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NONLINEAR MULTIVALUED PROBLEMS WITH VARIABLE EXPONENT AND DIFFUSE MEASURE DATA IN ANISOTROPIC SPACE

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ABSTRACT. We study a nonlinear elliptic problem governed by a general anisotropic operator with variable exponents and Radon diffuse measure data. We start by proving the existence of a renormalized solution and we establish that the notion of renormalized solution is equivalent to the notion of entropy solution. Finally, we show the uniqueness of the entropy solution.

1. Introduction and preliminaries

The goal of this paper is to establish the existence and uniqueness of entropy solution to the following nonlinear multivalued elliptic anisotropic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) + \beta(u) \ \ni \ \mu \ \text{ in } \ \Omega \\ \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$, β a maximal monotone graph on $\mathbb{R}$ such that $0 \in \beta(0)$, μ a bounded Radon diffuse measure.
Note that the space in which we work is the anisotropic Sobolev space $W^{1,\overrightarrow{p}(.)}(\Omega)$, where $\overrightarrow{p}(.) = (p_1(.), ..., p_N(.))$ is a vector with variable components. We denote by

$$p_M(x) := \max(p_1(x), ..., p_N(x)) \ \text{ and } \ p_m(x) := \min\left(p_1(x), ..., p_N(x)\right).$$

In the classical Sobolev and Lebesgue spaces, many authors have studied problems with a maximal monotone graph and measure data ([4, 5, 6, 10, 12, 16]). Note that these kinds of problems have been extended in Sobolev space with variable exponents in the context of isotropic operators (see [21]).
The interest in transposing the problems into new problems with variable exponents is linked to a large scale of applications that involve some nonhomogeneous materials (blood for example). It is already known that for an appropriate treatment of these materials, classical Sobolev and Lebesgue spaces are not adequate,

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so we have to allow the exponent to vary. We can refer here to electrorheological fluids (see [1, 11, 23]) thermorheological fluids, modelling of propagation of epidemic disease (see [2]), image restoration (see [9]). In order to answer to the preoccupation for the nonhomogeneous materials that behave differently on different spaces direction, the anisotropic space with variable exponents are introduced.

It is not a surprise to meet new difficulties when passing from isotropic variable exponents to anisotropic variable exponent. To overcome these difficulties, we combine the classical techniques with the recent techniques that have appeared when treating anisotropic problems with variable exponents.

All papers tackling the issues about (1.1) have considered particular cases (see [15, 17, 18] and the references therein).

Our aim is to prove the existence and uniqueness of renormalized and entropy solutions to the general elliptic problem (1.1). The main interest in our work is that we are dealing with general non-linearities β.

Note that some authors have already studied multivalued elliptic problem in the context of isotropic variable exponents (see [21] and the references therein). Indeed, the first work on multivalued elliptic problems with measure data in the context of isotropic variable exponent was done by Nyanquini et *als* (see [21]) under homogeneous Dirichlet boundary condition. In [21], the authors proved first, a decomposition theorem for the measure data (more precisely, as a sum of an element in $W^{-1,p(.)}(\Omega)$ (the dual space of $W_0^{1,p(.)}(\Omega)$), and a function in $L^1(\Omega)$) and used it to prove following [16], a result of the existence and uniqueness of an entropy solution to the problem

$$\begin{cases} -\nabla . a(x, \nabla u) + \beta(u) \ \ni \ \mu \ \text{ in } \ \Omega \\ \\ u = 0 \qquad\qquad\qquad\qquad\qquad \text{ on } \ \partial\Omega. \end{cases} \tag{1.2}$$

The problem (1.1) is the anisotropic case of the nonlinear isotropic problem (1.2). In this paper, we base our ideas on the decomposition theorem of measure done by the authors in [21], and following them we prove both the existence and the unqueness of an entropy solution to the nonlinear multivalued elliptic anisotropic problem (1.1).

We denote by $\mathcal{L}^N$ the N-dimensional Lebesgue measure of $\mathbb{R}^N$ and by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measures in Ω, equiped with its standard norm $||.||_{\mathcal{M}_b(\Omega)}$. Note that, if μ belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|(\Omega)$(the total variation of μ) is a bounded positive measure on Ω.

