HOMOMORPHISM OF CUTS OF MULTIGROUPS

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ABSTRACT. The notion of multigroups via multisets is of interest in non-classical groups. In this paper, we propose the notion of homomorphism of cuts of multigroups and obtain some related results. Some homomorphic properties of upper cut of multigroups are also discussed.

1. INTRODUCTION

In crisp set theory, repetition of elements is not allowed in a collection. The theory of groups was developed from this point of view as the algebraic structure of sets. Matching the notion of set with real-life situations, repetition of objects cannot be ignored. Sequel to this, the term multiset was first suggested by N.G. De Bruijn to Knuth as the generalization of crisp set theory (that is, in a multiset, elements are allowed to repeat)[14]. With this, one can conveniently say that, every set is a multiset but the reverse is not true [6]. Elaborate work on multiset and its applications can be found in [7, 24, 25, 26].

Since multiset is the generalization of set, it is then natural to generalize group as multigroup. The idea of multigroups was proposed in [17] as an algebraic structure of multisets that generalized groups. The notion is consistent with other non-classical groups in [1, 5, 16, 18, 20, 22, 23], etc.

Although other researchers in [4, 8, 15, 19, 21] earlier used the term multigroup as an extension of group theory (with each of them having a divergent view), the notion of multigroup via multiset in [17] is quite acceptable because it is in consonant with the aforementioned non-classical groups. Further studies on multigroups via multisets can be found in [2, 3, 13].

In [9], the algebraic perspective of n-level sets of multisets discussed in [12, 17] was proposed and called, cuts of multigroups. This paper introduce the notion of homomorphism of cuts of multigroups and deduce some results. Some homomorphic properties of upper cut of multigroups are also discussed.

2. PRELIMINARIES

In this section, we review some definitions and results for the sake of completeness and reference.

Definition 2.1 ([24]). Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A, that is, a counting function of A (where $C_A(x) = 0$, implies $x \notin Dom(A)$). The set X is called the ground or generic set of the class of all multisets (for short, msets) containing objects from X.

We denote the set of all multisets by $MS(X)$.

Definition 2.2 ([25]). Let A and B be two multisets over X, then A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x) \forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.

Definition 2.3 ([11]). Let $\{A_i\}_{i \in I}$ be a family of multisets over X. Then

    (i) $C_{\bigcap_{i \in I} A_i}(x) = \bigwedge_{i \in I} C_{A_i}(x) \forall x \in X$.
    (ii) $C_{\bigcup_{i \in I} A_i}(x) = \bigvee_{i \in I} C_{A_i}(x) \forall x \in X$.

Definition 2.4 ([26]). Let $A, B \in MS(X)$. Then A and B are comparable to each other if $A \subseteq B$ or $B \subseteq A$.

Definition 2.5 ([17]). Let X be a group. A multiset G over X is called a multigroup of X if the count function of G, that is, $C_G : X \to \mathbb{N}$ satisfies the following conditions:

    (i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$,
    (ii) $C_G(x^{-1}) = C_G(x) \forall x \in X$.

By implication, a multiset G over X is called a multigroup of a group X if

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y), \forall x, y \in X.$$

It follows immediately from the definition that,

$$C_G(e) \geq C_G(x) \forall x \in X,$$

where e is the identity element of X. We denote the set of all multigroups of X by $MG(X)$.

Definition 2.6 ([9]). Let $\{A_i\}_{i \in I}, I = 1, ..., n$ be an arbitrary family of multigroups of X. Then $\{A_i\}_{i \in I}$ is said to have inf/sup assuming chain if either $A_1 \subseteq A_2 \subseteq ... \subseteq A_n$ or $A_1 \supseteq A_2 \supseteq ... \supseteq A_n$, respectively.

Remark 2.7 ([9]). Every multigroup is a multiset but the converse is not necessarily true.

Definition 2.8 ([9]). Let $A \in MG(X)$. Then the sets $A_{[n]}$ and $A_{(n)}$ defined by

$$A_{[n]} = \{x \in X \mid C_A(x) \geq n, n \in \mathbb{N}\}$$

and

$$A_{(n)} = \{x \in X \mid C_A(x) > n, n \in \mathbb{N}\}$$

are called strong and weak upper cuts of A respectively.

Definition 2.9 ([9]). Let $A \in MG(X)$. Then the sets $A^{[n]}$ and $A^{(n)}$ defined by

$$A^{[n]} = \{x \in X \mid C_A(x) \leq n, n \in \mathbb{N}\}$$

and

$$A^{(n)} = \{x \in X \mid C_A(x) < n, n \in \mathbb{N}\}$$

are called strong and weak lower cuts of A respectively.

Remark 2.10 ([9]). We observe that $A_{[n]} = B_{[n]}, A_{(n)} = B_{(n)}, A^{[n]} = B^{[n]}$, and $A^{(n)} = B^{(n)}$ iff $A = B$. Also, $A_{(n)} \subseteq A_{[n]}$ and $A^{(n)} \subseteq A^{[n]}$.

