Gulf Journal of Mathematics
Vol 6, Issue 1 (2018) 46-60

SOME RESULTS FOR CLASSES OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. In this paper, a boundary version of the Schwarz lemma for classes $\mathcal{K}(\alpha)$ is investigated. For the function $f(z) = 1 + c_1 z + c_2 z^2 + ...$ defined in the unit disc such that $f(z) \in \mathcal{K}(\alpha)$, we estimate a modulus of the angular derivative of $f(z)$ function at the boundary point b with $f(b) = \alpha$. Also, we shall give an estimate below $|f'(b)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_0 \neq 0$. The sharpness of these inequalities is also proved.

1. INTRODUCTION

The most classical version of the Schwarz lemma involves the behavior at the origin of a bounded, holomorphic function on the unit disc $D = \{z : |z| < 1\}$. Also, the Schwarz lemma in one of the most important results in the complex analysis and is widely applied in many branches of mathematical research. In its most basic form, the familiar Schwarz lemma says this ([8], p.329):

Let D be the unit disc in the complex plane $\mathbb{C}$. Let $f : D \to D$ be a holomorphic function with $f(0) = 0$. Under these circumstances $|f(z)| \leq |z|$ for all $z \in D$, and $|f'(0)| \leq 1$. In addition, if the equality $|f(z)| = |z|$ holds for any $z \neq 0$, or $|f'(0)| = 1$ then f is a rotation, that is, $f(z) = ze^{i\theta}$, θ real.

In order to show our main results, we need the following lemma due to Jack's Lemma [9].

Lemma 1.1 (Jack's Lemma)**.** *Let $f(z)$ be a non-constant and holomorphic function in the unit disc D with $f(0) = 0$. If $|f(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0, then*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k,$$

where $k \geq 1$ is a real number.

Let $\mathcal{M}$ denote the class of functions $f(z)$ that are holomorphic in the unit disc D. That is,

$$f(z) = 1 + c_1 z + c_2 z^2 + ...$$

Date: Received: May 21, 2017; Accepted: Jan 18, 2018.
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2000 *Mathematics Subject Classification.* 30C80, 32A10.
Key words and phrases. Holomorphic function, Jack's lemma, Angular derivative.

Also, let $\mathcal{K}(\alpha)$ be the subclass of $\mathcal{M}$ consisting of all functions $f(z)$ which satisfy

$$\Re\left(1+\frac{zf'(z)}{f(z)}\right) > \frac{3\alpha-1}{2\alpha}\ (z \in D),$$

where $\frac{1}{2} \leq \alpha < 1$.

Let $f(z) \in \mathcal{K}(\alpha)$ and consider the function

$$\varphi(z) = \frac{f(z)-1}{1-2\alpha+f(z)}.$$

Obviously, $\varphi(z)$ is holomorphic function in D and $\varphi(0) = 0$.

We want to prove that $|\varphi(z)| < 1$ in D. From the definition of $\varphi(z)$, we take

$$f(z) = \frac{1+(1-2\alpha)\,\varphi(z)}{1-\varphi(z)}$$

and by logarithmic differentiations

$$\frac{zf'(z)}{f(z)} = \frac{(1-2\alpha)\,z\varphi'(z)}{1+(1-2\alpha)\,\varphi(z)} + \frac{z\varphi'(z)}{1-\varphi(z)}.$$

Assume that there exists a point $z_0 \in D$ such that

$$\max_{|z|\leq|z_0|} |\varphi(z)| = |\varphi(z_0)| = 1.$$

From the Jack's lemma, we obtain

$$\varphi(z_0) = e^{i\theta} \text{ and } \frac{z_0\varphi'(z_0)}{\varphi(z_0)} = k.$$

Therefore, we have

$$1+\frac{z_0 f'(z_0)}{f(z_0)} = 1+\frac{(1-2\alpha)\,z_0\varphi'(z_0)}{1+(1-2\alpha)\,\varphi(z_0)} + \frac{z_0\varphi'(z_0)}{1-\varphi(z_0)}$$

and

$$1+\frac{z_0 f'(z_0)}{f(z_0)} = 1+\frac{(1-2\alpha)\,ke^{i\theta}}{1+(1-2\alpha)\,e^{i\theta}} + \frac{ke^{i\theta}}{1-e^{i\theta}}.$$

Since

$$\begin{aligned}
\frac{ke^{i\theta}}{1-e^{i\theta}} &= \frac{k}{e^{-i\theta}\left(1-e^{i\theta}\right)} = \frac{k}{e^{-i\theta}-1} \\
&= k\frac{-1+\cos\theta+i\sin\theta}{2\left(1-\cos\theta\right)} \\
&= k\frac{-(1-\cos\theta)}{2\left(1-\cos\theta\right)} + k\frac{i\sin\theta}{2\left(1-\cos\theta\right)} \\
&= -\frac{k}{2} + k\frac{i\sin\theta}{2\left(1-\cos\theta\right)}
\end{aligned}$$

and

$$\begin{aligned}\frac{(1-2\alpha)\,ke^{i\theta}}{1+(1-2\alpha)\,e^{i\theta}} &= \frac{(1-2\alpha)\,k}{e^{-i\theta}\left(1+(1-2\alpha)\,e^{i\theta}\right)}=\frac{(1-2\alpha)\,k}{e^{-i\theta}+(1-2\alpha)} \\ &= k\,(1-2\alpha)\,\frac{1-2\alpha+\cos\theta+i\sin\theta}{1+2\,(1-2\alpha)\cos\theta+(1-2\alpha)^2},\end{aligned}$$

we obtain

$$\Re\left(1+\frac{z_0f'(z_0)}{f(z_0)}\right)=1-\frac{k}{2}+k\,(1-2\alpha)\,\frac{1-2\alpha+\cos\theta}{1+2\,(1-2\alpha)\cos\theta+(1-2\alpha)^2}. \quad (1.1)$$