Given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,p(.)}(\Omega)$ ($p(.)$-capacity for short) if $\mu(A) = 0$, for every set A such that $Cap_{p(.)}(A, \Omega) = 0$. For every $A \subset \Omega$, we denote

$$S_{p(.)}(A) = \{u \in W_0^{1,p(.)}(\Omega) \cap C_0(\Omega) : u = 1 \text{ on } A, u \geq 0 \text{ on } \Omega\}.$$

The $p(.)$-capacity of every subset A with respect to Ω is defined by

$$Cap_{p(.)}(A, \Omega) = \inf_{u \in S_{p(.)}(A)} \left\{ \int_\Omega |\nabla u|^{p(x)} dx \right\}.$$

In the case $S_{p(.)}(A) = \emptyset$, we set $Cap_{p(.)}(A, \Omega) = +\infty$.
The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$.
We will use the following decomposition result of bounded Radon diffuse measure proved by Nyanquini et *al.* (see [21]).

Theorem 1.1. *Let $p(.) : \overline{\Omega} \longrightarrow (1, +\infty)$ be a continuous function and $\mu \in \mathcal{M}_b(\Omega)$. Then $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$ if and only if $\mu \in L^1(\Omega) + W^{-1,p'(.)}(\Omega)$.*

For any $l_0 > 0$, we consider a function h_0 such that.

(i) $h_0 \in C_c^1(\mathbb{R})$, $h_0(r) \geq 0$, for all $r \in \mathbb{R}$,
(ii) $h_0(r) = 1$ if $|r| \leq l_0$ and $h_0(r) = 0$ if $|r| \geq l_0 + 1$.

If γ is a maximal monotone operator defined on $\mathbb{R}$, by γ_0 we denote the main section of γ; i.e.,

$$\gamma_0(s) = \begin{cases} \text{minimal absolute value of } \ \gamma(s) & \text{if } \gamma(s) \ \neq \emptyset \\ +\infty & \text{if } [s, +\infty) \cap D(\gamma) = \emptyset \\ -\infty & \text{if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

We also use a useful convergence result (see [21]).

Lemma 1.2. *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of the graph (for $(x, y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$.*
We suppose that: $\forall n \geq 1, w_n \in \beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then,

$$z \in dom(\beta).$$

The remaining part of this paper is organized as follows: in Section 2, we introduce some preliminary results and in Section 3 we prove the existence (via a renormalized solution) and uniqueness of an entropy solution of (1.1).
We study problem (1.1) under the following assumptions on the data.
Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary domain $\partial\Omega$ and
$\overrightarrow{p}(.) = (p_1(.), ..., p_N(.))$ such that for any $i = 1, ..., N$, $p_i(.) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous function with

$$1 < p_i^- := \inf_{x\in\Omega} p_i(x) \leq p_i^+ := \sup_{x\in\Omega} p_i(x) < \infty. \tag{1.3}$$

For any $i = 1, ..., N$, let $a_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Caratheodory function satisfaying the following.
• there exists a positive constant C_1 such that

$$|a_i(x, \xi)| \leq C_1\bigg(j_i(x) + |\xi|^{p_i(x)-1}\bigg), \tag{1.4}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p_i'(.)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p_i'(x)} = 1$;

• for $\xi,\ \eta\ \in \mathbb{R}$ with $\xi \neq \eta$ and for every $x \in \Omega$, there exists a positive constant C_2 such that

$$(a_i(x,\xi) - a_i(x,\eta))(\xi - \eta) \geq \begin{cases} C_2|\xi - \eta|^{p_i(x)} & \text{if } \ |\xi - \eta| \geq 1 \\ \\ C_2|\xi - \eta|^{p_i^-} & \text{if } \ |\xi - \eta| < 1 \end{cases} \tag{1.5}$$

and,
• there exists a positive constant C_3 such that

$$a_i(x,\xi).\xi \geq C_3|\xi|^{p_i(x)}, \tag{1.6}$$

for $\xi \in \mathbb{R}$ and almost every $x \in \Omega$.
The hypotheses on a_i are classical in the study of nonlinear problems (see [3]).
Throughout this paper, we assume that

$$\frac{\overline{p}(N-1)}{N(\overline{p}-1)} < p_i^- < \frac{\overline{p}(N-1)}{N-\overline{p}}, \quad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\overline{p} - N}{\overline{p}(N-1)} \tag{1.7}$$

