Gulf Journal of Mathematics
Vol 6, Issue 1 (2018) 24-45

STEPANOV-LIKE ALMOST AUTOMORPHIC MILD SOLUTIONS FOR SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. This work is concerned with the existence and uniqueness of Stepanov-like almost automorphic mild solutions for a class of semilinear fractional differential equations

$$\mathrm{D}_t^{\alpha} x(t) = Ax(t) + \mathrm{D}_t^{\alpha-1} F(t, x(t)), \quad t \in \mathbb{R},$$

where $1 < \alpha < 2$, A is a linear densely defined operator of sectorial type of $\omega < 0$ on a complex Banach space X and F is an appropriate function defined on phase space. The fractional derivative is understood in the Riemann-Liouville sense. The results obtained are utilized to study the existence and uniqueness of Stepanov-like almost automorphic mild solutions for a fractional relaxation-oscillation equation.

1. INTRODUCTION

In this paper, we are concerned with the existence and uniqueness of Stepanov-like almost automorphic mild solutions for the following semilinear fractional differential equations

$$\mathrm{D}_t^{\alpha} x(t) = Ax(t) + \mathrm{D}_t^{\alpha-1} F(t, x(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type of $\omega < 0$ on a complex Banach space X and $F : \mathbb{R} \times X \to X$ is an appropriate function. The fractional derivative is understood in the Riemann-Liouville sense.

In the earlier sixties, Bochner introduced the concept of almost automorphic function in his papers [1, 2, 3] in relation to some aspects of differential geometry. The notion of almost automorphic function was introduced to avoid some assumptions of uniform convergence that arise when using almost periodic function, it is an important generalization of the classical almost periodic function which is one of the most attractive topics in the qualitative theory of differential equations because of its significance and applications in physics, mathematical biology, control theory, and other related fields. From that time the theory of

Date: Received: Oct 26, 2017; Accepted: Jan 14, 2018.
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2010 *Mathematics Subject Classification.* Primary 34A08; Secondary 43A60, 26A33.
Key words and phrases. Stepanov-like almost automorphic function, Semilinear fractional differential equation, Fractional relaxation-oscillation equation, Solution operator, Fractional integral.

almost automorphic function has been studied by numerous authors, and it also becomes one of the most attractive topics in the qualitative theory of differential equations because of its significance and applications. Meanwhile, stimulated by the work [1, 2, 3], many interesting generalizations of the almost automorphic function have been introduced including asymptotic almost automorphy by N'Guérékata [4], p-almost automorphy by Diagana [5], pseudo almost automorphy by Liang et al.[6], and so on. These concepts are important and of interest because of their significance and applications in physics, mechanics, mathematical biology, and many others. Stepanov-like almost automorphy is also one of the most important generalizations. The concept of Stepanov-like almost automorphy was introduced by N'Guérékata and Pankov in [7], and subsequently was applied to study the existence of Stepanov-like almost automorphic solutions to some parabolic evolution equations. In connection with differential equations, the great importance from both the applied and theoretical points of view of the existence of periodic solutions is well known. However, either because models are only an approximation of reality or due to numerical errors, in practice it is impossible to verify whether a solution is exactly periodic. The concept of Stepanov-like almost automorphic function allows relaxing some assumptions to obtain solutions that have properties similar to those of a periodic function. Meanwhile, the applications of the new theory for these generalized functions, especially the Stepanov-like almost automorphic function, to various types of linear, semilinear as well as nonlinear differential equations were studied extensively (see, e.g., [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and references therein).

In recent years, fractional differential equations have gained considerable interest due to their applications in various fields of science such as physics, mechanics, chemistry engineering etc. Significant development has been made in ordinary and partial differential equations involving fractional derivatives, we only enumerate here the monographs of Kilbas et al. [24, 25], Diethelm [26], Hilfer [27], Podlubny [28] and the papers of Agarwal et al. [29, 30], Benchohra et al. [31, 32], El-Borai [33], Lakshmikantham et al. [34, 35, 36, 37], Mophou et al. [38, 39, 40, 41], N'Guérékata [42] and the reference therein.

Meanwhile due to their applications in fields of science where characteristics of anomalous diffusion are presented, type (1.1) equations are attracting increasing interest (cf. [43, 44, 45] and references therein). For example, anomalous diffusion in fractals [44] or in macroeconomics [46] has been recently well studied in the setting of fractional Cauchy problems like equation (1.1). While the study of almost automorphic mild solutions to equation (1.1) in the borderline case $\alpha = 1$ was well studied in [47, 48]. In [49] Cuevas and Lizama considered equation (1.1) when $1 < \alpha < 2$ and A is a linear operator of sectorial negative type on a complex Banach space, under suitable conditions on F, the authors proved the existence and uniqueness of an almost automorphic mild solution to equation (1.1). Cuevas et al. [50] and [51] study respectively the pseudo almost periodic and pseudo almost periodic of class infinity mild solutions to equation (1.1) assuming that $F : \mathbb{R} \times X \to X$ is a pseudo almost periodic and pseudo almost periodic of class infinity functions satisfying some appropriate conditions in $x \in X$. See also

[52, 53] where the S-asymptotically ω-periodic solutions to equation (1.1) are studied. Recently, Agarwal et al. [54] study the existence and uniqueness of a weighted pseudo-almost periodic mild solution to equation (1.1), and Cao et al. [55] study the existence of anti-periodic mild solutions to equation (1.1).

To the best of our knowledge, the existence of Stepanov-like almost automorphic mild solutions for the semilinear fractional differential equation (1.1) is a subject that has not been treated in the literature. Our purpose in this paper is to establish some results concerning the existence and uniqueness of Stepanov-like almost automorphic mild solutions for equations that can be modelled in the form (1.1). Upon making some appropriate assumptions, some sufficient conditions for the existence and uniqueness of Stepanov-like almost automorphic mild solutions to equation (1.1) are given. In particular, as application, and to illustrate our main results, we will examine some sufficient conditions for the existence and uniqueness of Stepanov-like almost automorphic mild solution to the fractional relaxation-oscillation equation given by

$$\partial_t^\alpha u(t,x) = \partial_x^2 u(t,x) - pu(t,x) + \partial_t^{\alpha-1} F(t, u(t,x)), \quad t \in \mathbb{R},\ x \in [0,\pi],$$

with boundary conditions

$$u(t,0) = u(t,\pi) = 0,\ t \in \mathbb{R},$$

where F satisfies some additional conditions.

The rest of this paper is organized as follows. In Section 2 we recall some concepts and prove some preliminary results. The section that follows contains the main results of this paper with four existence and uniqueness theorems. In the last section, we prove the existence and uniqueness of Stepanov-like almost automorphic mild solution for a fractional relaxation-oscillation equation as an example to illustrate our main results.