Theorem 2.11 ([9]). *Let $A, B \in MG(X)$. For all $n \in \mathbb{N}$, if $A \subseteq B$, then $A_{[n]} \subseteq B_{[n]}$ and $A^{[n]} \subseteq B^{[n]}$.*

Theorem 2.12 ([9]). *Let $A \in MG(X)$. For all $n_1, n_2 \in \mathbb{N}$ and $n_1 \leq n_2$, then $A_{(n_2)} \subseteq A_{[n_2]} \subseteq A_{(n_1)}$ and $A^{(n_1)} \subseteq A^{(n_2)} \subseteq A^{[n_2]}$.*

Theorem 2.13 ([9]). *Let $A \in MG(X)$. Then $A_{[n]}, n \in \mathbb{N}$ is a subgroup of X for $n \leq C_A(e)$ and $A^{[n]}, n \in \mathbb{N}$ is a subgroup of X for $n \geq C_A(e)$, where e is the identity element of X.*

Definition 2.14 ([10]). Let X and Y be groups and let $f : X \to Y$ be a homomorphism. If A and B be multigroups of X and Y, respectively. Then
   (i) the image of A under f, denoted by $f(A)$, is a multigroup of Y defined by

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

      for each $y \in Y$.
   (ii) the inverse image of B under f, denoted by $f^{-1}(B)$, is a multigroup of X defined by

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \forall x \in X.$$

Theorem 2.15 ([10]). *Let f be a homomorphic mapping from a group X onto a group Y.*
   (i) *For $A, B \in MG(X)$, if $A \subseteq B$, then $f(A) \subseteq f(B)$.*
   (ii) *For $A, B \in MG(Y)$, if $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.*

Proposition 2.16 ([17]). *Let X, Y be two groups and $f : X \to Y$ be a homomorphism. If $A \in MG(X)$ and $B \in MG(Y)$, respectively, then $f(A) \in MG(Y)$ and $f^{-1}(B) \in MG(X)$.*

3. Homomorphism of upper and lower cuts of multigroups

Definition 3.1. Let X, Y be groups, $A \in MG(X)$, $B \in MG(Y)$ and $f : X \to Y$ be a homomorphic mapping. Then for any $n \in \mathbb{N}$, if f is a homomorphic mapping from $A_{[n]}$ to $B_{[n]}$, f is called upper cut homomorphic mapping from A to B.

Definition 3.2. Let X, Y be groups, $A \in MG(X)$, $B \in MG(Y)$ and $f : X \to Y$ be a homomorphic mapping. Then for at least one $n \in \mathbb{N}$, if f is a homomorphic mapping from $A^{[n]}$ to $B^{[n]}$, f is called lower cut homomorphic mapping from A to B.

Example 3.3. Let $X = \{0, 1, 2, 3\}$ be a group of $\mathbb{Z}_4, +$ and $Y = \{0, 2, 4, 6\} \subseteq \mathbb{Z}_8, +$. Then $\exists$ a homomorphism $f : X \to Y$ defined by $f(x) = 2x \ \forall x \in X$. Suppose that A and B are multigroups of X and Y, respectively, given as

$$A = [0^4, 1^3, 2^2, 3^3] \text{ and}$$
$$B = [0^4, 2^3, 4^2, 6^3].$$

Then

$$f(A) = [0^4, 2^3, 4^2, 6^3] \text{ and}$$
$$f^{-1}(B) = [0^4, 1^3, 2^2, 3^3]$$

satisfying

$$C_{f(A)}(y) = C_A(f^{-1}(y)) \text{ and } C_{f^{-1}(B)}(x) = C_B(f(x)) \ \forall x \in X \text{ and } \forall y \in Y.$$

For $n = 1, 2, 3, 4$, we get

$$A_{[1]} = A_{[2]} = \{0, 1, 2, 3\}, A_{[3]} = \{0, 1, 3\}, A_{[4]} = \{0\}$$

and

$$B_{[1]} = B_{[2]} = \{0, 2, 4, 6\}, B_{[3]} = \{0, 2, 6\}, B_{[4]} = \{0\}.$$

Since $\exists$ a homomorphism f from $A_{[n]}$ to $B_{[n]}$ defined by $f(x) = 2x \ \forall x \in X$ for $n = 1, 2, 3, 4$, so f is an upper cut homomorphic mapping from A to B.

Similarly,

$$A^{[1]} = \{\}, A^{[2]} = \{2\}, A^{[3]} = \{1, 2, 3\}, A^{[4]} = \{0, 1, 2, 3\}$$

and

$$B^{[1]} = \{\}, A^{[2]} = \{4\}, A^{[3]} = \{2, 4, 6\}, A^{[4]} = \{0, 2, 4, 6\}.$$

So $\exists f : A^{[n]} \to B^{[n]}$ for only $n = 4$ defined by $f(x) = 2x \ \forall x \in X$. Consequently, f is a lower cut homomorphism from A to B.

