{p\}$ and $L = \langle v \rangle$, where $\langle\ \rangle$ denote the linear span. Let $\mathcal{Z}(X,b)$ be the set of all degree b schemes $v \cup \{p_1, \dots, p_{b-2}\}$ with v a degree 2 connected zero-dimensional scheme contained in X_{reg} and $p_1, \dots p_{b-2}$ distinct points of $X \setminus v_{\mathrm{red}}$. Call $\tau(X,b)'$ the set of all $q \in \tau(X,b)$ such that $q \in \langle Z \rangle$ for some $Z \in \mathcal{Z}(X,b)$. By the definition of join and the case $b=2$ just done the constructible set $\tau(X,b)'$ contains a non-empty open subset of $\tau(X,b)$. For each $q \in \tau(X,b)'$ let $\mathcal{Z}(X,b,q)$ denote the set of all $Z \in \mathcal{Z}(X,b)$ such that $q \in \langle Z \rangle$. The aim of this note is to give conditions assuring that $\sharp(\mathcal{Z}(X,q,b)) = 1$ for a general $q \in \tau(X,b)'$. We were inspired by the notion of weak defectivity introduced by L. Chiantini and C. Ciliberto in [3] and used for many important generic uniqueness theorems ([4, Corollary 2.7], [5]).

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Let $\mathcal{H}$ be the dual variety of $\tau(X,b)$, i.e. the closure in $\mathbb{P}^{r\vee}$ of the set of al hyperplanes tangent to $\tau(X,b)_{\mathrm{reg}}$. $\mathcal{H}$ is an irreducible variety and it is easy to check that $\dim \mathcal{H} \le r-b$ (Lemma 2.1). Fix a general $q \in \tau(X,b)$ and take $o \in \tau(X)_{\mathrm{reg}}$ and $(p_1,\dots,p_{b-2}) \in X_{\mathrm{reg}}^{b-2}$ such that q is a general element of $\langle T_o\tau(X) \cup T_{p_1}X \cup \cdots \cup \cdots \cup T_{p_{b-2}}X\rangle$. Since o is general in $\tau(X)$, there is $p \in X_{\mathrm{reg}}$ and a line $L \subset \mathbb{P}^r$ tangent to X at p and with $o \in L$. The contact locus of H and $\tau(X)_{\mathrm{reg}}$ contains $L \cap \tau(X)_{\mathrm{reg}}$ (we will say that the contact locus of H and $\tau(X)$ contain L) and the contact locus of H and X_{reg} contains $\{p_1,\dots,p_{b-2}\}$.

Definition 1.1. X is said to be *not weakly tangentially b-defective* if for a general $H \in \mathcal{H}$ the following conditions hold:

(1) $p_1,\dots,p_{b-2}$ are isolated points of the contact locus of H and X;
(2) L is an isolated line of the contact locus of H and $\tau(X)$.

If $\dim \tau(X,b) < b(n+1)-2$, then X is weakly tangentially b-defective (Lemma 2.2) and if $\dim \mathcal{H} < r-b$, then X is weakly tangentially b-defective (Lemma 2.4).

We prove the following result.

Theorem 1.2. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate projective variety, which is non-singular in codimension 1. Set $n := \dim(X)$. Fix an integer $b \ge 2$ and assume $r > b(n+1)-2$ and that X is not weakly tangentially b-defective. Fix a general $q \in \tau(X,b)$ and let $H \subset \mathbb{P}^r$ be a general hyperplane containing $T_q\tau(X,b)$. Then $\mathcal{Z}(X,b,q) = \{Z\}$ is a singleton and H is tangent to X_{reg} only at the points of Z_{red}.*

2. Proof of Theorem 1.2

Lemma 2.1. *We have $\dim \mathcal{H} \le r-b$.*

Proof. It is sufficient to prove that for a general $q \in \tau(X,b)$ there is a $(b-1)$-dimensional linear space $V \subseteq \tau(X,b)$ such that $V \cap \tau(X,b)_{\mathrm{reg}} \neq \emptyset$, $q \in V$ and $V \subseteq T_o\tau(X,b)$ for all $o \in V \cap \tau(X,b)_{\mathrm{reg}}$. Fix a general $q \in \tau(X,b)$. By Terracini's lemma for joins ([1, Corollary 1.10]) and a dimensional count there are $o \in \tau(X)_{\mathrm{reg}}$ and $(p_1,\dots,p_{b-2}) \in X_{\mathrm{reg}}^{b-2}$ such that $T_q\tau(X,b) = \langle T_o\tau(X) \cup T_{p_1}X \cup \cdots \cup T_{p_{b-2}}X\rangle$ and $(o,p_1,\dots,p_{b-2})$ is general in $\tau(X) \times X^{b-2}$. In particular $o \in \tau(X)'$. Since $o \in \tau(X)'$, there is a line $L \subset \tau(X)$ tangent to X_{reg}. $T_o\tau(X)$ is tangent to $\tau(X)_{\mathrm{reg}}$ at all points of $\tau(X)_{\mathrm{reg}} \cap L$. Hence by Terracini's lemma for joins $T_q\tau(X,b)$ is tangent to $\tau(X,b)_{\mathrm{reg}}$ at all points of $V := \langle L \cup \{p_1,\dots,p_{b-2}\}\rangle$. Since $r \ge b$, for a general $(o,p_1,\dots,p_{b-2})$ we have $\dim V = b-1$. □