and

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{1.8}$$

where $\displaystyle\frac{N}{\overline{p}} = \sum_{i=1}^{N} \frac{1}{p_i^-}$.
In this section we also recall some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces.
Set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \ \ a.e. \ x \in \Omega \right\}.$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(.)}(\Omega) := \left\{ u : u \text{ is a measurable real valued function such that } \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

endowed with the so-called Luxembourg norm

$$|u|_{p(.)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

The $p(.)$-modular of the $L^{p(.)}(\Omega)$ space is the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(.)}(\Omega)$, the following inequality (see [13], [14]) will be used later.

$$\min \left\{ |u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+} \right\} \leq \rho_{p(.)}(u) \leq \max \left\{ \ |u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+} \right\}. \tag{1.9}$$

For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{q(.)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ in Ω, we have the Helder type inequality

$$\left| \int_{\Omega} uvdx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(.)}|v|_{q(.)}. \tag{1.10}$$

If Ω is bounded and $p,\ q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous (see [19], Theorem 2.8).

Herein we need the anisotropic Sobolev space

$$W_0^{1,\overrightarrow{p}(.)}(\Omega) := \left\{ u \in W_0^{1,1}(\Omega) : \ \frac{\partial u}{\partial x_i} \in L^{p_i(.)}(\Omega),\ i = 1, ..., N \right\},$$

which is a separable and reflexive Banach space (see [20]) under the norm

$$||u||_{\overrightarrow{p}(.)} = \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|_{p_i(.)}.$$

We introduce the numbers

$$q = \frac{N(\overline{p} - 1)}{N - 1};\ q^* = \frac{N(\overline{p} - 1)}{N - \overline{p}} = \frac{Nq}{N - q}$$

and define $P_-^*,\ P_-^+,\ P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-} - 1},\ P_-^+ = \max\left\{p_1^-, ..., p_N^-\right\} \text{ and } P_{-,\infty} = \max\left\{P_-^+,\ P_-^*\right\}.$$

Remark 1.3. Thanks to Theorem 1.1, there exist $f \in L^1(\Omega)$ and $F \in (L^{p_m'(.)}(\Omega))^N$ such that

$$\mu = f - divF, \tag{1.11}$$

where $\dfrac{1}{p_m(x)} + \dfrac{1}{p_m'(x)},\ \forall\ x \in\ \Omega.$

We have the following embedding results (see [20], Theorem 1).

Theorem 1.4. *Assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with a smooth boundary. Assume also that the relation (1.8) is fulfilled. For any $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < P_{-,\infty} \text{ for any } x \in \overline{\Omega},$$

the embedding

$$W_0^{1,\overrightarrow{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$$

is continuous and compact.

The following result is due to Troisi (see [25]).

Theorem 1.5. *Let $p_1, ..., p_N \in [1, +\infty)$; $g \in W^{1,(p_1,...,p_N)}(\Omega)$ and*

$$q = \begin{cases} (\overline{p})^* & \text{if } (\overline{p})^* < N \\ \in [1, +\infty) & \text{if } (\overline{p})^* \geq N. \end{cases}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, ..., p_N$ if $\overline{p} < N$ and also on q and $meas(\Omega)$ if $\overline{p} \geq N$ such that

$$||g||_{L^q(\Omega)} \leq C_4 \prod_{i=1}^{N} \left\| \frac{\partial g}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}. \tag{1.12}$$

In this paper, we will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)(1 < q < +\infty)$ as the set of measurable function $g : \Omega \longrightarrow \mathbb{R}$ for which the distribution

$$\lambda_g(k) = meas(\{x \in \Omega : |g(x)| > k\}), \ k \geq 0 \tag{1.13}$$

satisfies an estimate of the form

$$\lambda_g(k) \leq Ck^{-q}, \quad \text{for some finite constant } C > 0. \tag{1.14}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$.

$$||g||_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q}, \ \forall k > 0\}. \tag{1.15}$$

Finally, throughout the paper we use the truncation function T_k, $(k > 0)$ by

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{1.16}$$

It is clear that $\lim_{k \to \infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.

We define $\mathcal{T}_0^{1,\overrightarrow{p}(.)}(\Omega)$ as the set of the measurable function $u : \Omega \longrightarrow \mathbb{R}$ such that $T_k(u) \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$.

