Gulf Journal of Mathematics
Vol 6, Issue 2 (2018) 7-12

RINGS IN WHICH EVERY NONZERO PROJECTIVE IDEAL IS REGULAR

CHAHRAZADE BAKKARI[1*] AND ABDOULJABBAR YAHIA OMARI[2]

ABSTRACT. In this paper we introduce and investigate a class of those rings in which every projective ideal is regular. We establish the transfer of this notion to trivial ring extension, direct product, pullbacks, and amalgamation of rings along an ideal and then generate new and original families of rings satisfying this property.

1. Introduction and preliminaries

All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary. We denote by $qf(R)$ the quotient field of an integral domain R, by $pd_R(M)$ the projective dimension of an R-module M and by $Jac(A)$ the jacobson radical of a ring A.

In this paper, we are interested in those rings in which every nonzero projective ideal is regular and which will be called PIR-rings. A local rings is example of $PIR-$rings. Also, every von Neumann regular ring which is not a field is example of a non $PIR-$ring.

Let A be a ring and E be an A-module. The trivial ring extension of A by E (also called the idealization of E over A) is the ring $R = A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$. Recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J = I \propto E'$ is an ideal of R. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A [5, Theorem 3.2]. Suitable background on commutative trivial ring extensions is [5, 6, 20, 23].

Let T be a ring and let M be an ideal of T. Denote by π the natural surjection $\pi : T \longrightarrow T/M$. Let D be a subring of T/M . Then, $R := \pi^{-1}(D)$ is a subring of T and M is a common ideal of R and T, such that $D = R/M$. R is known by

Date: Received: Mar 14, 2018; Accepted: May 12, 2018.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 46L55; Secondary 44B20.
Key words and phrases. $PIR-$ring, projective ideal, regular ideal, trivial ring extension, direct product, pullbacks, amalgamation.

a pullback ring associated to the following pullback diagram:

$$\begin{array}{ccc} R := \pi^{-1}(D) & \xrightarrow{\pi/R} & D = R/M \\ \downarrow i & & \downarrow j \\ T & \xrightarrow{\pi} & T/M \end{array}$$

where i and j are the natural injections.
As a particular case of this pullback is the $D+M$-construction, when the ring T is of the form $K+M$, where K is a field and M is a maximal ideal of T, and R becomes the form $D+M$. See for instance [17, 20].

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$

$$A \bowtie^f B = \{(a, f(a)+j) | a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f. Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation (see [11, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagata's idealization, and the CPI extensions (in the sense of Boisen and Sheldon [7]) are strictly related to it (see [11, Example 2.7 and Remark 2.8]). A particular case of this construction is the amalgamated duplication of a ring along an ideal I (introduced and studied by D'Anna and Fontana in [10, 15, 14]). Let A be a ring, and let I be an ideal of A. $A \bowtie I := \{(a, a+i) : a \in A,\ i \in i\}$ is called the amalgamated duplication of A along the ideal I. See for instance [10, 11, 12, 13, 14, 15].

The purpose of this paper is to give some simple methods in order to construct PIR-rings. For this, we investigate the transfer of $PIR-$property to trivial ring extensions, direct product, pullbacks, and amalgamation of rings.

2. Main results

A ring R is called a $PIR-$ring if every nonzero projective ideal is regular. We start with examples of PIR-rings.

Example 2.1. (1) *Every domain is a $PIR-$domain.*
(2) *Every local ring is a $PIR-$ring.*
(3) *Every von Neumann regular ring which is not a field is a non $PIR-$ring.*

A ring R is called semihereditary if every finitely generated ideal is projective. In this context, we have:

Proposition 2.2. *Let R be a semihereditary ring. Then, R is a $PIR-$ring if and and only if R is a domain.*

Proof. It is clear that if R is a domain, then R is a PIR-ring. Conversely, assume that R is a PIR-ring and let a be a nonzero element of R. Then Ra is projective (since R is semihereditary) and so Ra is a regular ideal (since R is a PIR-ring). This means that a is a non-zerodivisors, as desired. □

Now, we study the transfer of the PIR-property to the trivial ring extension of A by an A-module E.