2. Preliminaries

We begin this section by giving some notations. Throughout this paper, let $p \in [1,\infty)$, denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the set of positive integers, the set of integers and the set of real numbers, respectively. Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces. Let $BC(\mathbb{R}, X)$ (respectively, $BC(\mathbb{R}\times Y, X)$) denote the space of bounded continuous functions with supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in \mathbb{R}\}$$

(respectively, the space of jointly bounded continuous functions). By $L(Y,X)$ we denote the Banach space of all bounded linear operators from Y to X. If $Y = X$, it is simply denoted by $L(X)$.

Now, let us recall some basic definitions and results on almost automorphic functions.

Definition 2.1. (Bochner) [3] A continuous function $f : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $\{s'_n\}_{n=1}^\infty$, one can extract a subsequence $\{s_n\}_{n=1}^\infty$ such that

$$g(t) = \lim_{n\to\infty} f(t + s_n),$$

is well defined in $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t),$$

for each $t \in \mathbb{R}$.

Denote by $AA(X)$ the set of all such functions.

Remark 2.2. The function g in Definition 2.1 is measurable, but not necessarily continuous. Moreover, if g is continuous, then f is uniformly continuous (cf., e.g., [56], Theorem 2.6). If the convergence in Definition 2.1 is uniform in $t \in \mathbb{R}$, then f is almost periodic. A classical example of almost automorphic function (not almost periodic) is (cf. [14, 15])

$$f(t) = \sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right), \quad t \in \mathbb{R}.$$

Definition 2.3. [3] A continuous function $f : \mathbb{R} \times Y \to X$ is said to be almost automorphic if $f(t,x)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$, where K is any bounded subset of Y.

Denote by $AA(\mathbb{R} \times Y, X)$ the set of all such functions.

Next, let us recall some definitions and basic results on Stepanov-like almost automorphic functions (for more details, see [7]).

Definition 2.4. The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, of a function $f : \mathbb{R} \to X$ is defined by

$$f^b(t,s) := f(t+s).$$

Definition 2.5. Let $p \in [1,\infty)$. The space $BS^p(X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to X$ such that

$$f^b \in L^\infty(\mathbb{R}, L^p([0,1],X)).$$

This is a Banach space with the norm

$$\|f\|_{S^p} := \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \mathrm{d}\tau\right)^{\frac{1}{p}}.$$

Definition 2.6. The space $AS^p(X)$ of Stepanov-like almost automorphic functions consists of all $f \in BS^p(X)$ such that

$$f^b \in AA(L^p([0,1],X)).$$

That is, a function $f \in L^p_{loc}(\mathbb{R}, X)$ is said to be Stepanov-like almost automorphic if its Bochner transform

$$f^b : \mathbb{R} \to L^p([0,1],X)$$

is almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n=1}^\infty$, there exist a subsequence $\{s_n\}_{n=1}^\infty$ and a function $g \in L^p_{loc}(\mathbb{R}, X)$ such that

$$\left(\int_0^1 \|f(t+s_n+s) - g(t+s)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0,$$

and

$$\left(\int_0^1 \|g(t-s_n+s)-f(t+s)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0,$$

as $n \to \infty$ for all $t \in \mathbb{R}$.

Definition 2.7. A function

$$f : \mathbb{R} \times Y \to X,\ (t,x) \to f(t,x)$$

with $f(\cdot,x) \in L^p_{loc}(\mathbb{R},X)$ for each $x \in Y$, is said to be Stepanov-like almost automorphic in $t \in \mathbb{R}$ uniformly for $x \in Y$, if $t \to f(t,x)$ is Stepanov-like almost automorphic for each $x \in Y$. That is, for every sequence of real numbers $\{s'_n\}_{n=1}^{\infty}$, there exist a subsequence $\{s_n\}_{n=1}^{\infty}$ and a function $g(\cdot,x) \in L^p_{loc}(\mathbb{R},X)$ such that

$$\left(\int_0^1 \|f(t+s_n+s,x)-g(t+s,x)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0,$$

and

$$\left(\int_0^1 \|g(t-s_n+s,x)-f(t+s,x)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0,$$

as $n \to \infty$ for all $t \in \mathbb{R}$ and $x \in Y$.

Denote by $AS^p(\mathbb{R}\times Y,X)$ the set of all such functions.

Remark 2.8. It is clear that, if $x : \mathbb{R} \to X$ is an almost automorphic function, then x is a Stepanov-like almost automorphic function, that is, $AA(X) \subset AS^p(X)$.

Now we give a lemma for Stepanov-like almost automorphic functions.

Lemma 2.9. *Let $\{x_n(t)\}_{n\in\mathbb{N}}$ be a sequence of Stepanov-like almost automorphic functions such that*

$$\int_0^1 \|x_n(t+s)-x(t+s)\|^p\, ds \to 0, \tag{2.1}$$

as $n \to \infty$ for each $t \in \mathbb{R}$, then $x \in AS^p(X)$.

Proof. For any $i \in \mathbb{N}$ fixed, since $x_i(t) \in AS^p(X)$, then for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $y_i \in L^p_{loc}(\mathbb{R},X)$ such that

$$\left(\int_0^1 \|x_i(t+s_n+s)-y_i(t+s)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0, \tag{2.2}$$

and

$$\left(\int_0^1 \|y_i(t-s_n+s)-x_i(t+s)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0, \tag{2.3}$$

as $n \to \infty$ for all $t \in \mathbb{R}$.

On the other hand, from (2.1), one can easily deduce that $\{x_n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\|\cdot\|_{S^p}$. Observe that, for each $t \in \mathbb{R}$, the sequence y_i is also a Cauchy sequence in $L^p_{loc}(\mathbb{R},X)$. Indeed, if we write

$$y_i(t)-y_j(t) = y_i(t)-x_i(t+s_n)+x_i(t+s_n)-x_j(t+s_n)+x_j(t+s_n)-y_j(t),$$

then for a sufficiently large n, one gets

$$\begin{aligned}
&\Big(\int_0^1 \|y_i(t+s)-y_j(t+s)\|^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
\leq&\Big(\int_0^1 \Big(\|y_i(t+s)-x_i(t+s+s_n)\|+\|x_i(t+s+s_n)-x_j(t+s+s_n)\|\\
&+\|x_j(t+s+s_n)-y_j(t+s)\|\Big)^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
\leq&3\Big(\int_0^1 \Big(\|y_i(t+s)-x_i(t+s+s_n)\|^p+\|x_i(t+s+s_n)-x_j(t+s+s_n)\|^p\\
&+\|x_j(t+s+s_n)-y_j(t+s)\|^p\Big)\mathrm{d}s\Big)^{\frac{1}{p}}.
\end{aligned}$$

By (2.1), (2.2) and (2.3), the sequence of y_i is a Cauchy sequence in $L^p_{loc}(\mathbb{R},X)$. Using the completeness of $L^p_{loc}(\mathbb{R},X)$, we denote by $y(t)$ the pointwise limit of $y_i(t)$.

Now let us prove that $x(t)\in AS^p(X)$.