Since the right handside of (1.1) takes its minimum value for $\cos\theta=-1$ and $k\geq 1$, we have that

$$\Re\left(1+\frac{z_0f'(z_0)}{f(z_0)}\right)\leq 1-\frac{1}{2}-\frac{1-2\alpha}{2\alpha}=\frac{3\alpha-1}{2\alpha}.$$

This contradicts the condition $f(z)\in\mathcal{K}(\alpha)$. This means that there is no point $z_0\in D$ such that $|\varphi(z_0)|=1$ for all $z\in D$. Therefore, $|\varphi(z)|<1$ for $|z|<1$. By the Schwarz lemma, we obtain

$$|f'(0)|\leq 2\,(1-\alpha)\,.$$

The result is sharp and the extremal function is

$$f(z)=\frac{1+z\,(1-2\alpha)}{1-z}.$$

That proves

Lemma 1.2. *If $f(z)\in\mathcal{K}(\alpha)$, then we have*

$$|f'(0)|\leq 2\,(1-\alpha)\,. \quad (1.2)$$

The result is sharp and the extremal function is

$$f(z)=\frac{1+z\,(1-2\alpha)}{1-z}.$$

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b|=1$, and if $|f(b)|=1$ and $f'(b)$ exists, then $|f'(b)|\geq 1$, which is known as the Schwarz lemma on the boundary. The equality in $|f'(b)|\geq 1$ holds if and only if $f(z)=ze^{i\theta}$, θ real. This result of Schwarz lemma and its generalization are described as Schwarz lemma at the boundary in the literature.

More than the last decade, there have been tremendous studies on Schwarz lemma at the boundary (see,[1], [2], [5], [6], [13], [16], [17], [18], [22] and references therein). Some of them are about the below boundary of modulus of the functions derivation at the points (contact points) which satisfies $|f\,(b)|=1$ condition of the boundary of the unit circle.

Osserman [16] offered the following boundary refinement of the classical Schwarz lemma. It is very much in the spirit of the sort of result we wish to consider here. In other words,

$$|f'(b)|\geq\frac{2}{1+|f'(0)|} \quad (1.3)$$

and

$$|f'(b)| \geq 1 \tag{1.4}$$

under the assumption $f(0) = 0$, where f is a holomorphic function mapping the unit disc into itself and b is a boundary point which f extends continuosly, and $|f(b)| = 1$. In addition, the equality in (1.4) holds if and only if $f(z) = ze^{i\theta}$, where θ is a real number. Also, the equality in (1.3) holds if and only if f is of the form $f(z) = -z\frac{\xi - z}{1-\bar{\xi}z}$, $\forall z \in D$, for some constant $\xi \in (-1, 0]$.

The following set is called a Stolz angle at $b \in \partial D$

$$\triangle = \left\{ z \in D : \left|\arg\left(1 - \bar{b}z\right)\right| < \varsigma, \ |z - b| < r \right\}, \ \left(0 < \varsigma < \frac{\pi}{2}, \ r < 2\cos\varsigma\right).$$

Let f be a function from D to $\overline{\mathbb{C}}$. It is said that f has an angular limit $a \in \overline{\mathbb{C}}$ at $b \in \partial D$ if

$$f(z) \to a \text{ as } z \to b, \ z \in \triangle$$

for each stolz angle $\triangle$ at b. The number 2ς which is length of $\triangle$ can be any number less than π. It is said that f has the unrestricted limit $a \in \overline{\mathbb{C}}$ at b if

$$f(z) \to a \text{ as } z \to b, \ z \in D.$$

Clearly, in the last fact, if the function f which is continuous in D is defined at the point b as $f(b) = a$ then f becomes continuous in $D \cup \{b\}$.

Let f be a functionfrom D to D and ν be its angular limit at the point b. If there exists a point ζ such that

$$\lim_{z \to b, \ z \in \triangle} \frac{f(z) - \nu}{z - b} = \zeta$$

for every Stolz anagle $\triangle$ at the point b then ζ is called the angular derivative of the function f at b and iy is shown with $f'(b)$.

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [19]).

Lemma 1.3 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in D, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$.*

D. M. Burns and S. G. Krantz [3] and D. Chelst [4] studied the uniqueness part of the Schwarz lemma. In M. Mateljević's papers, for more general results and related estimates, see also ([12], [13], [14] and [15]).

The inequality (1.3) is a particular case of a result due to Vladimir N. Dubinin in [5], who strengthened the inequality $|f'(b)| \geq 1$ by involving zeros of the function f.