Lemma 2.2. *If $\dim \tau(X,b) < b(n+1)-2$, then X is weakly tangentially b-defective.*

Proof. First assume $\dim \tau(X) < 2n$ and call o a general point of $\tau(X)$. We get that every irreducible component of the set of all lines tangent to X_{reg} and containing o is positive-dimensional and so X is weakly tangential 2-degenerate. If $b > 2$ a lemma of Terracini giving the tangent space of a join at its general points ([1, Corollary 1.10]) shows that condition (2) of Definition 1.1 is not satisfied. Now assume $b > 2$ and that $\dim \tau(X) = 2n$. By Terracini's lemma we see that least one of the conditions (1) or (2) of Definition 1.1 is not satisfied. □

Remark 2.3. Fix $q \in \tau(X,b)'$. Since the image of algebraic set by a morphism is constructible, $\mathcal{Z}(X,b,q)$ is constructible. Thus it makes sense to speak about the irreducible components of $\mathcal{Z}(X,b,q)$. We have $\dim \tau(X,b) = b(n+1)-2$ if and only if $\mathcal{Z}(X,b,q)$ is finite for a general $q \in \tau(X,b)$.

Lemma 2.4. *If X is not weakly tangentially b-defective, we have* $\dim \mathcal{H} = r-b$.

Proof. By Lemma 2.1 it is sufficient to prove that $\dim \mathcal{H} \ge r-b$. Since $\dim \tau(X,b) = b(n+1)-2$, for a general $q \in \tau(X,b)$ the set $\mathcal{Z}(X,b,q)$ is finite. Fix a general $q \in \tau(X,b)$ and take $Z \in \mathcal{Z}(X,b,q)$. Write $Z = v \cup \{p_1,\dots,p_{b-2}$ and set $L := \langle v\rangle$. Since X is not weakly tangentially b-defective, L is an isolated line in the contact locus of H with $\tau(X)$ and $\{p_1,\dots,p_{b-2}\}$ are isolated points of the contact locus of H and X. Since $\dim \tau(X,b) = b(n+1)-2$, a dimensional count and Terracini's lemma gives $\dim \mathcal{H} \ge r-b$. □

Remark 2.5. Let $X \subset \mathbb{P}^r$, $r \ge \dim X + 2$, be and integral and non-degenerate variety, which is non-singular in codimension 1. Let L be a general tangent line of X_{reg}. Call $v \subset L$ the degree 2 effective divisor with as its support the point of tangency of L and X. Since a general curve section of X is a smooth curve, [6, Theorem 3.1] gives $L \cap X = v$ as schemes.

We recall the following lemma ([2, Lemma 2.4] and Remark 2.5).

Lemma 2.6. *Fix an integer $b \ge 2$. Let $X \subset \mathbb{P}^r$, $r \ge 1+b+\dim X$, be an integral and non-degenerate variety non-singular in codimension 1. Fix a general $Z \in \mathcal{Z}(X,b)$. Then $\dim\langle Z\rangle = b-1$ and $\langle Z\rangle \cap X = Z$ (scheme-theoretic intersection).*

It is the use of Lemma 2.6, which force us to assume that X is non-singular in codimension 1.