Note that the inequality below holds for any index i and any $t\in\mathbb{R}$,

$$\begin{aligned}
&\Big(\int_0^1 \|x(t+s+s_n)-y(t+s)\|^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
\leq&\Big(\int_0^1 \Big(\|x(t+s+s_n)-x_i(t+s+s_n)\|\\
&+\|x_i(t+s+s_n)-y_i(t+s)\|+\|y_i(t+s)-y(t+s)\|\Big)^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
\leq&3\Big(\int_0^1 \Big(\|x(t+s+s_n)-x_i(t+s+s_n)\|^p+\|x_i(t+s+s_n)-y_i(t+s)\|^p\\
&+\|y_i(t+s)-y(t+s)\|^p\Big)\mathrm{d}s\Big)^{\frac{1}{p}}.
\end{aligned}$$

So, from (2.1) and the fact that $y(t)$ is the pointwise limit of $y_i(t)$, for any sufficiently small $\varepsilon>0$ there exists a sufficiently large i, such that for each $t\in\mathbb{R}$,

$$\int_0^1 \|x_i(t+s+s_n)-y_i(t+s)\|^p \mathrm{d}s<\frac{\varepsilon^p}{3^{p+1}},$$
$$\int_0^1 \|x(t+s+s_n)-x_i(t+s+s_n)\|^p \mathrm{d}s<\frac{\varepsilon^p}{3^{p+1}}.$$

Now for this sufficiently large i, from (2.2) and (2.3), there exists a sufficient N such that for any $n>N$ one has

$$\int_0^1 \|y_i(t+s)-y(t+s)\|^p \mathrm{d}s<\frac{\varepsilon^p}{3^{p+1}}.$$

Thus

$$\Big(\int_0^1 \|x(t+s+s_n)-y(t+s)\|^p \mathrm{d}s\Big)^{\frac{1}{p}}<\varepsilon,\quad \text{for } n>N,$$

which implies

$$\left(\int_0^1 \|x(t+s+s_n)-y(t+s)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0,$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

One can use the same steps to prove that

$$\left(\int_0^1 \|y(t+s-s_n)-x(t+s)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0,$$

as $n \to \infty$ pointwise on $\mathbb{R}$. That is, $x(t) \in AS^p(X)$. The proof is finished. □

We need some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.10. [24]. The fractional integral of order $\alpha > 0$ with the lower limit t_0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s)\mathrm{d}s, \quad t > t_0, \quad \alpha > 0,$$

provided the right-hand side is pointwise defined on $[t_0, \infty)$, where Γ is the Gamma function.

Definition 2.11. [24]. Riemann-Liouville derivative of order $\alpha > 0$ with the lower limit t_0 for a function $f : [t_0, \infty) \to \mathbb{R}$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^t (t-s)^{-\alpha} f(s)\mathrm{d}s, \quad t > t_0, \quad n-1 < \alpha < n.$$

The first and maybe the most important property of Riemann-Liouville fractional derivative is that for $t > t_0$ and $\alpha > 0$, one has

$$D_t^\alpha (I^\alpha f(t)) = f(t)$$

which means that Riemann-Liouville fractional differentiation operator is a left inverse to the Riemann-Liouville fractional integration operator of the same order α.

In the following, we give the definitions of sectorial linear operators and their associated solution operators.

Recall that a closed and linear operator A is said to be sectorial of type ω and angle θ if there exist $0 < \theta < \frac{\pi}{2}$, $M > 0$ and $\omega \in \mathbb{R}$ such that its resolvent exists outside the sector

$$\omega + S_\theta := \Big\{\omega + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\Big\},$$

and

$$\|(\lambda - A)^{-1}\| \le \frac{M}{|\lambda - \omega|}, \quad \lambda \notin \omega + S_\theta.$$

Sectorial operators are well studied in the literature, usually for the case $\omega = 0$. For a recent reference including several examples and properties we refer the reader to [57]. Note that an operator A is sectorial of type ω if and only if $\omega I - A$ is sectorial of type 0.

Definition 2.12. [58] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A is the generator of a solution operator if there are $\omega \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to L(X)$ such that

$$\Big\{\lambda^\alpha : \mathrm{Re}\lambda > \omega\Big\} \subseteq \rho(A),$$

and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\mathrm{d}t, \quad \mathrm{Re}\lambda > \omega, \quad x \in X.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

We note that if A is sectorial of type ω with $0 \leq \theta \leq \pi(1-\frac{\alpha}{2})$, then A is the generator of a solution operator given by

$$S_\alpha(t) := \frac{1}{2\pi i}\int_\gamma e^{-\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}\mathrm{d}\lambda, \tag{2.4}$$

where γ is a suitable path lying outside the sector $\omega + \Sigma_\theta$ (cf. [57]).

Very recently, Cuesta in [57](Theorem 1) has proved that if A is a sectorial operator of type $\omega < 0$ for some $M > 0$ and $0 \leq \theta < \pi(1-\frac{\alpha}{2})$, then there exists $C > 0$ such that

$$\|S_\alpha(t)\|_{L(X)} \leq \frac{CM}{1+|\omega|t^\alpha}, \tag{2.5}$$

for $t \geq 0$. In the border case $\alpha = 1$, this is analogous to saying that A is the generator of a exponentially stable C_0-semigroup. The main difference is that in the case $\alpha > 1$ the solution family $S_\alpha(t)$ decays like $t^{-\alpha}$. Cuesta's result proves that $S_\alpha(t)$ is, in fact, integrable.

Now we give other two lemmas.

Lemma 2.13. *Let $f : \mathbb{R}\times X \to X$, $(t,x) \to f(t,x)$ be Stepanov-like almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in X$, and assume that f satisfies the following Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|,$$

for all $x, y \in X$ and $t \in \mathbb{R}$, where $L > 0$ is a constant which is independent of t. Then for any Stepanov-like almost automorphic function $x : \mathbb{R} \to X$, the function $F : \mathbb{R} \to X$ given by $F(t) = f(t, x(t))$ is Stepanov-like almost automorphic.

Proof. Let $\{s'_m\}_{m\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $x \in AS^p(X)$, there exist a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ and a function $\tilde{x} \in AS^p(X)$ such that

$$\left(\int_0^1 \|x(t+s_n+s) - \tilde{x}(t+s)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0, \tag{2.6}$$

and

$$\left(\int_0^1 \|\tilde{x}(t-s_n+s) - x(t+s)\|^p \mathrm{d}s\right)^{\frac{1}{p}} \to 0, \tag{2.7}$$

as $m \to \infty$ pointwise on $\mathbb{R}$. On the other hand, since $(t,x) \to f(t,x)$ is Stepanov-like almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in X$, one can extract a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ (for convenience, we also denote it by $\{s_m\}_{m\in\mathbb{N}}$) and a function $\tilde{f}(\cdot,x) \in L^p_{loc}(\mathbb{R},X)$ such that

$$\Big(\int_0^1 \|f(t+s_n+s,x)-\tilde{f}(t+s,x)\|^p \mathrm{d}s\Big)^{\frac{1}{p}} \to 0, \tag{2.8}$$

and

$$\Big(\int_0^1 \|\tilde{f}(t-s_n+s,x)-f(t+s,x)\|^p \mathrm{d}s\Big)^{\frac{1}{p}} \to 0, \tag{2.9}$$

as $m \to \infty$ pointwise on $\mathbb{R}$ for each $x \in X$.