X. Tang, T. Liu and J. Lu [20] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

Also, M. Jeong [11] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [10] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc. Furthermore, X. Tang, T. Liu and W. Zhang [21] established a new type of the classical Schwarz lemma at the boundary for holomorphic self-mappings of the unit ball in $\mathbb{C}^n$, and then give the boundary version of the rigidity theorem. S.L. Wail and W.M. Shah [22] established some results by using a boundary refinement of the classical Schwarz lemma. For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [2], [7]).

2. Main Results

In this section, for holomorphic function $f(z)$ belong to the class of $\mathcal{K}(\alpha)$, it has been estimated from below the modulus of the angular derivative of the function on the boundary point of the unit disc. It has been proved that these results are sharp. Also, we derive an improvement of the above of inequalities (1.3) and (1.4) as the special cases of our main results.

Theorem 2.1. *Let $f(z) \in \mathcal{K}(\alpha)$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = \alpha$. Then we have the inequality*

$$|f'(b)| \geq \frac{1-\alpha}{2}. \tag{2.1}$$

The inequality (2.1) *is sharp with extremal function*

$$f(z) = \frac{1 + z(1-2\alpha)}{1-z}.$$

Proof. Let us consider the following function

$$\varphi(z) = \frac{f(z)-1}{1-2\alpha+f(z)}.$$

Then $\varphi(z)$ is holomorphic function in the unit disc D and $\varphi(0) = 0$. By the Jack's lemma and since $f(z) \in \mathcal{K}(\alpha)$, we take $|\varphi(z)| < 1$ for $|z| < 1$. Also, we have $|\varphi(b)| = 1$ for $b \in \partial D$. It is clear that

$$\varphi'(z) = 2(1-\alpha)\frac{f'(z)}{(1-2\alpha+f(z))^2}$$

and

$$\begin{aligned} \varphi'(b) &= 2(1-\alpha)\frac{f'(b)}{(1-2\alpha+f(b))^2} = 2(1-\alpha)\frac{f'(b)}{(1-2\alpha+\alpha)^2} \\ &= 2(1-\alpha)\frac{f'(b)}{(1-\alpha)^2} = 2\frac{f'(b)}{1-\alpha} \end{aligned}$$

since $f(b) = \alpha$.

Therefore, we take from (1.4), we obtain

$$1 \leq |\varphi'(b)| = 2\frac{|f'(b)|}{1-\alpha}$$

and

$$|f'(b)| \geq \frac{1-\alpha}{2}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z)=\frac{1+z\left(1-2\alpha\right)}{1-z}.$$

Then

$$f'(z)=\frac{\left(1-2\alpha\right)\left(1-z\right)+1+z\left(1-2\alpha\right)}{\left(1-z\right)^{2}},$$

$$\left|f'(-1)\right|=\frac{1-\alpha}{2}.$$

□

Theorem 2.2. *Let $f(z)\in\mathcal{K}(\alpha)$. Suppose that, for some $b\in\partial D$, f has an angular limit $f(b)$ at b, $f(b)=\alpha$. Then we have the inequality*

$$\left|f'(b)\right|\geq\frac{2\left(1-\alpha\right)^{2}}{2\left(1-\alpha\right)+\left|f'(0)\right|}. \tag{2.2}$$

The inequality (2.2) *is sharp with extremal function*

$$f(z)=\frac{1+z\left(1-2\alpha\right)}{1-z}.$$

Proof. Let $\varphi(z)$ be the same as in the proof of Theorem2.1. Therefore, we take from (1.3), we obtain

$$\frac{2}{1+\left|\varphi'(0)\right|}\leq\left|\varphi'(b)\right|=2\frac{\left|f'(b)\right|}{1-\alpha}.$$

Since

$$\varphi'(z)=2\left(1-\alpha\right)\frac{f'(z)}{\left(1-2\alpha+f(z)\right)^{2}},$$

it is clear that

$$\left|\varphi'(0)\right|=\frac{\left|f'(0)\right|}{2\left(1-\alpha\right)}.$$

Then, we obtain

$$\frac{2}{1+\frac{\left|f'(0)\right|}{2(1-\alpha)}}\leq2\frac{\left|f'(b)\right|}{1-\alpha}.$$

The last inequality shows that the inequality intended is satisfied.

Now, we shall show that the inequality (2.2) is sharp. Let

$$f(z)=\frac{1+z\left(1-2\alpha\right)}{1-z}.$$

Then

$$f'(z)=2\frac{1-\alpha}{\left(z-1\right)^{2}},$$

$$\left|f'(-1)\right|=\frac{1-\alpha}{2}.$$

Since$|f'(0)|=2(1-\alpha)$, (2.2) is satisfied with equality. That is;

$$\begin{aligned}\frac{2(1-\alpha)^2}{2(1-\alpha)+|f'(0)|} &= \frac{2(1-\alpha)^2}{2(1-\alpha)+2(1-\alpha)}\\ &= \frac{2(1-\alpha)^2}{4(1-\alpha)}=\frac{1-\alpha}{2}.\end{aligned}$$

□

The inequality (2.2) can be strengthened as below by taking into account c_2 which is second coefficient in the expansion of the function $f(z)$.