Proof of Theorem 1.2: Fix a general $q \in \tau(X,b)$. Since $\dim \tau(X,b) = b(n+1)-2$ and q is general, $\mathcal{Z}(X,b,q)$ is finite (Remark 2.2). Assume the existence of $Z, W \in \mathcal{Z}(X,b,q)$ with $Z \ne W$. Since q is general, we may see Z and W as general elements of $\mathcal{Z}(X,b)$. Write $Z = v \cup \{p_1,\dots,p_{b-2}\}$, $W = w \cup \{o_1,\dots,o_{b-2}\}$ and set $\{p\} := v_{\mathrm{red}}$, $L := \langle v\rangle$, $\{o\} := w_{\mathrm{red}}$ and $R := \langle w\rangle$. Take $e \in L$ and $f \in R$ such that $q \in \langle\{e,p_1,\dots,p_{b-2}\}\rangle \cap \langle\{f,o_1,\dots,o_{b-2}\}\rangle$. Let $H \subset \mathbb{P}^r$ be a general hyperplane containing $T_e\tau(X) \cup T_{p_1}X \cup \dots \cup T_{p_{b-2}}X$; H exists, because $r > b(n+1)-2$. Note that $\mathcal{H} \subseteq \tau(X)^\vee$. Call $\mathcal{H}'$ the set $\mathcal{H}$ seen as a subset of the hyperplane sections of $\tau(X)$ (i.e. we are using the set-up of [3, §2] for $\tau(X)$, not for X). Let $N_{H\cap\tau(X),\mathcal{H}'}$ denote the normal sheaf of $H \cap \tau(X)$ in $\mathcal{H}' \subset \mathbb{P}^{r\vee}$.

Claim 1: $H^0(N_{H,\mathcal{H}'}) \subseteq H^0(\mathcal{I}_L(1))$.

Proof of Claim 1: Let $E \subset \mathbb{P}^r$ be a general hyperplane and let $E^\vee$ denote the dual $(r-1)$-dimensional projective space. Call $\mathcal{H}_E \subseteq E^\vee$ the family induced by $\mathcal{H}$. Since E is general and $\dim \mathcal{H} < r-1$, $\tau(X,b) \cap E$ is integral, $\dim \mathcal{H}_E = \dim \mathcal{H}$ and $\mathcal{H}_E = (\tau(X,b) \cap E)^\vee$. Since $\dim \mathcal{H} \le r-2$ and E is general, it is easy to check that $\dim \mathcal{H}_E = \dim H_E$. Since E is general, we have $H \ne E$ and so $H \cap E \in \mathcal{H}_E$. Let $N_{H\cap E,\mathcal{H}_E}$ denote the normal sheaf of $H \cap E$ in $\mathcal{H}_E$. By the infinitesimal Bertini's theorem ([3, Theorem 2.2]) applied to $\mathcal{H}_E$ and $\tau(X) \cap E$ we get $H^0(N_{H\cap E,\mathcal{H}_E}) \subseteq H^0(E, \mathcal{I}_{E\cap L}(1))$. Hence $H^0(N_{H,\mathcal{H}'}) \subseteq H^0(\mathcal{I}_{E\cap L}(1))$. Since this is true for a general hyperplane $E \subset \mathbb{P}^r$, we get $H^0(N_{H,\mathcal{H}'}) \subseteq H^0(\mathcal{I}_L(1))$. □

Call $\mathcal{H}''$ the variety $\mathcal{H}$ seen as a subset of the set of all hyperplane sections of X. Let $N_{H\cap X,\mathcal{H}''}$ denote the normal sheaf of $H\cap X$ in $\mathcal{H}''$.

Claim 2: $H^0(N_{H\cap X,\mathcal{H}''})\subseteq H^0(\mathcal{I}_{\{p_1,\dots,p_{b-2}\}}(1))$.

Proof of Claim 2: This is an obvious consequence of the infinitesimal Bertini's theorem ([3, Theorem 2.2]). □

By Claims 1 and 2 we have $H^0(N_{H,\mathcal{H}})\subseteq H^0(\mathcal{I}_Z(1))$. Since $\dim\langle Z\rangle = b-1$ and $\dim\mathcal{H}=r-b$, we get $H^0(N_{H,\mathcal{H}})=H^0(\mathcal{I}_Z(1))$. Since $\langle Z\rangle\cap X=Z$, we see that H cannot be tangent to $\tau(X)_{\mathrm{reg}}$ outside L and it cannot be tangent to X_{reg} outside Z_{red}.

Let M be another general hyperplane containing $T_q(\tau,X,b)$. We just proved that $H^0(N_{M,\mathcal{H}})=H^0(\mathcal{I}_Z(1))$. By Terracini's lemma we have $\langle T_e\tau(X)\cup T_{p_1}X\cup\cdots\cup T_{p_{b-2}}X\rangle = T_q\tau(X,b)=\langle T_f\tau(X)\cup T_{o_1}X\cup\cdots\cup T_{o_{b-2}}X\rangle$. Hence using W instead of Z we get $H^0(\mathcal{I}_W(1))=H^0(\mathcal{I}_Z(1))$. Since $\langle Z\rangle\cap H=Z$ (as schemes) by Lemma 2.6, we get $W=Z$, a contradiction.

The last assertion of Theorem 1.2 follows from the proof of the first one and the infinitesimal Bertini's theorem ([3, Theorem 2.2]). □

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