Now, let us consider the function $\tilde{F}:\mathbb{R}\to X$ defined by

$$\tilde{F}(t) := \tilde{f}(t,\tilde{x}(t)),\ t \in \mathbb{R}.$$

Note that

$$\begin{aligned}
&F(t+s+s_m)-\tilde{F}(t+s)\\
=&f(t+s+s_m,x(t+s+s_m))-f(t+s+s_m,\tilde{x}(t+s))\\
&+f(t+s+s_m,\tilde{x}(t+s))-\tilde{f}(t+s,\tilde{x}(t+s)).
\end{aligned}$$

So, one obtains

$$\begin{aligned}
&\Big(\int_0^1 \|F(t+s+s_m)-\tilde{F}(t+s)\|^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
\le&\Big(\int_0^1 \Big(\|f(t+s+s_m,x(t+s+s_m))-f(t+s+s_m,\tilde{x}(t+s))\|\\
&+\|f(t+s+s_m,\tilde{x}(t+s))-\tilde{f}(t+s,\tilde{x}(t+s))\|\Big)^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
\le&2\Big(\int_0^1 \Big(\|f(t+s+s_m,x(t+s+s_m))-f(t+s+s_m,\tilde{x}(t+s))\|^p\\
&+\|f(t+s+s_m,\tilde{x}(t+s))-\tilde{f}(t+s,\tilde{x}(t+s))\|^p\Big)\mathrm{d}s\Big)^{\frac{1}{p}}\\
\le&2\Big(\int_0^1 \Big(L^p\|x(t+s+s_m)-\tilde{x}(t+s)\|^p\\
&+\|f(t+s+s_m,\tilde{x}(t+s))-\tilde{f}(t+s,\tilde{x}(t+s))\|^p\Big)\mathrm{d}s\Big)^{\frac{1}{p}}.
\end{aligned}$$

One can deduce from (2.6), (2.7), (2.8) and (2.9) that

$$\Big(\int_0^1 \|F(t+s+s_m)-\tilde{F}(t+s)\|^p \mathrm{d}s\Big)^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Similarly one can prove that

$$\Big(\int_0^1 \|\tilde{F}(t+s-s_m)-F(t+s)\|^p \mathrm{d}s\Big)^{\frac{1}{p}} \to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. The proof is finished. $\square$

Lemma 2.14. *Assume that (2.5) is true. Given a function $F(t) \in AS^p(X)$. Let*

$$[\Phi F](t) := \int_{-\infty}^{t} S_\alpha(t-s)F(s)ds.$$

Then $[\Phi F](t)$ is Stepanov-like almost automorphic.

Proof. Firstly, note that

$$\int_0^\infty \frac{1}{1+|\omega|s^\alpha}\mathrm{d}s = \frac{\omega^{-\frac{1}{\alpha}}\pi}{\alpha\sin\frac{\pi}{\alpha}} \quad \text{for} \quad 1<\alpha<2. \tag{2.10}$$

By condition (2.5), one has

$$\begin{aligned}
&\left(\int_t^{t+1}\left\|\int_{-\infty}^{\sigma} S_\alpha(\sigma-s)F(s)\mathrm{d}s\right\|^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
=&\left(\int_t^{t+1}\left\|\int_0^\infty S_\alpha(\tau)F(\sigma-\tau)\mathrm{d}\tau\right\|^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
\le& CM\left(\int_t^{t+1}\int_0^\infty\left(\frac{1}{1+|\omega|\tau^\alpha}\right)^p\left\|F(\sigma-\tau)\right\|^p\mathrm{d}\tau\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
\le& CM\|F\|_{S^p}\left(\int_0^\infty\left(\frac{1}{1+|\omega|\tau^\alpha}\right)^p\mathrm{d}\tau\right)^{\frac{1}{p}}\\
\le& CM\|F\|_{S^p}\left(\int_0^\infty\frac{1}{1+|\omega|\tau^\alpha}\mathrm{d}\tau\right)^{\frac{1}{p}}\\
=& CM\|F\|_{S^p}\left[\frac{\omega^{-\frac{1}{\alpha}}\pi}{\alpha\sin\frac{\pi}{\alpha}}\right]^{\frac{1}{p}}.
\end{aligned}$$

Thus, Φ is well defined and ΦF is bounded. On the other hand, for any $t,\ h \in \mathbb{R}$,

$$\begin{aligned}
&\left(\int_t^{t+1}\left\|[\Phi F](\sigma+h)-[\Phi F](\sigma)\right\|^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
=&\left(\int_t^{t+1}\left\|\int_{-\infty}^{\sigma+h} S_\alpha(\sigma+h-s)F(s)\mathrm{d}s-\int_{-\infty}^{\sigma} S_\alpha(\sigma-s)F(s)\mathrm{d}s\right\|^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
=&\left(\int_t^{t+1}\left\|\int_{-\infty}^{\sigma} S_\alpha(\sigma-s)[F(s+h)-F(s)]\mathrm{d}s\right\|^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
=&\left(\int_t^{t+1}\left\|\int_0^\infty S_\alpha(\tau)[F(\sigma-\tau+h)-F(\sigma-\tau)]\mathrm{d}\tau\right\|^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
\le& CM\|F(t+h)-F(t)\|_{S^p}\left(\int_0^\infty\left[\frac{1}{1+|\omega|\tau^\alpha}\right]^p\mathrm{d}\tau\right)^{\frac{1}{p}}\\
\le& CM\|F(t+h)-F(t)\|_{S^p}\left(\int_0^\infty\frac{1}{1+|\omega|\tau^\alpha}\mathrm{d}\tau\right)^{\frac{1}{p}}
\end{aligned}$$

$$=CM\left[\frac{\omega^{-\frac{1}{\alpha}}\pi}{\alpha\sin\frac{\pi}{\alpha}}\right]^{\frac{1}{p}}\|F(t+h)-F(t)\|_{S^p},$$

which yields that ΦF is continuous.