Theorem 2.3. *Let $f(z)\in\mathcal{K}(\alpha)$. Suppose that, for some $b\in\partial D$, f has an angular limit $f(b)$ at b, $f(b)=\alpha$. Then we have the inequality*

$$|f'(b)|\geq\frac{1-\alpha}{2}\left(1+\frac{2(2(1-\alpha)-|c_1|)^2}{4(1-\alpha)^2-|c_1|^2+|2(1-\alpha)c_2-c_1^2|}\right). \tag{2.3}$$

The equality in (2.3) *occurs for the function*

$$f(z)=\frac{1+z(1-2\alpha)}{1-z}.$$

Proof. Let $\varphi(z)$ be the same as in the proof of Theorem2.1. Let us consider the function

$$\mu(z)=\frac{\varphi(z)}{B(z)},$$

where $B(z)=z$. The function $\varphi(z)$ is holomorphic in D. According to the maximum principle, we have $|\varphi(z)|<1$ for each $z\in D$. In particular, we have

$$|\mu(0)|=\frac{|c_1|}{2(1-\alpha)}\leq 1 \tag{2.4}$$

and

$$|\mu'(0)|=\frac{|2(1-\alpha)c_2-c_1^2|}{4(1-\alpha)^2}.$$

Furthermore, it can be seen that

$$\frac{b\varphi'(b)}{\varphi(b)}=|\varphi'(b)|\geq|B'(b)|=\frac{bB'(b)}{B(b)}.$$

Consider the function

$$\Psi(z)=\frac{\mu(z)-\mu(0)}{1-\overline{\mu(0)}\mu(z)}.$$

That function is holomorphic in D, $|\Psi(z)|\leq 1$ for $|z|<1$, $\Psi(0)=0$, and $|\Psi(b)|=1$ for $b\in\partial D$. From (1.3), we have

$$\begin{aligned}\frac{2}{1+|\Psi'(0)|} &\leq |\Psi'(b)|=\frac{1-|\mu(0)|^2}{\left|1-\overline{\mu(0)}\mu(b)\right|^2}|\mu'(b)|\\ &\leq \frac{1+|\mu(0)|}{1-|\mu(0)|}\left\{|\varphi'(b)|-|B'(b)|\right\}.\end{aligned}$$

Since

$$\Psi'(z)=\frac{1-|\mu(0)|^2}{\left(1-\overline{\mu(0)}\mu(z)\right)^2}\mu'(z),$$

$$\Psi'(0)=\frac{\mu'(0)}{1-|\mu(0)|^2},$$

and

$$|\Psi'(0)|=\frac{\frac{\left|2(1-\alpha)c_2-c_1^2\right|}{4(1-\alpha)^2}}{1-\left(\frac{|c_1|}{2(1-\alpha)}\right)^2}=\frac{\left|2\left(1-\alpha\right)c_2-c_1^2\right|}{4\left(1-\alpha\right)^2-|c_1|^2},$$

we obtain

$$\begin{aligned}
\frac{2}{1+\frac{\left|2(1-\alpha)c_2-c_1^2\right|}{4(1-\alpha)^2-|c_1|^2}} &\leq \frac{1+\frac{|c_1|}{2(1-\alpha)}}{1-\frac{|c_1|}{2(1-\alpha)}}\left\{2\frac{|f'(b)|}{1-\alpha}-1\right\},\\
\frac{2\left(4\left(1-\alpha\right)^2-|c_1|^2\right)}{4\left(1-\alpha\right)^2-|c_1|^2+\left|2\left(1-\alpha\right)c_2-c_1^2\right|} &\leq \frac{2\left(1-\alpha\right)+|c_1|}{2\left(1-\alpha\right)-|c_1|}\left\{2\frac{|f'(b)|}{1-\alpha}-1\right\},\\
\frac{2\left(2\left(1-\alpha\right)-|c_1|\right)^2}{4\left(1-\alpha\right)^2-|c_1|^2+\left|2\left(1-\alpha\right)c_2-c_1^2\right|} &\geq 2\frac{|f'(b)|}{1-\alpha}-1
\end{aligned}$$

and

$$|f'(b)|\geq\frac{1-\alpha}{2}\left(1+\frac{2\left(2\left(1-\alpha\right)-|c_1|\right)^2}{4\left(1-\alpha\right)^2-|c_1|^2+\left|2\left(1-\alpha\right)c_2-c_1^2\right|}\right).$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z)=\frac{1+z\left(1-2\alpha\right)}{1-z}.$$

Then

$$|f'(1)|=\frac{1-\alpha}{2}.$$

Since $|c_1|=2\left(1-\alpha\right)$, (2.3) is satisfied with equality. □

If $f(z)-1$ has no zeros different from $z=0$ in Theorem2.3, the inequality (2.3) can be further strengthened. This is given by the following Theorem.