Theorem 2.3. *Let $R := A \propto E$ be the trivial ring extension of a ring A by an A-module E. Then:*

1. *Assume that E is flat. Then if R is a $PIR-$ring, then so is A.*
2. *Assume that $A := D$ is a domain, $K = qf(D)$ and E is a K-vector space. Then R is a PIR-ring.*
3. *Assume that A is a local ring. Then $R = A \propto E$ is a $PIR-$ring.*

The proof of this theorem involves the following lemma.

Lemma 2.4. *Let $T = K \propto E$ be the trivial ring extension of a field K by $K-$vector space E. Then there exists no proper flat ideal of T.*

Proof. Let $J = 0 \propto E'$ be a proper ideal of T, where E' $(\subseteq E)$is a $K-$vector space. We claim that J is not flat. Deny. Let $\{f_i\}_{i\in I}$be a basis of the $K-$vector space E' and consider the $T-$map $T^{(I)} \to J$ defined by $v\left((a_i, e_i)_{i\in I}\right) = \left(0, \sum_{i\in I} a_i f_i\right)$.We have $Ker\,(v) = 0 \propto E^{(I)} = (0 \propto E)^{(I)}$. Hence, by [?, Theorem 3.55] , we obtain

$$(0 \propto E)^{(I)} = \left(0 \propto E^{(I)}\right) \cap (0 \propto E)\, T^{(I)} = (0 \propto E)^{(I)}\,(0 \propto E) = 0$$

a contradiction. Hence, J is not flat. □

Proof of Theorem 2.3.

1) Assume that R is a $PIR-$ring and let I be a nonzero projective ideal of A. Then $J = I \otimes_A R = IR = I \propto IE$ is a nonzero projective ideal of R. Hence J is regular since R is a $PIR-$ring. Let $(a, e) \in J = I \propto IE$ be a regular element of J, then $a \in I$. We claim that a is regular.

Indeed, let $b \in A$ such that $ab = 0$, and prove that $b = 0$. We have $(a, e)\,(b, 0) = (ab, eb) = (0, eb)$

Case 1: $eb = 0$.

In this case, $(a, e)\,(b, 0) = 0$, but (a, e) is a regular element of R, then $b = 0$.

Case 2: $eb \neq 0$.

In this case, we have,

$$\begin{aligned}(a, e)\,(0, eb) &= (0, abe)\\ &= 0\end{aligned}$$

impossible since (a, e) is a regular element.

Hence, $a \in I$ is a regular element. Therefore, I is regular, as desired.

2) Let J be a nonzero projective ideal of R. Set $T = K \propto E = S^{-1}R$ which is a flat R- module, where $S = D \setminus \{0\}$. Hence $JT\,(= J \otimes_R T)$ is a nonzero projective ideal of T and so $JT = T = K \propto E$ by Lemma 2.4. Therefore, $I = \{a \in D/\,(a, e) \in J\} \neq \{0\}$ which implies that $J = I \propto E$. Hence $a \in I \setminus \{0\}$, then $(a, e) \in J$ is a regular element of R and so R is a $PIR-$ring.

3) Clear since R is local, and this completes the proof of Theorem 2.3. □

The following example which is not a domain, show that the condition "semi-hereditary" in Proposition 2.2 is necessary:

Example 2.5. *The ring* $R = \mathbb{Z} \propto \mathbb{R}$ *is a* $PIR-$*ring by Theorem 2.3(2).*

Now, we give a class of a non-PIR-ring by Theorem 2.3:

Example 2.6. *Let* A *be a von Neumann regular ring which is not a field and* E *be a flat* A*-module (for example, we may take* $E = A$*). Then, the ring* $R = A \propto E$ *is not a* $PIR-$*ring by Theorem 2.3(1).*

Now, we show that the direct product of rings is never a $PIR-$ring.

Proposition 2.7. *Let* $(R_i)_{i=1,\dots,m}$ *be a family of rings, where* $m \geq 2$*. Then* $\prod_{i=1}^{m} R_i$ *is never a* $PIR-$*ring.*

Proof. Let $I := \prod_{i=1}^{m} I_i$, where $I_1 = R_1$ and $I_i = 0$ for $i = 2, ..., m$. Then, I is a projective ideal of $R := \prod_{i=1}^{m} R_i$ by [**?**, Lemma 2.5(2)] since I_i is a projective ideal of R_i for $i = 1, ..., m$. On the other hand, $(0, 1, ...1)I = 0_R$. Therefore, $R := \prod_{i=1}^{m} R_i$ is never a $PIR-$ring, as desired. □

Next, we study the transfer of the PIR-property to pullbacks.