Let $\{s'_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $F\in AS^p(X)$, there exist a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ and a function $\tilde{F}\in AS^p(X)$ such that

$$\left(\int_0^1\|F(t+s_m+s)-\tilde{F}(t+s)\|^p\mathrm{d}s\right)^{\frac{1}{p}}\to 0, \tag{2.11}$$

and

$$\left(\int_0^1\|\tilde{F}(t-s_m+s)-F(t+s)\|^p\mathrm{d}s\right)^{\frac{1}{p}}\to 0, \tag{2.12}$$

as $m\to\infty$ pointwise on $\mathbb{R}$ for each $x\in X$. Let

$$[\Phi\tilde{F}](t):=\int_{-\infty}^t S_\alpha(t-s)\tilde{F}(s)\mathrm{d}s.$$

Thus

$$\begin{aligned}
&\left(\int_0^1\|[\Phi F](t+s+s_m)-[\Phi\tilde{F}](t+s)\|^p\mathrm{d}s\right)^{\frac{1}{p}}\\
=&\left(\int_0^1\left\|\int_{-\infty}^t S_\alpha(\sigma)F(t+s+s_m-\sigma)\mathrm{d}\sigma-\int_{-\infty}^t S_\alpha(\sigma)\tilde{F}(t+s-\sigma)\mathrm{d}\sigma\right\|^p\mathrm{d}s\right)^{\frac{1}{p}}\\
=&\left(\int_0^1\left\|\int_{-\infty}^t S_\alpha(\sigma)\Big[F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\Big]\mathrm{d}\sigma\right\|^p\mathrm{d}s\right)^{\frac{1}{p}}\\
\le&\left(\int_0^1\left(\int_{-\infty}^t\|S_\alpha(\sigma)\|\|F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\|\mathrm{d}\sigma\right)^p\mathrm{d}s\right)^{\frac{1}{p}}\\
\le&\left(\int_0^1\int_{-\infty}^t\|S_\alpha(\sigma)\|^p\|F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\|^p\mathrm{d}\sigma\mathrm{d}s\right)^{\frac{1}{p}}\\
\le&\left(\int_0^1\int_0^\infty\left[\frac{CM}{1+|\omega|\sigma^\alpha}\right]^p\|F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\|^p\mathrm{d}\sigma\mathrm{d}s\right)^{\frac{1}{p}}\\
=&\left(\int_0^\infty\left[\frac{CM}{1+|\omega|\sigma^\alpha}\right]^p\int_0^1\|F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\|^p\mathrm{d}s\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
\le&\left(\int_0^\infty\left[\frac{C^pM^p}{1+|\omega|\sigma^\alpha}\right]\int_0^1\|F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\|^p\mathrm{d}s\mathrm{d}\sigma\right)^{\frac{1}{p}}.
\end{aligned}$$

From (2.10), (2.11) and (2.12), obviously, the last inequality goes to 0 as $m\to\infty$ pointwise on $\mathbb{R}$. Similarly one can prove that

$$\left(\int_0^1\|[\Phi\tilde{F}](t+s-s_m)-[\Phi F](t+s)\|^p\mathrm{d}s\right)^{\frac{1}{p}}\to 0,$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Thus we conclude that $[\Phi F] \in AS^p(X)$. The proof is now complete. □

3. Stepanov-like almost automorphic mild solutions

Let $1 < \alpha < 2$. We first consider the linear version for equation (1.1), that is

$$\mathrm{D}_t^\alpha x(t) = Ax(t) + \mathrm{D}_t^{\alpha-1} F(t), \quad t \in \mathbb{R}. \tag{3.1}$$

Observe that equation (3.1) can be viewed as the limiting equation for the equation

$$y'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Ay(s) + F(t), \quad t \geq 0, \ y(0) = x_0 \in X, \tag{3.2}$$

in the sense that the solutions $x(t)$ of equation (3.1) and $y(t)$ of equation (3.2) are asymptotic to each other as $t \to \infty$. In fact, if we assume that A is sectorial of type ω with $0 \leq \theta < \pi(1 - \frac{\alpha}{2})$, then equation (3.2) is well posed (cf.[57]) and the variation of parameters formula allows us to write the solution of equation (3.2) as

$$y(t) = S_\alpha(t)x_0 + \int_0^t S_\alpha(t-s)F(s)\mathrm{d}s, \ t \geq 0,$$

where the family of operators $S_\alpha(t)$ is given by (2.4). On the other hand, if $S_\alpha(t)$ is integrable, then the solution of equation (3.1) is given by

$$x(t) = \int_{-\infty}^t S_\alpha(t-s)F(s)\mathrm{d}s. \tag{3.3}$$

Hence

$$y(t) - x(t) = S_\alpha(t)x_0 - \int_t^\infty S_\alpha(s)F(t-s)\mathrm{d}s,$$

which shows that

$$y(t) - x(t) \to 0 \text{ as } t \to \infty$$

whenever $F \in L^p(\mathbb{R}^+, X)$ for some $p \in [1, +\infty)$.

From Cuesta's result, it follows that $S_\alpha(t)$ is integrable. Thus the above considerations motivate the following definition.

Definition 3.1. A function $x : \mathbb{R} \to X$ is said to be a mild solution to equation (3.1) if the function $s \to S_\alpha(t-s)F(s)$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and

$$x(t) = \int_{-\infty}^t S_\alpha(t-\sigma)F(\sigma)\mathrm{d}\sigma.$$

Similarly, a function $x : \mathbb{R} \to X$ is said to be a mild solution to equation (1.1) if the function $s \to S_\alpha(t-s)F(s, x(s))$ is integrable on $(-\infty, t)$ for each $t \in \mathbb{R}$ and

$$x(t) = \int_{-\infty}^t S_\alpha(t-\sigma)F(\sigma, x(\sigma))\mathrm{d}\sigma.$$

To study the existence and uniqueness of Stepanov-like almost automorphic mild solutions to equation (1.1), we first consider the existence and uniqueness of Stepanov-like almost automorphic mild solutions to the linear fractional differential equation (3.1) with $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type of $\omega < 0$ on a complex Banach space X and $F : \mathbb{R} \to X$ is a Stepanov-like almost automorphic function. The fractional derivative is understood in the Riemann-Liouville sense.

The following are the main result for the linear fractional differential equation (3.1).

Theorem 3.2. *Assume that A is sectorial of type $\omega < 0$. Then equation (3.1) admits a unique Stepanov-like almost automorphic mild solution.*

Proof. Let us first prove the uniqueness. Assume that $x : \mathbb{R} \to X$ is a bounded function and satisfies the homogeneous equation

$$\mathrm{D}_t^{\alpha} x(t) = Ax(t), \quad t \in \mathbb{R}. \tag{3.4}$$

Then

$$x(t) = S_\alpha(t-s)x(s), \text{ for any } t \geq s.$$

Thus

$$\|x(t)\| \leq \frac{CMK}{1 + |\omega|(t-s)^\alpha}$$

with $\|x(s)\| \leq K$ for $s \in \mathbb{R}$. Take a sequence of real numbers $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n \to -\infty$ as $n \to \infty$. For any $t \in \mathbb{R}$ fixed, one can find a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}} \subset \{s_n\}_{n\in\mathbb{N}}$ such that $s_{n_k} < t$ for all $k = 1, 2, \cdots$. By letting $k \to \infty$, we get $x(t) = 0$.

Now, if $x_1,\ x_2 : \mathbb{R} \to X$ are bounded solutions to equation (3.1), then $x = x_1 - x_2$ is a bounded solution to equation (3.4). In view of the above, $x = x_1 - x_2 = 0$, that is, $x_1 = x_2$.