Theorem 2.4. *Let $f(z)\in\mathcal{K}(\alpha)$ 0, $f(z)-1$ has no zeros in D except $z=0$ and $c_1>0$. Suppose that, for some $b\in\partial D$, f has an angular limit $f(b)$ at b, $f(b)=\alpha$. Then we have the inequality*

$$|f'(b)|\geq\frac{1-\alpha}{2}\left(1-\frac{4\left(1-\alpha\right)|c_1|\ln^2\left(\frac{|c_1|}{2(1-\alpha)}\right)}{4\left(1-\alpha\right)|c_1|\left|\ln\left(\frac{|c_1|}{2(1-\alpha)}\right)\right|-\left|2\left(1-\alpha\right)c_2-c_1^2\right|}\right). \tag{2.5}$$

The equality in (2.5) *occurs for the function*

$$f(z)=\frac{1+z\left(1-2\alpha\right)}{1-z}.$$

Proof. Let $c_1 > 0$ and let us consider the function $\mu(z)$ as in Theorem2.3. Taking account of the equality (2.4), we denote by $\ln \mu(z)$ the holomorphic branch of the logarithm normed by condition

$$\ln \mu(0) = \ln\left(\frac{|c_1|}{2(1-\alpha)}\right) < 0.$$

Take the following auxiliary function

$$\phi(z) = \frac{\ln \mu(z) - \ln \mu(0)}{\ln \mu(z) + \ln \mu(0)}.$$

It is obvious that $\phi(z)$ is a holomorphic function in D, $\phi(0) = 0$, $|\phi(z)| \le 1$ for $|z| < 1$, and also $|\phi(b)| = 1$ for $b \in \partial D$. So, we can apply (1.3) to the function $\phi(z)$. Since

$$\phi'(z) = 2\ln \mu(0) \frac{\mu'(z)}{\mu(z)\left(\ln \mu(z) + \ln \mu(0)\right)^2},$$

and

$$\phi'(b) = 2\ln \mu(0) \frac{\mu'(b)}{\mu(b)\left(\ln \mu(b) + \ln \mu(0)\right)^2},$$

we obtain

$$\begin{aligned}
\frac{2}{1+|\phi'(0)|} &\le |\phi'(b)| = \frac{2\,|\ln \mu(0)|}{|\ln \mu(b) + \ln \mu(0)|^2}\left|\frac{\mu'(b)}{\mu(b)}\right|, \\
&= \frac{-2\ln \mu(0)}{\ln^2 \mu(0) + \arg^2 \mu(b)}\left|\frac{\varphi'(b)}{B(b)} - \frac{\varphi(b)B'(b)}{B(b)^2}\right| \\
&= \frac{-2\ln \mu(0)}{\ln^2 \mu(0) + \arg^2 \mu(b)}\left|\frac{\varphi(b)}{b^2}\right|\left|\frac{b\varphi'(b)}{\varphi(b)} - \frac{bB'(b)}{B(b)}\right| \\
&= \frac{-2\ln \mu(0)}{\ln^2 \mu(0) + \arg^2 \mu(b)}\left\{|\varphi'(b)| - |B'(b)|\right\} \\
&\le \frac{-2\ln \mu(0)}{\ln^2 \mu(0)}\left\{2\frac{|f'(b)|}{1-\alpha} - 1\right\} \\
&= \frac{-2}{\ln \mu(0)}\left\{2\frac{|f'(b)|}{1-\alpha} - 1\right\}.
\end{aligned}$$

Since

$$\phi'(0) = \frac{\mu'(0)}{2\mu(0)\ln \mu(0)}$$

and thus,

$$|\phi'(0)| = \frac{\frac{\left|2(1-\alpha)c_2 - c_1^2\right|}{4(1-\alpha)^2}}{-2\frac{|c_1|}{2(1-\alpha)}\ln\left(\frac{|c_1|}{2(1-\alpha)}\right)},$$

we have

$$\frac{2}{1 - \frac{\left|2(1-\alpha)c_2 - c_1^2\right|}{4(1-\alpha)|c_1|\ln\left(\frac{|c_1|}{2(1-\alpha)}\right)}} \le \frac{-2}{\ln\left(\frac{|c_1|}{2(1-\alpha)}\right)}\left\{2\frac{|f'(b)|}{1-\alpha} - 1\right\}.$$

By getting elementary arrangements, we obtain

$$\frac{4\left(1-\alpha\right)\left|c_1\right|\ln\left(\frac{\left|c_1\right|}{2(1-\alpha)}\right)}{4\left(1-\alpha\right)\left|c_1\right|\left|\ln\left(\frac{\left|c_1\right|}{2(1-\alpha)}\right)\right|-\left|2\left(1-\alpha\right)c_2-c_1^2\right|}\leq\frac{-1}{\ln\left(\frac{\left|c_1\right|}{2(1-\alpha)}\right)}\left\{2\frac{\left|f'(b)\right|}{1-\alpha}-1\right\},$$

$$-\frac{4\left(1-\alpha\right)\left|c_1\right|\ln^2\left(\frac{\left|c_1\right|}{2(1-\alpha)}\right)}{4\left(1-\alpha\right)\left|c_1\right|\left|\ln\left(\frac{\left|c_1\right|}{2(1-\alpha)}\right)\right|-\left|2\left(1-\alpha\right)c_2-c_1^2\right|}\leq 2\frac{\left|f'(b)\right|}{1-\alpha}-1,$$

$$\left|f'(b)\right|\geq\frac{1-\alpha}{2}\left(1-\frac{4\left(1-\alpha\right)\left|c_1\right|\ln^2\left(\frac{\left|c_1\right|}{2(1-\alpha)}\right)}{4\left(1-\alpha\right)\left|c_1\right|\left|\ln\left(\frac{\left|c_1\right|}{2(1-\alpha)}\right)\right|-\left|2\left(1-\alpha\right)c_2-c_1^2\right|}\right)$$

Since $\left|c_1\right|=2\left(1-\alpha\right)$, (2.5) is satisfied with equality. □

In the following Theorem, we shall give an estimate below $|f'(b)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z=0$ and $z_0\neq 0$.