Proposition 3.4. *Let $f : X \to Y$ be a homomorphism, $A \in MG(X)$ and $B \in MG(Y)$, respectively. For any $n \in \mathbb{N}$, we have*

(i) $f(A_{[n]}) \subseteq (f(A))_{[n]}$,
(ii) $f^{-1}(B_{[n]}) = (f^{-1}(B))_{[n]}$,
(iii) $f(A_{(n)}) \subseteq f(A_{[n]}) \subseteq (f(A))_{[n]}$,
(iv) $f^{-1}(B_{(n)}) \subseteq f^{-1}(B_{[n]}) = (f^{-1}(B))_{[n]}$.

Proof. (i) Let $y \in f(A_{[n]})$, then $\exists x \in A_{[n]}$ such that $f(x) = y$ and

$$C_A(x) \geq n, n \in \mathbb{N}.$$

Consequently, we get

$$C_A(f^{-1}(y)) \geq n, n \in \mathbb{N} \text{ implies } C_{f(A)}(y) \geq n, n \in \mathbb{N}$$

and so, $y \in (f(A))_{[n]}$. Hence, $f(A_{[n]}) \subseteq (f(A))_{[n]}$.

(ii) For every x, $x \in f^{-1}(B_{[n]}) \Leftrightarrow f(x) \in B_{[n]} \Leftrightarrow C_B(f(x)) \geq n, n \in \mathbb{N}$. By Definition 2.8 and Definition 2.14, we see that

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \geq n, n \in \mathbb{N},$$

that is, $x \in (f^{-1}(B))_{[n]}$. Hence, $f^{-1}(B_{[n]}) = (f^{-1}(B))_{[n]}$.

(iii) Since $A_{(n)} \subseteq A_{[n]}$ by Remark 2.10, then $f(A_{(n)}) \subseteq f(A_{[n]})$ by Theorem 2.15. Hence, the result follows from (i).

(iv) Also, $B_{(n)} \subseteq B_{[n]}$ and so, $f^{-1}(A_{(n)}) \subseteq f^{-1}(A_{[n]})$ by the same reasons as in (iii). The proof is completed by (ii). $\qquad\square$

Corollary 3.5. *Let $f : X \to Y$ be a homomorphism. Suppose $A \in MG(X)$ and $B \in MG(Y)$, respectively, then for at least one $n \in \mathbb{N}$,*

(i) $f(A^{[n]}) \subseteq (f(A))^{[n]}$,
(ii) $f^{-1}(B^{[n]}) = (f^{-1}(B))^{[n]}$,
(iii) $f(A^{(n)}) \subseteq f(A^{[n]}) \subseteq (f(A))^{[n]}$,
(iv) $f^{-1}(B^{(n)}) \subseteq f^{-1}(B^{[n]}) = (f^{-1}(B))^{[n]}$.

Proof. Similar to Proposition 3.4. $\qquad\square$

Theorem 3.6. *Let X, Y be groups, $A \in MG(X), B \in MG(Y)$ and $f : X \to Y$, respectively. Then $(f(A))_{[n]} = f(A_{[n]})$ for any $n \in \mathbb{N}$ if and only if for each $y \in Y$ there exists $x_0 \in f^{-1}(y)$ such that $C_{f(A)}(y) = C_A(x_0)$.*

Proof. Suppose $(f(A))_{[n]} = f(A_{[n]})$. For arbitrary $y \in Y$, let $C_{f(A)}(y) = n$, then $y \in (f(A))_{[n]} = f(A_{[n]})$. It follows that there exist $x_0 \in A_{[n]}$ such that $y = f(x_0)$. Hence, we have $x_0 \in f^{-1}(y)$ which satisfies $C_A(x_0) \geq n$. Consequently, we have

$$C_A(x_0) \geq C_{f(A)}(y) = \bigvee_{x \in f^{-1}(y)} C_A(x) \geq C_A(x_0).$$

Therefore, $C_{f(A)}(y) = C_A(x_0)$.

Conversely, for each $y \in Y \, \exists \, x_0 \in f^{-1}(y)$ such that $C_{f(A)}(y) = C_A(x_0)$. For $n \in \mathbb{N}$, let $y \in (f(A))_{[n]}$. We show that $y \in f(A_{[n]})$. Since

$$C_A(x_0) = C_{f(A)}(y) = \bigvee_{x \in f^{-1}(y)} C_A(x) \geq n,$$

we have $f(x_0) = y$ and so $x_0 \in A_{[n]}$ implies $y \in f(A_{[n]})$. Hence, $(f(A))_{[n]} = f(A_{[n]})$. $\qquad\square$

Corollary 3.7. *Let $f : X \to Y$ be a homomorphic mapping, $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then $(f(A))^{[n]} = f(A^{[n]})$ for at least one $n \in \mathbb{N}$ if and only if for each $y \in Y$ there exists $x_0 \in f^{-1}(y)$ such that $C_{f(A)}(y) = C_A(x_0)$.*

Proof. Similar to Theorem 3.6. $\qquad\square$

Definition 3.8. Let X, Y be groups, $f : X \to Y$ and $A \in MG(X)$. Then for every $y \in Y$, if there exists $x_0 \in f^{-1}(y)$ such that $C_{f(A)}(y) = C_A(x_0)$, then f is said to be quasi-surjective.