Now let us investigate the existence. Consider for each $n = 1, 2, \cdots$, the integrals

$$x_n(t) = \int_{n-1}^{n} S_\alpha(\sigma)F(t-\sigma)\mathrm{d}\sigma,$$

for each $t \in \mathbb{R}$. Firstly, by using Hölder inequality, one gets

$$\begin{aligned}
\|x_n(t)\|_{S^p} &= \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|x_n(\tau)\|^p \mathrm{d}\tau \right)^{\frac{1}{p}} \\
&= \sup_{t\in\mathbb{R}} \left(\int_0^1 \|x_n(t+s)\|^p \mathrm{d}s \right)^{\frac{1}{p}} = \sup_{t\in\mathbb{R}} \left(\int_0^1 \|x_n(t+s)\|^p \mathrm{d}s \right)^{\frac{1}{p}} \\
&= \sup_{t\in\mathbb{R}} \left(\int_0^1 \left\| \int_{n-1}^{n} S_\alpha(\sigma)F(t+s-\sigma)\mathrm{d}\sigma \right\|^p \mathrm{d}s \right)^{\frac{1}{p}} \\
&\leq \sup_{t\in\mathbb{R}} \left(\int_0^1 \left(\int_{n-1}^{n} \|S_\alpha(\sigma)\| \|F(t+s-\sigma)\| \mathrm{d}\sigma \right)^p \mathrm{d}s \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{t\in\mathbb{R}}\left(\int_0^1\int_{n-1}^n \|S_\alpha(\sigma)\|^p\|F(t+s-\sigma)\|^p\mathrm{d}\sigma\mathrm{d}s\right)^{\frac{1}{p}}\\
&\leq \sup_{t\in\mathbb{R}}\left(\int_0^1\int_{n-1}^n \left[\frac{CM}{1+|\omega|\sigma^\alpha}\right]^p\|F(t+s-\sigma)\|^p\mathrm{d}\sigma\mathrm{d}s\right)^{\frac{1}{p}}\\
&= \sup_{t\in\mathbb{R}}\left(\int_{n-1}^n \left[\frac{CM}{1+|\omega|\sigma^\alpha}\right]^p\int_0^1\|F(t+s-\sigma)\|^p\mathrm{d}s\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
&= \sup_{t\in\mathbb{R}}\left(\int_{n-1}^n \left[\frac{CM}{1+|\omega|\sigma^\alpha}\right]^p\|F\|_{S^p}^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
&= CM\|F\|_{S^p}\left(\int_{n-1}^n \left[\frac{1}{1+|\omega|\sigma^\alpha}\right]^p\mathrm{d}\sigma\right)^{\frac{1}{p}}\\
&\leq CM\|F\|_{S^p}\frac{1}{1+|\omega|(n-1)^\alpha}\\
&\leq \frac{CM\|F\|_{S^p}}{|\omega|}\frac{1}{(n-1)^\alpha}.
\end{aligned}$$

From $1<\alpha<2$, it follows that

$$\frac{CM\|F\|_{S^p}^p}{|\omega|}\sum_{n=1}^\infty\frac{1}{(n-1)^\alpha}<\infty,$$

one can deduce from the well-known Weierstrass test that the series $\sum_{n=1}^\infty x_n(t)$ is convergent in the sense of the norm $\|\cdot\|_{S^p}$ uniformly on $\mathbb{R}$. Now let

$$\Phi(t):=\sum_{n=1}^\infty x_n(t),\quad \text{for each}\quad t\in\mathbb{R}.$$

Observe that

$$\Phi(t)=\int_{-\infty}^t S_\alpha(t-\sigma)F(\sigma)\mathrm{d}\sigma,\quad \text{for each}\quad t\in\mathbb{R}.$$

Clearly, $\Phi(t)\in C(\mathbb{R},X)$.

Now let us show that each $x_n\in AS^p(X)$. Indeed, let $\{s'_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $F\in AS^p(X)$, there exist a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ and a function $\tilde{F}\in AS^p(X)$ such that

$$\left(\int_0^1\|F(t+s_n+s)-\tilde{F}(t+s)\|^p\mathrm{d}s\right)^{\frac{1}{p}}\to 0,$$

and

$$\left(\int_0^1\|\tilde{F}(t-s_n+s)-F(t+s)\|^p\mathrm{d}s\right)^{\frac{1}{p}}\to 0,$$

as $m\to\infty$ pointwise on $\mathbb{R}$. Moreover, if we let

$$\tilde{x}_n(t)=\int_{n-1}^n S_\alpha(\sigma)\tilde{F}(t-\sigma)\mathrm{d}\sigma,$$

one has

$$
\begin{aligned}
&\Big(\int_0^1 \|x_n(t+s+s_m)-\tilde{x}_n(t+s)\|^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
&=\Big(\int_0^1 \Big\|\int_{n-1}^n S_\alpha(\sigma)F(t+s+s_m-\sigma)\mathrm{d}\sigma-\int_{n-1}^n S_\alpha(\sigma)\tilde{F}(t+s-\sigma)\mathrm{d}\sigma\Big\|^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
&=\Big(\int_0^1 \Big\|\int_{n-1}^n S_\alpha(\sigma)\Big[F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\Big]\mathrm{d}\sigma\Big\|^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
&\le\Big(\int_0^1 \Big(\int_{n-1}^n \|S_\alpha(\sigma)\|\|F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\|\mathrm{d}\sigma\Big)^p \mathrm{d}s\Big)^{\frac{1}{p}}\\
&\le\Big(\int_0^1 \int_{n-1}^n \|S_\alpha(\sigma)\|^p\|F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\|^p\mathrm{d}\sigma\mathrm{d}s\Big)^{\frac{1}{p}}\\
&\le\Big(\int_0^1 \int_{n-1}^n \Big[\frac{CM}{1+|\omega|\sigma^\alpha}\Big]^p \|F(t+s-\sigma)\|^p\mathrm{d}\sigma\mathrm{d}s\Big)^{\frac{1}{p}}\\
&=\Big(\int_{n-1}^n \Big[\frac{CM}{1+|\omega|\sigma^\alpha}\Big]^p \int_0^1 \|F(t+s+s_m-\sigma)-\tilde{F}(t+s-\sigma)\|^p\mathrm{d}s\mathrm{d}\sigma\Big)^{\frac{1}{p}}.
\end{aligned}
$$

Obviously, the last inequality goes to 0 as $m\to\infty$ pointwise on $\mathbb{R}$. Similarly one can prove that

$$\Big(\int_0^1 \|\tilde{x}_n(t+s-s_m)-x_n(t+s)\|^p\mathrm{d}s\Big)^{\frac{1}{p}}\to 0,$$

as $m\to\infty$ pointwise on $\mathbb{R}$. Thus we conclude that each $x_n\in AS^p(X)$ and consequently their uniform limit $\Phi(t)\in AS^p(X)$, by using Lemma 2.9.

In view of the above, it follows that x is the only bounded Stepanov-like almost automorphic mild solution to equation (3.1). The proof is now complete. □

Now we investigate the Stepanov-like almost automorphic mild solutions to the nonlinear fractional differential equation (1.1), the following are the main results.