To give our notion of solution and the main results, we set

$$\text{int}(\text{dom}(\beta)) = (m, M) \text{ with } -\infty < m \leq 0 \leq M < +\infty.$$

For any $r \in \mathbb{R}$ and any measurable function u on Ω, $[u = 0]$, $[u \leq r]$ and $[u \geq r]$ denote the set $\{x \in \Omega : u(x) = r\}$, $\{x \in \Omega : u(x) \leq r\}$, $\{x \in \Omega : u(x) \geq r\}$ respectively.

2. Main results

Our main results are stated as follows.

Theorem 2.1. *For any $\mu \in \mathcal{M}_b^{p_m(.)}(\Omega)$, the problem (1.1) has at least one solution (u, w) in the sense that $(u, w) \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \times L^1(\Omega)$, $u \in dom(\beta)$ $\mathcal{L}^N$- a.e. in Ω, $w \in \beta(u)$ $\mathcal{L}^N$- a.e. in Ω, there exists $\nu \in \mathcal{M}_b^{p_m(.)}(\Omega)$ such that $\nu \perp \mathcal{L}^N$,*

$$\nu^+ \text{ is concentred on } [u = M], \ \nu^- \text{ is concentred on } [u = m]$$

and

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial v}{\partial x_i} dx + \int_\Omega wv dx + \int_\Omega v d\nu = \int_\Omega v d\mu, \tag{2.1}$$

for any $v \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$. *Moreover,*

$$\nu^+ \leq \mu_s \lfloor\ [u = M] \tag{2.2}$$

and

$$\mu^- \leq -\mu_s \lfloor\ [u = m]. \tag{2.3}$$

The connexion between our notion of entropy solution and the entropic formulation (see [3]) of the solution is given in the following theorem.

Theorem 2.2. *If* (u, w) *is a solution of (1.1) in the sense of Theorem 2.1, then* (u, w) *is a solution in the following sense : for any* $v \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$ *such that* $v \in dom\beta$,

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u-v)dx + \int_\Omega w T_k(u-v)dx \leq \int_\Omega T_k(u-v)d\mu, \text{ for any } k > 0. \tag{2.4}$$

Proof of the main result

Throughout this section, $\mu \in \mathcal{M}_b^{p_m(.)}(\Omega)$. By Remark 1, we have $\mu = f - div(F)$, with $f \in L^1(\Omega)$ and $F \in \left(L^{p'_m(.)}(\Omega)\right)^N$. This section is devoted to the proof of Theorem 2.1 and Theorem 2.2.

For every $\epsilon > 0$, we consider the Yosida regularization β_ϵ of β (see [8]), given by

$$\beta_\epsilon = \frac{1}{\epsilon}(I - (I + \epsilon\beta)^{-1}).$$

Thanks to [8], there exists a non negative, convex and l.s.c. function j defined on $\mathbb{R}$ such that

$$\beta = \partial j.$$

To regularise β, we consider

$$j_\epsilon(s) = \min_{r \in \mathbb{R}} \left\{ \frac{1}{2\epsilon}|s - r|^2 + j(r) \right\},\ \forall\ s \in \mathbb{R},\ \forall \epsilon\ > 0.$$

By Proposition 2.11 in [8] we have

$$\begin{cases} dom(\beta) \subset dom(j) \subset \overline{dom(j)} = \overline{dom(\beta)}, \\ j_\epsilon(s) = \frac{\epsilon}{2}|\beta_\epsilon(s)|^2 + j(J_\epsilon) \text{ where } J_\epsilon := (I + \epsilon\beta)^{-1}, \\ j_\epsilon \text{ is a convex, Frechet-differentiable function and } \beta_\epsilon = \partial j_\epsilon, \\ j_\epsilon \uparrow j \text{ as } \epsilon \downarrow 0. \end{cases}$$

Moreover, for any $\epsilon > 0$, β_ϵ is a nondecreasing and Lipschitz-continuous function. To regularize μ, for any $\epsilon > 0$, we define the functions

$$f_\epsilon(x) = T_{\frac{1}{\epsilon}}(f(x)) \text{ for } a.e.\ x \in \Omega.$$

and

$$\mu_\epsilon = f_\epsilon - \nabla.\ F \text{ for any } \epsilon > 0.$$

Then, we consider the following approximating scheme problem.