Lemma 3.9. *Let X, Y be groups, $f : X \to Y$ and $A \in MG(X)$. Then for at least one $n \in \mathbb{N}$, we have $(f(A))_{[n]} = f(A_{[n]})$ or $(f(A))^{[n]} = f(A^{[n]})$ if and only if f is quasi-surjective.*

Proof. Combining Theorem 3.6, Corollary 3.7 and Definition 3.8, the result follows. $\qquad\square$

Theorem 3.10. *Let X, Y be groups, $A \in MG(X), B \in MG(Y)$ and $f : X \to Y$ be quasi-surjective. Then f is an upper cut homomorphic mapping from A to B if and only if f is a homomorphic mapping from X to Y, and $(f(A))_{[n]} \subseteq B_{[n]}$ for any $n \in \mathbb{N}$.*

Proof. Suppose f is an upper cut homomorphic mapping from A to B. Then for every $n \in \mathbb{N}$, we can infer that f is a homomorphic mapping from $A_{[n]}$ to $B_{[n]}$. Actually, $X = A_{[1]}, Y = B_{[1]}$, thus f is an homomorphic mapping from X to Y. As f is quasi-surjective, then in light of Lemma 3.9, we get

$$(f(A))_{[n]} = f(A_{[n]}) \subseteq B_{[n]}.$$

Conversely, suppose f is a homomorphic mapping from X to Y and $(f(A))_{[n]} \subseteq B_{[n]}$ for $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, because f is quasi-surjective, for any $x \in A_{[n]} \subseteq X$, we have

$$f(x) \in f(A_{[n]}) = (f(A))_{[n]} \subseteq B_{[n]}.$$

Therefore, f is a homomorphism from $A_{[n]}$ to $B_{[n]}$. Since f is a homomorphic mapping from X to Y, then

$$f(xy) = f(x)f(y) \text{ holds for arbitrary } x, y \in A_{[n]} \subseteq X,$$

where $f(x), f(y) \in B_{[n]}$. According to Theorem 2.13, this indicates that $B_{[n]}$ is a subgroup of Y. Hence, $f(x)f(y) \in B_{[n]}$, that is to say, f preserves the operation. Synthesizing this discussion, f is a homomorphic mapping from $A_{[n]}$ to $B_{[n]}$. Hence, by Definition 3.1, we obtain that f is an upper cut homomorphic mapping from A to B. □

Corollary 3.11. *Let X, Y be groups such that $f : X \to Y$ is quasi-surjective, $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then f is a lower cut homomorphic mapping from A to B if and only if f is a homomorphic mapping from X to Y, and $(f(A))^{[n]} \subseteq B^{[n]}$ for at least one $n \in \mathbb{N}$.*

Proof. Similar to Theorem 3.10. □

Definition 3.12. Let X and Y be groups, $f : X \to Y$, $A \in MG(X)$ and $B \in MG(Y)$, respectively. For any $n \in \mathbb{N}$, if f is a surjective homomorphic mapping from $A_{[n]}$ to $B_{[n]}$, then f is called a surjective upper cut homomorphic mapping from A to B. Moreover, A and B are upper cuts homomorphic with respect to f.

Theorem 3.13. *Let X and Y be groups, A and B be multigroups of X and Y, respectively, and $f : X \to Y$ with f quasi-surjective. Then f is a surjective upper cut homomorphic mapping from A to B if and only if f is a surjective homomophic mapping from X to Y with $f(A_{[n]}) = B_{[n]}$ for any $n \in \mathbb{N}$.*

Proof. Suppose f is a surjective upper cut homomorphic mapping from A to B. Then by Definition 3.12, for any $n \in \mathbb{N}$ it follows that f is a surjective homomorphic mapping from $A_{[n]}$ to $B_{[n]}$. Observe that, $X = A_{[1]}$ and $Y = B_{[1]}$, evidently, f is a surjective homomorphic mapping from X to Y. Clearly, we have $f(A_{[n]}) \subseteq B_{[n]}$. Similarly, $B_{[n]} \subseteq f(A_{[n]})$ is obvious. Hence, $f(A_{[n]}) = B_{[n]}$.

Conversely, for any $n \in \mathbb{N}$ and $y \in B_{[n]}$, then $f(A_{[n]}) = B_{[n]}$ implies that $\exists\, x \in A_{[n]}$ such that $f(x) = y$, that is, f is a surjection from $A_{[n]}$ to $B_{[n]}$. To prove that f preserves the operation follows from the converse proof of Theorem 3.10, so we omit it. Hence, for any $n \in \mathbb{N}$, it follows that f is a surjective upper cut homomorphic mapping from A to B. $\square$

Definition 3.14. Let X and Y be groups, $f : X \to Y$, $A \in MG(X)$ and $B \in MG(Y)$, respectively. For at least one $n \in \mathbb{N}$, if f is a surjective homomorphic mapping from $A^{[n]}$ to $B^{[n]}$, then f is called a surjective lower cut homomorphic mapping from A to B. Moreover, A and B are lower cuts homomorphic with respect to f.