Theorem 3.3. *Assume that A is sectorial of type $\omega<0$. Let $F:\mathbb{R}\times X\to X$ satisfy $F(t,x)\in AS^p(\mathbb{R}\times X,X)$ and the following Lipschitz condition*

$$\|F(t,x)-F(t,y)\|\le L(t)\|x-y\|,\quad \text{for all } x,\ y\in X,\quad t\in\mathbb{R}, \tag{3.5}$$

where $L(t)\in L^p(\mathbb{R})$ is bounded. Then equation (1.1) admits a unique Stepanov-like almost automorphic mild solution.

Proof. Define operator Γ on $AS^p(X)$ by

$$\Gamma x(t)=\int_{-\infty}^t S_\alpha(t-\sigma)F(\sigma,x(\sigma))\mathrm{d}\sigma.$$

Since $L(t)$ is bounded, from Lemma 2.9, it follows that

$$F(\cdot)=f(\cdot,x(\cdot))\in AS^p(X).$$

From the function $\frac{1}{1+|\omega|t^{\alpha}}$ is integrable on $\mathbb{R}^+$ ($\alpha > 1$) and the proof of Lemma 2.14, one can easily see that Γx is well-defined and continuous. Then by using the proof of Theorem 3.2 with the above Lemma 2.14, one has that $\Gamma x \in AS^p(X)$ whenever $x \in AS^p(X)$. Thus Γ maps $AS^p(X)$ into itself. It suffices now to show that this operator Γ has a unique fixed point in $AS^p(X)$. For this, let x, y be in $AS^p(X)$ and define

$$C_\alpha := \sup_{t\in\mathbb{R}} \|S_\alpha(t)\|,$$

one has

$$\begin{aligned}
&\|\Gamma x(t) - \Gamma y(t)\|_{S^p} \\
=&\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \Big\| \int_{-\infty}^{\tau} S_\alpha(\tau-\sigma)\Big[F(\sigma, x(\sigma)) - F(\sigma, y(\sigma))\Big]\mathrm{d}\sigma\Big\|^p \mathrm{d}\tau\right)^{\frac{1}{p}} \\
\le&\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \int_{-\infty}^{\tau} L^p(\sigma)\|S(\tau-\sigma)\|^p \|x(\sigma)-y(\sigma)\|^p \mathrm{d}\sigma\mathrm{d}\tau\right)^{\frac{1}{p}} \\
\le& C_\alpha \|L\|_p \|x-y\|_{S^p}.
\end{aligned}$$

In general we get

$$\begin{aligned}
&\|[\Gamma^n x](t) - [\Gamma^n y](t)\|_{S^p} \\
\le&\frac{C_\alpha^n}{(n-1)!} \left(\int_{-\infty}^t L^p(\sigma) \left(\int_{-\infty}^{\sigma} L^p(\tau)\mathrm{d}\tau\right)^{n-1} \mathrm{d}\sigma\right)^{\frac{1}{p}} \|x-y\|_{S^p} \\
\le&\frac{C_\alpha^n}{n!} \left(\left(\int_{-\infty}^t L(\sigma)\mathrm{d}\sigma\right)^{\frac{1}{p}}\right)^n \|x-y\|_{S^p} \\
\le&\frac{(C_\alpha\|L\|_p)^n}{n!} \|x-y\|_{S^p}.
\end{aligned}$$

Hence, since

$$\frac{(C_\alpha\|L\|_p)^n}{n!} < 1$$

for n sufficiently large, by the contraction principle Γ has a unique fixed point $x \in AS^p(X)$. The proof is complete. □

We note that conditions of type (3.5) have been previously considered in the literature for almost automorphic functions [59]. Our motivation comes from their use in the study of pseudo-almost periodic solutions of semilinear Cauchy problems [60]. Now we consider a more general case of equations introducing a new class of functions L which do not necessarily belong to $L^p(\mathbb{R})$. We have the following result.

Theorem 3.4. *Assume that A is sectorial of type $\omega < 0$. Let $F : \mathbb{R} \times X \to X$ satisfy $F(t, x) \in AS^p(\mathbb{R} \times X, X)$ and the Lipschitz condition (3.5) where the integral $\int_{-\infty}^t L(\sigma)d\sigma$ exists for all $t \in \mathbb{R}$ and $L(t)$ is bounded. Then equation (1.1) admits a unique Stepanov-like almost automorphic mild solution.*

Proof. Define a new norm

$$|||x||| := \sup_{t\in\mathbb{R}} \Big\{ v(t)\|x(t)\|_{S^p} \Big\},$$

where

$$v(t) := \Big[e^{-k\int_{-\infty}^{t} L(\sigma)\mathrm{d}\sigma} \Big]^{\frac{1}{p}}$$

and k is a fixed positive constant greater than $C_\alpha := \sup_{t\in\mathbb{R}} \|S_\alpha(t)\|$. Let x, y be in $AS^p(X)$, then one has

$$\begin{aligned}
& v(t)\|\Gamma x(t) - \Gamma y(t)\|_{S^p} \\
=& v(t)\sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \Big\| \int_{-\infty}^{\tau} S(\tau-\sigma)\Big[F(\sigma,x(\sigma)) - F(\sigma,y(\sigma))\Big]\mathrm{d}\sigma \Big\|^p \mathrm{d}\tau \right)^{\frac{1}{p}} \\
\le& C_\alpha \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \int_{-\infty}^{\tau} v^p(\tau) L^p(\sigma)\|x(\sigma) - y(\sigma)\|^p \mathrm{d}\sigma\mathrm{d}\tau \right)^{\frac{1}{p}} \\
=& C_\alpha \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \int_{-\infty}^{\tau} v^p(\tau) v^p(\sigma) L^p(\sigma)(v^p(\sigma))^{-1}\|x(\sigma) - y(\sigma)\|^p \mathrm{d}\sigma\mathrm{d}\tau \right)^{\frac{1}{p}} \\
\le& C_\alpha |||x-y||| \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \int_{-\infty}^{\tau} v^p(\tau)(v^p(\sigma))^{-1} L^p(\sigma)\mathrm{d}\sigma\mathrm{d}\tau \right)^{\frac{1}{p}} \\
=& \frac{C_\alpha}{k} |||x-y||| \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \int_{-\infty}^{\tau} k e^{k\int_\tau^\sigma L(\tau)\mathrm{d}\tau} L(\sigma)\mathrm{d}\sigma\mathrm{d}\tau \right)^{\frac{1}{p}} \\
=& \frac{C_\alpha}{k} |||x-y||| \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \int_{-\infty}^{\tau} \frac{\mathrm{d}}{\mathrm{d}\sigma}\Big(e^{k\int_\tau^\sigma L(\tau)\mathrm{d}\tau} \Big)\mathrm{d}\sigma\mathrm{d}\tau \right)^{\frac{1}{p}} \\
=& \frac{C_\alpha}{k} \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \Big(1 - e^{-k\int_{-\infty}^{\tau} L(\tau)\mathrm{d}\tau} \Big)\mathrm{d}\tau \right)^{\frac{1}{p}} |||x-y||| \\
\le& \frac{C_\alpha}{k} |||x-y|||.
\end{aligned}$$

Hence, since $\frac{C_\alpha}{k} < 1$, Γ has a unique fixed point $x \in AS^p(X)$. □

Note that the above result does not include the cases where L is a constant.