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u_\epsilon}{\partial x_i}) + \beta_\epsilon(u) = \mu_\epsilon \ \text{ in } \ \Omega \\ \\ u_\epsilon = 0 \qquad\qquad\qquad\qquad \text{on } \ \partial\Omega. \end{cases} \tag{2.5}$$

Theorem 2.3. *The problem (2.5) admits a unique weak solution u_ϵ in the sense that $u_\epsilon \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$, $\beta_\epsilon(u_\epsilon) \in L^1(\Omega)$ and $\forall v \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$,*

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial v}{\partial x_i} dx + \int_\Omega \beta_\epsilon(u_\epsilon) v dx = \int_\Omega f_\epsilon v dx + \int_\Omega F.\nabla v dx. \tag{2.6}$$

Proof. The techniques of this proof follow the proof of the Theorem 3.1 in [15]. For any $k > 0$, if b is a continuous nondecreasing function with $b(0) = 0$, the following problem

$$P(T_k(b), \Upsilon) \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) + T_k(b(u)) = \Upsilon \ \text{ in } \ \Omega \\ \\ u = 0 \qquad\qquad\qquad\qquad \text{on } \ \partial\Omega \end{cases}$$

admits at least one solution $u_k \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that $\forall v \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$,

$$\sum_{i=1}^{N} \int_\Omega a_i\left(x, \frac{\partial u_k}{\partial x_i}\right) \frac{\partial v}{\partial x_i} dx + \int_\Omega T_k(b(u_k)) v dx = \int_\Omega \Upsilon v dx, \tag{2.7}$$

where $\Upsilon \in L^\infty(\Omega)$.
Furthermore,

$$\forall k > ||\Upsilon||_\infty, |b(u_k)| \leq ||\Upsilon||_\infty \text{ a.e. in } \Omega. \tag{2.8}$$

Indeed, We define the operator A_k as follows.

$$\langle A_k u, v\rangle = \langle Au, v\rangle + \int_\Omega T_k(b(u)) v dx, \ \forall \ u, v \in W_0^{1,\overrightarrow{p}}(\Omega),$$

where

$$\langle Au, v\rangle = \int_\Omega \sum_{i=1}^{N} a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial v}{\partial x_i} dx.$$

According to [15], the operator A_k is of type M, bounded and coercive from $W_0^{1,\overrightarrow{p}(.)}(\Omega)$ into its dual $(W_0^{1,\overrightarrow{p}(.)}(\Omega))'$. Thus, A_k is surjective (see [24], Corollary 2.2). Therefore, for $B \in (W_0^{1,\overrightarrow{p}(.)}(\Omega))'$ defined by $\langle B, v\rangle = \displaystyle\int_\Omega \Upsilon v dx$, we can deduce the existence of a function $u_k \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$ such that $\langle A_k u_k, v\rangle = \langle B, v\rangle$, $\forall v \in W_0^{1,\overrightarrow{p}(.)}(\Omega)$.
Let us fix $k > ||\Upsilon||_\infty$, we get the existence of the solutions to problem $P(b, \Upsilon)$ for any b and Υ as above.
The proof of (2.8) is detailed in [15]. So we can set $b = \beta_\epsilon$ and $\Upsilon = \mu_\epsilon$ to get the result of the Theorem 2.3. □

We have the following results.

Proposition 2.4. (i) *For any* $k > 0$,

$$\sum_{i=1}^{N} \int_{\{|u_\epsilon| \le k\}} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_i(x)} dx \le \frac{k|\mu|(\Omega)}{C_3} \tag{2.9}$$

and

$$\int_{\{|u_\epsilon| \le k\}} |\nabla T_k(u_\epsilon)|^{p_m^-} dx \le C_5. \tag{2.10}$$

(ii) *The sequence* $(\beta_\epsilon(u_\epsilon))_{\epsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$.

(iii) *For any* $k > 0$, *the sequence* $(\beta_\epsilon(T_k(u_\epsilon)))_{\epsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$.