Corollary 3.15. *Let X and Y be groups, A and B be multigroups of X and Y, respectively, and $f : X \to Y$ with f quasi-surjective. Then f is a surjective lower cut homomorphic mapping from A to B if and only if f is a surjective homomophic mapping from X to Y with $f(A^{[n]}) = B^{[n]}$ for at least one $n \in \mathbb{N}$.*

Proof. Similar to Theorem 3.13. $\square$

Definition 3.16. Let X and Y be groups, $A \in MG(X)$, $B \in MG(Y)$ and $f : X \to Y$. Then for any $n \in \mathbb{N}$, if f is an isomorphic mapping from $A_{[n]}$ to $B_{[n]}$, then f is called an upper cut isomorphic mapping from A to B. In particular, A and B are said to be upper cuts isomorphic with respect to f.

Theorem 3.17. *Let X and Y be groups, $A \in MG(X), B \in MG(Y)$ and $f : X \to Y$ with f quasi-surjective. Then f is an upper cut isomorphic mapping from A to B if and only if f is an isomorphic mapping from X to Y with $f(A_{[n]}) = B_{[n]}$ for any $n \in \mathbb{N}$.*

Proof. The Proof follows by combining Theorem 3.10 and Theorem 3.13. $\square$

Definition 3.18. Let $f : X \to Y$ be homomorphism, $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then for at least one $n \in \mathbb{N}$, if f is an isomorphic mapping from $A^{[n]}$ to $B^{[n]}$, then f is called a lower cut isomorphic mapping from A to B. In particular, A and B are said to be lower cuts isomorphic with respect to f.

Corollary 3.19. *Let $f : X \to Y$ be homomorphism with f quasi-surjective, $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then f is a lower cut isomorphic mapping from A to B if and only if f is an isomorphic mapping from X to Y with $f(A^{[n]}) = B^{[n]}$ for at least one $n \in \mathbb{N}$.*

Proof. The Proof follows by combining Corollary 3.11 and Corollary 3.15. $\square$

Theorem 3.20. *Let $f : X \to Y$ be an isomorphism, $A \in MG(X)$ and $B \in MG(Y)$. Then $f(A_{[n]}) \in Y$ and $f^{-1}(B_{[n]}) \in X$ for all $n \leq (C_A(e), C_B(e'))$, where e, e' are the identities of X and Y, respectively.*

Proof. By Theorem 2.13, it is clear that $A_{[n]}$ is a subgroup of X. We show that $f(A_{[n]})$ is a subgroup of Y. Let $y_1, y_2 \in f(A_{[n]})$ be any two elements, then $C_{f(A)}(y_1) \geq n$ and $C_{f(A)}(y_2) \geq n$. By Proposition 3.4, $f(A_{[n]}) \subseteq (f(A))_{[n]}$, $n \in \mathbb{N}$. So, $\exists\, x_1, x_2 \in X$ such that

$$C_A(x_1) = C_{f(A)}(y_1) \geq n \text{ and } C_A(x_2) = C_{f(A)}(y_2) \geq n$$

imply

$$C_A(x_1) \geq n \text{ and } C_A(x_2) \geq n.$$

Then,

$$C_A(x_1) \wedge C_A(x_2) \geq n.$$

Again, $C_A(x_1 x_2^{-1}) \geq C_A(x_1) \wedge C_A(x_2) \geq n \Rightarrow C_A(x_1 x_2) \geq n$. Hence, $x_1 x_2^{-1} \in A_{[n]}$
$\Leftrightarrow f(x_1 x_2^{-1}) \in f(A_{[n]}) \subseteq (f(A))_{[n]}$
$\Leftrightarrow f(x_1) f(x_2^{-1}) \in (f(A))_{[n]} = f(x_1)(f(x_2))^{-1} \in (f(A))_{[n]}$
$\Leftrightarrow y_1 y_2^{-1} \in (f(A))_{[n]}$. Therefore, $f(A_{[n]}) \in Y$.
The proof of $f^{-1}(B_{[n]}) \in X$ is similar. $\qquad\square$

Corollary 3.21. *Let* $f : X \to Y$ *be an isomorphism,* $A \in MG(X)$ *and* $B \in MG(Y)$. *Then* $f(A^{[n]})$ *is a subgroup of* Y *and* $f^{-1}(B^{[n]})$ *is a subgroup of* X *for all* $n \geq (C_A(e), C_B(e'))$, *where* e, e' *are the identities of* X *and* Y, *respectively.*