Theorem 2.5. *Let $f(z)\in\mathcal{K}(\alpha)$ and $f(z_0)=1$ for $0<|z_0|<1$. Suppose that, for some $b\in\partial D$, f has an angular limit $f(b)$ at b, $f(b)=\alpha$. Then we have the inequality*

$$\left|f'(b)\right|\geq\frac{(1-\alpha)}{2}\left(1+\frac{1-|z_0|^2}{|b-z_0|^2}+\frac{2(1-\alpha)|z_0|-|f'(0)|}{2(1-\alpha)|z_0|+|f'(0)|}\right. \tag{2.6}$$
$$\left.\times\left[1+\frac{4(1-\alpha)^2|z_0|^2+|f'(0)||f'(z_0)|\left(1-|z_0|^2\right)-4(1-\alpha)\left(1-|z_0|^2\right)|f'(z_0)|-2(1-\alpha)|f'(0)|}{4(1-\alpha)^2|z_0|^2+|f'(0)||f'(z_0)|\left(1-|z_0|^2\right)+4(1-\alpha)\left(1-|z_0|^2\right)|f'(z_0)|+2(1-\alpha)|f'(0)|}\frac{1-|z_0|^2}{|b-z_0|^2}\right]\right).$$

The inequality (2.6) is sharp, with equality for each possible values $|f'(0)|=(1-\alpha)\,c$ and $|f'(z_0)|=(1-\alpha)\,d$ $\left(0\leq c\leq 2\left(1-\alpha\right)|z_0|,\ 0\leq d\leq 2\left(1-\alpha\right)\frac{|z_0|}{1-|z_0|^2}\right)$.

Proof. Let

$$\rho(z)=\frac{z-z_0}{1-\overline{z_0}z}.$$

Let $\kappa:D\to D$ be a holomorphic function and a point $z_0\in D$. Therefore, we have

$$|\kappa(z)|\leq\frac{|\kappa(z_0)|+|\rho(z)|}{1+|\kappa(z_0)|\,|\rho(z)|}. \tag{2.7}$$

If $\varpi:D\to D$ is holomorphic function and $0<|z_0|<1$, letting

$$\kappa(z)=\frac{\varpi(z)-\varpi(0)}{z\left(1-\overline{\varpi(0)}\varpi(z)\right)}$$

in (2.7), we obtain

$$\left|\frac{\varpi(z)-\varpi(0)}{\left(1-\overline{\varpi(0)}\varpi(z)\right)}\right|\leq|z|\,\frac{\left|\frac{\varpi(z_0)-\varpi(0)}{z_0\left(1-\overline{\varpi(0)}\varpi(z_0)\right)}\right|+|\rho(z)|}{1+\left|\frac{\varpi(z_0)-\varpi(0)}{z_0\left(1-\overline{\varpi(0)}\varpi(z_0)\right)}\right||\rho(z)|}$$

and

$$|\varpi(z)| \leq \frac{|\varpi(0)| + |z| \frac{|K|+|\rho(z)|}{1+|K||\rho(z)|}}{1 + |\varpi(0)| |z| \frac{|K|+|\rho(z)|}{1+|K||\rho(z)|}}, \tag{2.8}$$

where

$$K = \frac{\varpi(z_0) - \varpi(0)}{z_0 \left(1 - \overline{\varpi(0)}\varpi(z_0)\right)}.$$

Without loss of generality, we will assume that $b = 1$. If we take

$$\varpi(z) = \frac{\varphi(z)}{z \frac{z-z_0}{1-\overline{z_0}z}},$$

then

$$\varpi(0) = \frac{\varphi'(0)}{-z_0}, \ \varpi(z_0) = \frac{\varphi'(z_0)\left(1 - |z_0|^2\right)}{z_0}$$

and

$$K = \frac{\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0} + \frac{\varphi'(0)}{z_0}}{z_0 \left(1 + \frac{\overline{\varphi'(0)}}{z_0} \frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right)},$$

where $|K| \leq 1$. Let $|\varpi(0)| = \gamma$ and

$$M = \frac{\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right| + \left|\frac{\varphi'(0)}{z_0}\right|}{|z_0| \left(1 + \left|\frac{\varphi'(0)}{z_0}\right| \left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)}.$$

From (2.8), we take

$$|\varphi(z)| \leq |z| |\rho(z)| \frac{\gamma + |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|}}{1 + \gamma |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|}}$$

and

$$\frac{1 - |\varphi(z)|}{1 - |z|} \geq \frac{1 + \gamma |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|} - \gamma |z| |\rho(z)| - |z|^2 |\rho(z)| \frac{M+|\rho(z)|}{1+M|\rho(z)|}}{(1 - |z|) \left(1 + \gamma |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|}\right)} = \Sigma.$$

Let $G(z) = 1 + \gamma |z| \frac{M+|\rho(z)|}{1+M|\rho(z)|}$ and $H(z) = 1 + M |\rho(z)|$. Then

$$\Sigma = \frac{1 - |z|^2 |\rho(z)|^2}{(1 - |z|) G(z)H(z)} + M |\rho(z)| \frac{1 - |z|^2}{(1 - |z|) H(z)G(z)} + M\gamma |z| \frac{1 - |\rho(z)|^2}{(1 - |z|) G(z)H(z)}. \tag{2.9}$$