Theorem 3.5. *Assume that A is sectorial of type $\omega < 0$. Let $F : \mathbb{R}\times X \to X$ satisfy $F(t,x) \in AS^p(\mathbb{R}\times X, X)$ and the following Lipschitz condition*

$$\|F(t,x) - F(t,y)\| \le L\|x-y\|, \quad \textit{for all} \;\; x,\; y \in X, \quad t \in \mathbb{R}.$$

Then equation (1.1) admits a unique Stepanov-like almost automorphic mild solution whenever

$$CML\omega^{-\frac{1}{\alpha}}\pi < \alpha \sin\frac{\pi}{\alpha}.$$

Proof. For $x,\ y \in AS^p(X)$, one has

$$
\begin{aligned}
&\|\Gamma x(t) - \Gamma y(t)\|_{S^p} \\
=& \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \int_{-\infty}^{s} S_\alpha(s-\sigma)\Big[F(\sigma, x(\sigma)) - F(\sigma, y(\sigma))\Big]\mathrm{d}\sigma \right\|^p \mathrm{d}s\right)^{\frac{1}{p}} \\
=& \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\| \int_{0}^{\infty} S_\alpha(\tau)\Big[F(s-\tau, x(s-\tau)) - F(s-\tau, y(s-\tau))\Big]\mathrm{d}\tau \right\|^p \mathrm{d}s\right)^{\frac{1}{p}} \\
\leq& L \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \int_0^\infty \|S_\alpha(\tau)\|^p \|x(s-\tau)) - y(s-\tau)\|^p \mathrm{d}\tau \mathrm{d}s\right)^{\frac{1}{p}} \\
\leq& CML\|x-y\|_{S^p} \left(\int_0^\infty \left(\frac{1}{1+|\omega|\tau^\alpha}\right)^p \mathrm{d}\tau\mathrm{d}s\right)^{\frac{1}{p}} \\
\leq& CML\|x-y\|_{S^p} \left(\int_0^\infty \frac{1}{1+|\omega|\tau^\alpha} \mathrm{d}\tau\mathrm{d}s\right)^{\frac{1}{p}} \\
=& CML\left[\frac{\omega^{-\frac{1}{\alpha}}\pi}{\alpha\sin\frac{\pi}{\alpha}}\right]^{\frac{1}{p}} \|x-y\|_{S^p}.
\end{aligned}
$$

This proves that Γ is a strict contraction, so it follows from the Banach contraction mapping principle that Γ admits a unique fixed point $x \in AS^p(X)$, which is the unique Stepanov-like almost automorphic mild solution to equation (1.1) . □

Taking $A = -\rho^\alpha I$ with $\rho > 0$ in equation (1.1) , the above theorem gives the following corollary.

Corollary 3.6. *Let $F : \mathbb{R}\times X \to X$ satisfy $F(t,x) \in AS^p(\mathbb{R}\times X, X)$ and the following Lipschitz condition*

$$\|F(t,x) - F(t,y)\| \leq L\|x-y\|, \quad \textit{for all} \;\; x,\ y \in X, \quad t \in \mathbb{R}. \tag{3.6}$$

Then equation (1.1) admits a unique Stepanov-like almost automorphic mild solution whenever

$$CL < \frac{\alpha\sin(\frac{\pi}{\alpha})}{\rho\pi}.$$

Remark 3.7. It is interesting to note that the function

$$\alpha \to \frac{\alpha\sin(\frac{\pi}{\alpha})}{\rho\pi}$$

is increasing from 0 to $\frac{2}{\rho\pi}$ in the interval $1 < \alpha < 2$. Therefore, with respect to the Lipschitz condition (3.6), the class of admissible semilinear terms $F(t, x(t))$ is the best in the case $\alpha = 2$ and the worst in the case $\alpha = 1$. Note the direct relation with the term $\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}$ in equation (3.2) , where the singularity becomes better (smooth) when α goes from 1 to 2.

4. Applications

In this section we give an example to illustrate the above results.

Consider the following fractional relaxation-oscillation equation

$$\begin{aligned}&\partial_t^{\alpha} u(t,x) = \partial_x^2 u(t,x) - \mu u(t,x) + \partial_t^{\alpha-1} F(t,u(t,x)), \quad t\in\mathbb{R}, \quad x\in[0,\pi],\\ &u(t,0)=u(t,\pi)=0, \quad t\in\mathbb{R},\end{aligned} \tag{4.1}$$

where $\mu > 0$, $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given function.

Take $X = L^2([0,\pi])$ and define the operator A by

$$A\varphi := \varphi'' - \mu\varphi, \quad \varphi\in D(A),$$

where

$$D(A) := \Big\{\varphi\in L^2[0,\pi] : \varphi''\in L^2[0,\pi], \varphi(0)=\varphi(\pi)\Big\} \subset L^2[0,\pi].$$

It is well known that $Au = u''$ is the generator of an analytic semigroup on $L^2[0,\pi]$. Hence, $\mu I - A$ is sectorial of type $\omega = -\mu < 0$. Thus equation (4.1) can be formulated by the inhomogeneous problem (1.1), where $u(t) = u(t,\cdot)$.

Let us consider the nonlinearity

$$F(t,x)(s) = \beta b(t)\sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)\sin(x(s)),$$

for all $x\in X$ and $s\in[0,\pi]$, $t\in\mathbb{R}$, where $b(t)$ is a bounded function. Thus one has

$$F(t,x)\in AS^p(\mathbb{R}\times X, X)$$

and

$$\|F(t,x)-F(t,y)\|_2^2 \le \int_0^{\pi}\beta^2|b(t)|^2|\sin(x(s))-\sin(y(s))|ds \le \beta^2|b(t)|^2\|x(s)-y(s)\|_2^2.$$

In consequence, the fractional differential equation (1.1) has unique Stepanov-like almost automorphic mild solutions if either $b\in L^p(\mathbb{R})$ (Theorem 3.3) or $\int_{-\infty}^t b(\sigma)d\sigma$ exists for all $t\in\mathbb{R}$ (Theorem 3.4). If we assume that $b\in L(\mathbb{R})$ and

$$|\beta| < \frac{\alpha\sin(\frac{\pi}{\alpha})}{\pi CM\|b\|_\infty|\mu|^{-\frac{1}{\alpha}}},$$

then the same conclusion holds by Theorem 3.5.

Acknowledgement. This work is supported by the Characteristic Innovation Project (Natural Science) of Guangdong Province (No.2016KTSCX183 and No.2016KTSCX094), the Guangdong Province Natural Science Foundation (No.2015A030313896), the Science and Technology Program Project of Guangzhou (No.201707010230).

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