Proof. Similar to Theorem 3.20. $\qquad\square$

Corollary 3.22. *If* $f : X \to Y$ *be homomorphism of group* X *onto group* Y *and* $\{A_i\}_{i \in I}$ *be family of multigroups of* X. *Then for all* $n \leq (C_{A_i}(e), C_{B_i}(e'))$, *where* e, e' *are the identities of* X *and* Y, *respectively,*

    (i) $f(\bigcap_{i \in I} A_{i[n]})$ *is a subgroup of* Y.
    (ii) $f^{-1}(\bigcap_{i \in I} B_{i[n]})$ *is a subgroup of* X.
    (iii) $f(\bigcup_{i \in I} A_{i[n]})$ *is a subgroup of* Y *provided* $\{A_i\}_{i \in I}$ *have sup/inf assuming chain.*
    (iv) $f^{-1}(\bigcup_{i \in I} B_{i[n]})$ *is a subgroup of* X *provided* $\{B_i\}_{i \in I}$ *have sup/inf assuming chain.*

Proof. Similar to Theorem 3.20. $\qquad\square$

Corollary 3.23. *If* $f : X \to Y$ *be homomorphism of group* X *onto group* Y *and* $\{A_i\}_{i \in I}$ *be family of multigroups of* X. *Then for all* $n \geq (C_{A_i}(e), C_{B_i}(e'))$, *where* e, e' *are the identities of* X *and* Y, *respectively,*

    (i) $f(\bigcap_{i \in I} A_i^{[n]})$ *is a subgroup of* Y.
    (ii) $f^{-1}(\bigcap_{i \in I} B_i^{[n]})$ *is a subgroup of* X.
    (iii) $f(\bigcup_{i \in I} A_i^{[n]})$ *is a subgroup of* Y *provided* $\{A_i\}_{i \in I}$ *have sup/inf assuming chain.*
    (iv) $f^{-1}(\bigcup_{i \in I} B_i^{[n]})$ *is a subgroup of* X *provided* $\{B_i\}_{i \in I}$ *have sup/inf assuming chain.*

Proof. Similar to Theorem 3.20. $\qquad\square$

4. Upper cut homomorphic properties of multigroups

This section focuses on upper cut homomorphic properties of multigroups. We define a pre-surjective mapping f, introduce analogous concept of nested set and obtain some results.

Definition 4.1. Let X, Y be sets, $f : X \to Y$ be a mapping and $A \in MS(X)$, respectively. If for every $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$ we have $(f(A))_{[n_2]} \subseteq f(A_{[n_1]})$, then f is called pre-surjective, or it is said that f posses the pre-surjective property.

Theorem 4.2. *Let $f : X \to Y$ be a mapping, $A \in MS(X)$ and $n_1, n_2 \in \mathbb{N}$ satisfies $n_1 < n_2$. Then f is pre-surjective if and only if for every $y \in (f(A))_{[n_2]}$ there exists $x \in A_{[n_1]}$ such that $f(x) = y$.*

Proof. By hypothesis and Definition 4.1, it follows that f is pre-surjective $\Leftrightarrow$ $(f(A))_{[n_2]} \subseteq f(A_{[n_1]}) \Leftrightarrow y \in (f(A))_{[n_2]} \Rightarrow y \in f(A_{[n_1]}) \Leftrightarrow \exists\, x \in A_{[n_1]}$ such that $f(x) = y$. $\qquad\qquad\square$

Definition 4.3. Let $h : N \to P(X), n \mapsto h(n) \in P(X)$ be a mapping, T be an index set. Then the mapping h is called a nested set on X, if the following conditions are satisfied.

(i) $n_1 < n_2 \Rightarrow h(n_2) \subseteq h(n_1)$,
(ii) $\bigcap_{t \in T} h(n_t) \subseteq \bigcap \{h(n) \mid n < \bigvee_{t \in T} n_t\}$.

We depicts the sets that posses such conditions on X by $N(X)$.

Theorem 4.4. *Let $f : X \to Y$ be a mapping, $A \in MS(X)$ and for all $n \in \mathbb{N}$, let $h(n) = f(A_{[n]})$. Then $h \in N(Y)$ if and only if f is pre-surjective.*

Proof. Suppose $h \in N(Y)$. In order to prove hat f is pre-surjective, we only need to show that $(f(A))_{[n_2]} \subseteq f(A_{[n_1]})$, where $n_1, n_2 \in \mathbb{N}$ and $n_1 < n_2$. In fact, for any $y \in (f(A))_{[n_2]}$, we get

$$C_{f(A)}(y) = \bigvee_{x \in f^{-1}(y)} C_A(x) \geq n_2.$$

Putting $T = \{t \in T \mid f(t) = y\}$ and $C_A(t) = n_t$, we have

$$\bigvee_{t \in T} n_t = C_{f(A)}(y) \geq n_2.$$

For $t \in T$, we have $t \in A_{[n]}$ with $y = f(t)$, thus $y \in f(A_{[n_t]})$.
Since the mapping h is an analogous of nested set on Y, by Definition 4.3, it is straightforward to get

$$y \in \bigcap_{t \in T} f(A_{[n_t]}) \subseteq \{f(A_{[n]}) \mid \bigvee_{t \in T} n_t > n\}.$$

Considering $n_1 < n_2 \leqslant \bigvee_{t \in T} n_t$, we infer that $y \in f(A_{[n_1]})$, which implies $(f(A))_{[n_2]} \subseteq f(A_{[n_1]})$. In the light of Definition 4.1, f has pre-surjective property.