Since

$$\lim_{z\to 1} G(z) = 1 + \gamma, \ \lim_{z\to 1} H(z) = 1 + M$$

and

$$1 - |\rho(z)|^2 = 1 - \left|\frac{z - z_0}{1 - \overline{z_0}z}\right|^2 = \frac{\left(1 - |z_0|^2\right)\left(1 - |z|^2\right)}{|1 - \overline{z_0}z|^2},$$

passing to the angular limit in (2.9) gives

$$\begin{aligned} |\varphi'(1)| &\geq \frac{2}{(1+\rho)(1+M)}\left(1+\frac{1-|z_0|^2}{|1-z_0|^2}+M+\gamma M\frac{1-|z_0|^2}{|1-z_0|^2}\right) \\ &= 1+\frac{1-|z_0|^2}{|1-z_0|^2}+\frac{1-\gamma}{1+\gamma}\left(1+\frac{1-M}{1+M}\frac{1-|z_0|^2}{|1-z_0|^2}\right). \end{aligned}$$

Moreover, since

$$\begin{aligned} \frac{1-\gamma}{1+\gamma} &= \frac{1-|\varpi(0)|}{1+|\varpi(0)|}=\frac{1-\frac{|\varphi'(0)|}{|z_0|}}{1+\frac{|\varphi'(0)|}{|z_0|}}=\frac{|z_0|-|\varphi'(0)|}{|z_0|+|\varphi'(0)|}=\frac{|z_0|-\frac{|f'(0)|}{2(1-\alpha)}}{|z_0|+\frac{|f'(0)|}{2(1-\alpha)}} \\ &= \frac{2(1-\alpha)|z_0|-|f'(0)|}{2(1-\alpha)|z_0|+|f'(0)|} \end{aligned}$$

and

$$\begin{aligned} \frac{1-M}{1+M} &= \frac{1-\dfrac{\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|+\left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)(1-|z_0|^2)}{z_0}\right|\right)}}{1+\dfrac{\left|\frac{\varphi'(z_0)(1-|z_0|^2)}{z_0}\right|+\left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)(1-|z_0|^2)}{z_0}\right|\right)}} \\ &= \frac{|z_0|\left(1+\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|\right)-\left|\frac{\varphi'(z_0)\left(1-|z_0|^2\right)}{z_0}\right|-\left|\frac{\varphi'(0)}{z_0}\right|}{|z_0|\left(1+\left|\frac{\varphi'(0)}{z_0}\right|\left|\frac{\varphi'(z_0)(1-|z_0|^2)}{z_0}\right|\right)+\left|\frac{\varphi'(z_0)(1-|z_0|^2)}{z_0}\right|+\left|\frac{\varphi'(0)}{z_0}\right|} \\ &= \frac{|z_0|\left(1+\frac{\left|f'(0)\right|}{2(1-\alpha)|z_0|}\frac{\frac{|f'(z_0)|}{2(1-\alpha)}\left(1-|z_0|^2\right)}{|z_0|}\right)-\frac{\frac{|f''(z_0)|}{2(1-\alpha)}\left(1-|z_0|^2\right)}{|z_0|}-\frac{\left|f'(0)\right|}{2(1-\alpha)|z_0|}}{|z_0|\left(1+\frac{|f'(0)|}{2(1-\alpha)|z_0|}\frac{\frac{|f'(z_0)|}{2(1-\alpha)}(1-|z_0|^2)}{|z_0|}\right)+\frac{\frac{|f''(z_0)|}{2(1-\alpha)}(1-|z_0|^2)}{|z_0|}+\frac{|f'(0)|}{2(1-\alpha)|z_0|}} \\ &= \frac{4(1-\alpha)^2|z_0|^2+|f'(0)||f'(z_0)|\left(1-|z_0|^2\right)-4(1-\alpha)\left(1-|z_0|^2\right)|f'(z_0)|-2(1-\alpha)|f'(0)|}{4(1-\alpha)^2|z_0|^2+|f'(0)||f'(z_0)|\left(1-|z_0|^2\right)+4(1-\alpha)\left(1-|z_0|^2\right)|f'(z_0)|+2(1-\alpha)|f'(0)|}, \end{aligned}$$

we obtain

$$\begin{aligned} |\varphi'(1)| \geq 1 &+ \frac{1-|z_0|^2}{|1-z_0|^2}+\frac{2(1-\alpha)|z_0|-|f'(0)|}{2(1-\alpha)|z_0|+|f'(0)|}\times \\ &\left[1+\frac{4(1-\alpha)^2|z_0|^2+|f'(0)||f'(z_0)|\left(1-|z_0|^2\right)-4(1-\alpha)\left(1-|z_0|^2\right)|f'(z_0)|-2(1-\alpha)|f'(0)|}{4(1-\alpha)^2|z_0|^2+|f'(0)||f'(z_0)|\left(1-|z_0|^2\right)+4(1-\alpha)\left(1-|z_0|^2\right)|f'(z_0)|+2(1-\alpha)|f'(0)|}\frac{1-|z_0|^2}{|1-z_0|^2}\right]. \end{aligned}$$

From the definition of $\varphi(z)$, we have

$$\varphi'(z)=2(1-\alpha)\frac{f'(z)}{(1-2\alpha+f(z))^2}$$

and

$$|\varphi'(1)|=2\frac{|f'(1)|}{1-\alpha}.$$

Thus, we obtain the inequality (2.6).