Theorem 2.8. *Let* $T = K + M$*, where* K *is a field and* M *is a nonzero maximal ideal of* T*,* D *is a subring of* K *such that* $qf(D) = K$*, and set* $R = D + M$*. Assume that for every* $m \in M \setminus \{0\}$*,* m *is not regular. Then* R *is a* $PIR-$*ring.*

Proof. Let J be a nonzero projective ideal of R. Hence $J \otimes_R T = JT$ is a nonzero projetive ideal of T and so there exists $a + m \in T$ a regular element for some $a \in K$ and $m \in M$ such that $J \otimes_R T = JT = T(a + m)$ since T is local. we claim that $a \neq 0$. Deny. Then $m = a + m$ is regular, a contradiction. Therefore $a \neq 0$ and so $a + m \notin M$. Then $a + m$ is invertible element of T. Hence $J \otimes_R T = JT = T(a + m) = T$.

On the other hand,

$$
\begin{aligned}
J \otimes_R \frac{R}{M} &= \frac{J}{JM} \\
&= \frac{J}{J(TM)} \\
&= \frac{J}{(JT)M} \\
&= \frac{J}{M}
\end{aligned}
$$

Hence $M \subseteq J$ and so $J = I + M$, where $I = \{d \in D / d + m \in J\} \subseteq J$ is an ideal of D. We claim that $I \neq 0$. Deny. Then $I = 0$ and so $J = M$. Hence $T = JT = MT = M$, a contradiction. Therefore $I \neq 0$. Let $d \in I \setminus \{0\}$. We claim that d is a regular element of R.

Indeed, let $\alpha+m \in R\backslash\{0\}$ for some $\alpha \in D$ and $m \in M$ such that $d(\alpha+m)=0$. We have $0=d(\alpha+m)=d\alpha+dm$. Then

$$\begin{cases} d\alpha = 0 & for\ d \neq 0 \in D \\ dm = 0 & for\ d \neq 0 \in D \end{cases} \Longrightarrow \begin{cases} \alpha = 0 \\ d^{-1}dm = 0 & for\ d \neq 0 \in D \end{cases}$$
$$\Longrightarrow \begin{cases} \alpha = 0 \\ m = 0 \end{cases}$$

Hence $\alpha+m=0$. Then d is regular element of R and so J is regular. Therefore, R is $PIR-$ring. $\square$

Corollary 2.9. *Let K be a field, X an indeterminate and $T = \frac{K[[X]]}{<X^n>} = K + \overline{X}T$. Let D be a subring of K. Then $R = D + \overline{X}T$ is a $PIR-$ring.*

Proof. Clear since T is local and $\overline{X^{n-1}}M = \overline{X^{n-1}}\overline{X}T = \overline{X^n}T = 0$. $\square$

Example 2.10. *Let $T = \frac{\mathbb{Q}[[\mathbb{X}]]}{<X^3>} = \mathbb{Q} + \overline{X}T$ and $R = \mathbb{Z} + \overline{X}T$. Then $R = \mathbb{Z} + \overline{X}T$ is a $PIR-$ring.*

Next, we study the transfer of the PIR-property to the amalgamation of rings along an ideal. Recall that $Rad(B)$ means the jacobson radical of a ring B.

Proposition 2.11. *Let A and B be rings, $f : A \longrightarrow B$ be a ring homomorphism, J be a nonzero proper ideal of B, and set $R := A \bowtie^f J$ the amalgamation of A and B along J with respect to f. Then :*

1. *Assume that A is a local ring with maximal ideal M and $J \subseteq Rad(B)$. Then, R is a $PIR-$ring.*
2. *Assume that J and $f^{-1}(J)$ are regular ideals of B and A, respectively. Then $T(R)$ the total ring of quotient of R is not a $PIR-$ring.*
3. *Assume that A is a local ring and I is a regular proper ideal of A, and let S be the set of regular elements of $R = A \bowtie I$. Then $T(R)$ the total ring of quotient of R is not a $PIR-$ring.*

Proof. 1) Since (A, M) is local, $\mathrm{Max}\left(A \bowtie^f J\right) = \left\{M \bowtie^f J\right\} \cup \left\{\overline{Q}^f\right\}$ with $Q \in \mathrm{Max}\,(B)$ not containing J and $\overline{Q}^f = \{(a, f(a)+j) \mid a \in A, j \in J \text{ and } f(a)+j \in Q\}$. But since $J \subseteq Rad\,(B)$, the set $\left\{\overline{Q}^f\right\}$ is empty, and so $A \bowtie^f J$ is local with maximal ideal $M \bowtie^f J$. In particular $A \bowtie^f J$ is a $PIR-$ring.