Conversely, suppose f is pre-surjective. On the one hand, whenever $n_1 < n_2$, by using Theorem 2.11, Theorem 2.12 and Theorem 2.15, $f(A_{[n_2]}) \subseteq f(A_{[n_1]})$ is clear.
On the other hand, for any $y \in \bigcap_{t \in T} f(A_{[n_t]})\ \exists\, x_t \in A_{[n_t]}$ such that $f(x_t) = y$. Consequently, for arbitrary $t \in T$, we get

$$C_{f(A)}(y) \geq \bigvee_{t \in T} C_A(x_t) \geq \bigvee_{t \in T} n_t.$$

Therefore, $y \in (f(A))_{[\bigvee_{t \in T} n_t]}$.

As f is pre-surjective, for $n < \bigvee_{t \in T} n_t$, by Theorem 4.2, we deduce that there exists $x \in A_{[n]}$ such that $f(x) = y$. This implies that $y \in f(A_{[n]})$, thus

$$y \in \bigcap \{f(A_{[n]}) \mid n < \bigvee_{t \in T} n_t\}.$$

Hence,

$$\bigcap_{t \in T} f(A_{[n]}) \subseteq \bigcap \{f(A_{[n]}) \mid n < \bigvee_{t \in T} n_t\}.$$

By Definition 4.3, $h \in N(Y)$.   □

Corollary 4.5. *Let $f : X \to Y$ be a mapping and $A \in MS(X)$. For every $n \in \mathbb{N}$, we define $h(n) = f(A_{[n]})$, then $h \in N(X)$ if and only if $(f(A))_{[n]} \subseteq f(A_{[n]})$.*

Proof. Take any $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$. By Theorem 2.11, Theorem 2.12 and Theorem 2.15, we obtain

$$f(A_{[n_2]}) \subseteq f(A_{[n_1]}).$$

Combined with

$$(f(A))_{[n_2]} \subseteq f(A_{[n_2]}),$$

we get

$$(f(A))_{[n_2]} \subseteq f(A_{[n_1]}),$$

that is, f is pre-surjective. By Theorem 4.4, it follows that $h \in N(X)$.

Conversely, suppose $h \in N(X)$. The proof follows by adopting a similar method to the proof of the necessity part of Theorem 4.4.   □

Corollary 4.6. *Let $f : X \to Y$ be a quasi-surjective mapping, $A \in MS(X)$ and $h(n) = f(A_{[n]})$ for any $n \in \mathbb{N}$. Then $h \in N(X)$.*

Proof. Combining Lemma 3.9 and Corollary 4.5, the result follows.   □

Theorem 4.7. *Let X be a set and $A \in MS(X)$. Then for arbitrary $n_t \in \mathbb{N}$, $t \in T$,*

$$\bigcap_{t \in T} A_{[n_t]} = A_{[\bigvee_{t \in T} n_t]}.$$

Proof. For any $x \in \bigcap_{t \in T} A_{[n_t]}$, we have $x \in A_{[n_t]} \forall t \in T$, then we get $C_A(x) \geq n_t$. Consequently,

$$C_A(x) = \bigvee_{t \in T} C_A(x) \geq \bigvee_{t \in T} n_t.$$

Hence, $x \in A_{[\bigvee_{t \in T} n_t]}$, that is,

$$\bigcap_{t \in T} A_{[n_t]} \subseteq A_{[\bigvee_{t \in T} n_t]}.$$

Again, for all $x \in A_{[\bigvee_{t \in T} n_t]}$ and $t \in T$, we get

$$C_A(x) \geq \bigvee_{t \in T} n_t \geq n_t.$$

This implies that $C_A(x) \geq n_t$, that is, $x \in \bigcap_{t \in T} A_{[n_t]}$. Hence,

$$A_{[\bigvee_{t \in T} n_t]} \subseteq \bigcap_{t \in T} A_{[n_t]}.$$

Therefore, $\bigcap_{t \in T} A_{[n_t]} = A_{[\bigvee_{t \in T} n_t]}$. □

Theorem 4.8. *Let X and Y be groups, $f : X \to Y$ be a homomorphism, where f is quasi-surjective, $A \in MG(X)$ and $B \in MG(Y)$. If f is an upper cut homomorphic mapping from A to B with the pre-surjective property, then $f(A) \in MG(Y)$, and f is also an upper cut homomorphic mapping from A to $f(A)$.*

Proof. As f is pre-surjective, it follows that $h \in N(Y)$, where $h(n) = f(A_{[n]})$ for every $n \in \mathbb{N}$. Since f is quasi-surjective, by Lemma 3.9, for arbitrary $n \in N$, we have $(f(A))_{[n]} = f(A_{[n]})$. So, $f(A) \in MG(Y)$. In addition, for $A \in MG(X)$, it is clear that $A_{[n]}$ is a subgroup of X by Theorem 2.13. Consequently, $f(A_{[n]})$ is a subgroup of Y. Since

$$(f(A))_{[n]} = f(A_{[n]}) \Rightarrow (f(A))_{[n]}$$

is a subgroup of Y. Hence, $f(A) \in MG(Y)$.