Now, we shall show that the inequality (2.6) is sharp.

Since $\varpi(z)=\frac{\varphi(z)}{z\frac{z-z_0}{1-\overline{z_0}z}}$ is holomorphic function in the unit disc and $|\varpi(z)|\leq 1$ for $|z|<1$, we obtain

$$|\varphi'(0)|\leq|z_0|$$

and

$$|\varphi'(z_0)|\leq\frac{|z_0|}{1-|z_0|^2}.$$

We take $z_0\in(-1,0)$ and arbitrary two numbers c and d, such that $0\leq c\leq 2\left(1-\alpha\right)|z_0|$, $0\leq d\leq 2\left(1-\alpha\right)\frac{|z_0|}{1-|z_0|^2}$. Let

$$\mathrm{T}=\frac{\frac{d\left(1-|z_0|^2\right)}{z_0}+\frac{c}{z_0}}{z_0\left(1+cd\frac{1-|z_0|^2}{z_0^2}\right)}=\frac{1}{z_0^2}\frac{d\left(1-|z_0|^2\right)+c}{1+cd\frac{1-|z_0|^2}{z_0^2}}.$$

The composite function

$$m(z)=z\frac{z-z_0}{1-\overline{z_0}z}\frac{-\frac{c}{z_0}+z\frac{\mathrm{T}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathrm{T}\frac{z-z_0}{1-\overline{z_0}z}}}{1-\frac{c}{z_0}z\frac{\mathrm{T}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathrm{T}\frac{z-z_0}{1-\overline{z_0}z}}}$$

is holomorphic in D and $|m(z)|<1$ for $|z|<1$.Let

$$\frac{f(z)-1}{1-2\alpha+f(z)}=z\frac{z-z_0}{1-\overline{z_0}z}\frac{-\frac{c}{z_0}+z\frac{\mathrm{T}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathrm{T}\frac{z-z_0}{1-\overline{z_0}z}}}{1-\frac{c}{z_0}z\frac{\mathrm{T}+\frac{z-z_0}{1-\overline{z_0}z}}{1+\mathrm{T}\frac{z-z_0}{1-\overline{z_0}z}}}. \tag{2.10}$$

Therefore, we take $|f'(0)|=2\left(1-\alpha\right)c$ and

$$\begin{aligned}2\left(1-\alpha\right)\frac{f'(z)}{\left(1-2\alpha+f(z)\right)^2} &= -\frac{z_0}{1-z_0^2}\frac{-\frac{c}{z_0}+\mathrm{T}z_0}{1-\frac{c}{z_0}z_0\mathrm{T}}=\frac{z_0}{1-z_0^2}\frac{-\frac{c}{z_0}+\frac{1}{z_0^2}\frac{d\left(1-|z_0|^2\right)+c}{1+cd\frac{1-|z_0|^2}{z_0^2}}z_0}{1-\frac{c}{z_0}z_0\frac{1}{z_0^2}\frac{d\left(1-|z_0|^2\right)+c}{1+cd\frac{1-|z_0|^2}{z_0^2}}}\\ |f'(z_0)| &= 2\left(1-\alpha\right)d.\end{aligned}$$

From (2.10), with the simple calculations, we obtain

$$\begin{aligned}2\frac{f'(1)}{1-\alpha} &= 1+\frac{1-z_0^2}{\left(1-z_0\right)^2}+\frac{\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{1-\mathrm{T}^2}{(1+\mathrm{T})^2}\right)\left(1-\frac{c}{z_0}\right)+\frac{c}{z_0}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{1-\mathrm{T}^2}{(1+\mathrm{T})^2}\right)\left(-\frac{c}{z_0}+1\right)}{\left(-\frac{c}{z_0}+1\right)^2}\\ &= 1+\frac{1-z_0^2}{\left(1-z_0\right)^2}+\frac{1+\frac{c}{z_0}}{1-\frac{c}{z_0}}\left(1+\frac{1-z_0^2}{\left(1-z_0\right)^2}\frac{1-\mathrm{T}}{1+\mathrm{T}}\right)\\ &= 1+\frac{1-z_0^2}{\left(1-z_0\right)^2}+\frac{c+z_0}{-c+z_0}\left(1+\frac{1-z_0^2}{\left(1-z_0\right)^2}\frac{z_0^2+cd\left(1-z_0^2\right)-d\left(1-z_0^2\right)-c}{z_0^2+cd\left(1-z_0^2\right)+d\left(1-z_0^2\right)+c}\right)\end{aligned}$$

and

$$|f'(1)| = \frac{1-\alpha}{2}\left[1+\frac{1-z_0^2}{(1-z_0)^2}+\frac{c+z_0}{-c+z_0}\left(1+\frac{1-z_0^2}{(1-z_0)^2}\frac{z_0^2+cd\,(1-z_0^2)-d\,(1-z_0^2)-c}{z_0^2+cd\,(1-z_0^2)+d\,(1-z_0^2)+c}\right)\right].$$

Since $z_0 \in (-1,0)$, the last equality show that (2.6) is sharp. □

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