2) By [12, Proposition 3.1], we have that $T\left(A \bowtie^f J\right) = T(A) \times T(B)$ which is a non PIR-ring by Proposition 2.7.

3) By [14, Corollary 3.3(d)], we have that $T(R) = S^{-1}(A \bowtie I) = T(A) \times T(A)$ which is a non PIR-ring by Proposition 2.7 and this completes the proof. $\square$

References

[1] M. M. Ali, *Idealization and Theorems of D.D. Anderson*, Comm. Algebra **34** (2006), 4479-4501.

[2] M. M. Ali, *Idealization and Theorems of D.D. Anderson II*, Comm. Algebra **35** (2007), 2767-2792.

[3] M. M. Ali and D. J. Smith, *Pure Submodules of Multiplication Modules*, Beiträge Algebra Geom. **45** (2004), 61-74.

[4] D. D. Anderson, Some remarks on multiplication ideals II, Comm. in Algebra **28** (2000), 2577-2583.

[5] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra **1** (2009), no. 1, 3-56.

[6] C. Bakkari S. Kabbaj and N. Mahdou, *Trivial extension definided by Prûfer conditions*, J. Pure App. Algebra 214 (2010), 53-60.

[7] M. B. Boisen and P. B. Sheldon, *CPI-extension: Over rings of integral domains with special prime spectrum*, Canad. J. Math. **29** (1977), 722-737.

[8] F. Cheniour and N. Mahdou, *When every flat ideal is projective*, Comment. Math. Univ. Carolina **55**(1) (2014), 1-7.

[9] P. M. Cohn, *On the free product of associative rings*, Math. Z. **7**1 (1959), 380-398.

[10] M. D'Anna, *A construction of Gorenstein rings*, J. Algebra **306**(2) (2006), 507-519.

[11] M. D'Anna, C.A. Finocchiaro and M. Fontana, *Amalgamated algebra along an ideal*, Commmutative Algebra and Applications, Walter De Gruyter (2009), 155-172.

[12] M. D'Anna, C. A. Finocchiaro and M. Fontana, *Properties of chains of prime ideals in amalgamated algebras along an ideal*, J. Pure Appl. Algebra **214** (2010), 1633-1641.

[13] M. D'Anna, C. A. Finocchiaro and M. Fontana, *New algebraic properties of an amalgamated algebra along an ideal*, Communications in Algebra **44** (2016), 1836-1851.

[14] M. D'Anna and M. Fontana, *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6** (2007), 443-459.

[15] M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. **45**(2) (2007), 241-252.

[16] G. De Marco, *Projectivity of pure ideals*, Rend. Sem. Mat. Univ. Padova **68** (1983) 289-304.

[17] S. El Baghdadi, A. Jhilal and N. Mahdou, *On FF-rings*, J. Pure and Appl. Algebra **216** (2012), 71-76.

[18] C. Faith, Algebra: *Rings, Modules and Categories*, Springer-Verlag (1981).

[19] D. J. Fieldhouse, *Pure theories*, Math. Ann. **184** (1969) 1-18

[20] S. Glaz, *Commutative coherent rings*, Springer-Verlag, Lecture Notes in Mathematics,(1989), 13-71.

[21] J. S. Hu and N. Q. Ding, *Some results on torsionfree modules*, J. Algebra Appl. **12**(1) (2013) 1250138.

[22] J. S. Hu, H. Liu and Y. Geng, *When every pure ideal is projective*, J. Algebra Appl. Vol. 15, No. 2 (2016) 1650030.

[23] S. Kabbaj and N. Mahdou, *Trivial extensions defined by coherent-like conditions*, Comm. Algebra **32** (2004), no. 10, 3937-3953.

[1] Department of Mathematics and Computer Sciences, Faculty of Science, University Moulay Ismail, Meknes, Morocco

Email address: cbakkari@hotmail.com

[2] Department of Mathematics and Computer Sciences, Faculty of Science, University Moulay Ismail, Meknes, Morocco

Email address: omariyahia78@gmail.com