Again, since f is an upper cut homomorphic mapping from A and B, we see that f is a homomorphism from $A_{[n]}$ to $B_{[n]}$. Then for every $x, y \in A_{[n]}$, it is clear that $f(x), f(y) \in f(A_{[n]})$. By Theorem 3.6 and Theorem 3.10,

$$f(A_{[n]}) = (f(A))_{[n]} \subseteq B_{[n]}.$$

Since f is a homomorphism from X to Y, $f(xy) = f(x)f(y)$ and $(f(A))_{[n]}$ is a subgroup of Y, we have $f(x)f(y) \in (f(A))_{[n]}$, that is, f preserves the operation. Consequently, f is an upper cut homomorphism from A to $f(A)$. □

Corollary 4.9. *Let X and Y be groups, $f : X \to Y$ be a homomorphism, where f is quasi-surjective, $A \in MG(X)$ and $B \in MG(Y)$. If f is an upper cut homomorphic mapping from A to B with the pre-surjective property, then $f^{-1}(B) \in MG(X)$, and f is also a surjective homomorphism from $f^{-1}(B)$ to B.*

Proof. For any $n \in \mathbb{N}$, setting $h(n) = f^{-1}(B_{[n]})$, we show that $h \in N(X)$. Given $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$, by Theorem 2.12 and Theorem 2.15, we get

$$f^{-1}(B_{[n_2]}) \subseteq f^{-1}(B_{[n_1]}).$$

By Proposition 3.4 and the properties of $B_{[n]}$, we get

$$\bigcap_{t \in T} f^{-1}(B_{[n_t]}) = \bigcap_{t \in T} (f^{-1}(B))_{[n_t]}.$$

By Theorem 4.7, it follows that

$$\bigcap_{t \in T} (f^{-1}(B))_{[n_t]} = (f^{-1}(B))_{[\bigvee_{t \in T} n_t]}.$$

Since

$$(f^{-1}(B))_{[\bigvee_{t \in T} n_t]} = f^{-1}(B_{[\bigvee_{t \in T} n_t]}),$$

it is obvious that

$$\bigcap_{t\in T} f^{-1}(B_{[n_t]}) \subseteq \{f^{-1}(B_{[n]}) \mid n < \bigvee_{t\in T} n_t\}.$$

Hence, $h \in N(X)$. Since $(f^{-1}(B))_{[n]} \neq \emptyset$, for every $n \in \mathbb{N}$, then for any $x, y \in (f^{-1}(B))_{[n]} = f^{-1}(B_{[n]})$, there exists $x_0, y_0 \in B_{[n]}$ such that $f(x) = x_0$ and $f(y) = y_0$. As f is upper cut homomorphism from A to B, and $B_{[n]}$ is a subgroup of Y, we infer immediately that

$$f(xy^{-1}) = f(x)(f(y))^{-1} = x_0 y_0^{-1} \in B_{[n]}.$$

This implies that

$$xy^{-1} \in f^{-1}(B_{[n]}) = (f^{-1}(B))_{[n]},$$

and so, $(f^{-1}(B))_{[n]}$ is a subgroup of X. Hence, $f^{-1}(B) \in MG(X)$. Since f is a surjective upper cut homomophic mapping from A to B, from Definition 3.12, we know that f is a surjective homomorphism from $A_{[n]}$ to $B_{[n]}$. For all $x, y \in (f^{-1}(B))_{[n]}$, we notice that $B \in MG(Y)$, $f(xy) = f(x)f(y) \in B_{[n]}$. Hence, f is a surjective upper cut homomorphism from $f^{-1}(B)$ to B. $\qquad\square$

Remark 4.10. Let $f : X \to Y$ be homomorphism of groups where f is quasi-surjective, $A \in MG(X)$ and $B \in MG(Y)$. If f is an upper cut isomorphic mapping from A to B with the pre-surjective property, then

(i) $f(A) \in MG(Y)$, and f is also upper cut isomorphic mapping from A to $f(A)$,

(ii) $f^{-1}(B) \in MG(X)$, and f is also upper cut isomorphic mapping from $f^{-1}(B)$ to B.

This remark is a direct consequence of Theorem 4.8 and Corollary 4.9.

5. Conclusion

We have proposed the notion of homomorphism of cuts of multigroups, explicated its properties and obtained some related results. Some homomorphic properties of upper cut of multigroups were explored.

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