Proof. i) We take $v = T_k(u_\epsilon)$ as a test function in (2.6) to get

$$\sum_{i=1}^{N} \int_{\{|u_\epsilon| \le k\}} a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right) \frac{\partial u_\epsilon}{\partial x_i} dx + \int_\Omega \beta(u_\epsilon) T_k(u_\epsilon) dx = \int_\Omega f_\epsilon T_k(u_\epsilon) dx + \int_\Omega F \nabla T_k(u_\epsilon) dx.$$

Using relation (1.6) and the fact that $\int_\Omega \beta(u_\epsilon) T_k(u_\epsilon) dx \ge 0$, we obtain

$$\sum_{i=1}^{N} \int_{\{|u_\epsilon| \le k\}} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_i(x)} dx \le \frac{k|\mu|(\Omega)}{C_3}.$$

We make the following notation.
$\mathcal{I}_1 = \{i \in \{1, ..., N\} : |\frac{\partial u_\epsilon}{\partial x_i}| \le 1\}$, $\mathcal{I}_2 = \{i \in \{1, ..., N\} : |\frac{\partial u_\epsilon}{\partial x_i}| > 1\}$ and $A_h := \{|u_\epsilon| \le k\}$.
We have

$$\begin{aligned}
\sum_{i=1}^{N} \int_{A_h} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_i(x)} dx &= \sum_{i \in \mathcal{I}_1} \int_{A_h} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_i(x)} dx + \sum_{i \in \mathcal{I}_2} \int_{A_h} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_i(x)} dx \\
&\ge \sum_{i \in \mathcal{I}_2} \int_{A_h} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_i(x)} dx \\
&\ge \sum_{i \in \mathcal{I}_2} \int_{A_h} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_m^-} dx \\
&\ge \sum_{i=1}^{N} \int_{A_h} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_m^-} dx - \sum_{i \in \mathcal{I}_1} \int_{A_h} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_m^-} dx \\
&\ge \sum_{i=1}^{N} \int_{A_h} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^{p_m^-} dx - N meas(\Omega) \\
&\ge \sum_{i=1}^{N} \left|\left| \frac{\partial u_\epsilon}{\partial x_i} \right|\right|^{p_m^-}_{L^{p_m^-}(A_h)} - N meas(\Omega) \\
&\ge \alpha ||\nabla u_\epsilon||^{p_m^-}_{(L^{p_m^-}(A_h))^N} - N meas(\Omega),
\end{aligned}$$

where we use the Poincare inequality. We deduce that

$$\sum_{i=1}^{N}\int_{|u_\epsilon|\leq k}\left|\frac{\partial u_\epsilon}{\partial x_i}\right|^{p_i(x)}dx \geq \alpha\int_{\Omega}|\nabla T_k(u_\epsilon)|^{p_m^-}-Nmeas(\Omega). \tag{2.11}$$

By combinating (2.9) and (2.11) and setting $C_5=\frac{1}{\alpha}\left(\frac{k|\mu|(\Omega)}{C_3}+Nmeas(\Omega)\right)$, we get (2.10) □

The sequence $(u_\epsilon)_{\epsilon>0}$ satisfies the following inequalities.

Proposition 2.5. *For any $k>0$ large enough, u_ϵ a weak solution of (2.5), we have*

$$meas\{|u_\epsilon|>k\}\leq\frac{C(\mu,\Omega)}{\min\{\beta_\epsilon(k),|\beta_\epsilon(-k)|\}}, \tag{2.12}$$

$$meas\left\{\left|\frac{\partial u_\epsilon}{\partial x_i}\right|>k\right\}\leq\frac{C(\mu,\Omega)}{k^{\frac{1}{(p_M)'}}} \tag{2.13}$$

and

$$meas\left\{|\nabla u_\epsilon|>k\right\}\leq\frac{C_6(k+1)}{k^{p_m^-}}+\frac{C(\mu,\Omega)}{\min\{\beta_\epsilon(k),|\beta_\epsilon(-k)|}, \tag{2.14}$$

where C_6 is a positive constant.

The proof of Proposition 2.5 follows the proof of the Proposition 4.3 in [21] (see also [17, 22]).
We also have the following results (see [7, 15, 18]).

Lemma 2.6. *For any $k>0$, there exist some constants $C_7,\ C_8>0$ such that.*

(i) $||u_\epsilon||_{\mathcal{M}^{q^*}(\Omega)}\leq C_7$;

(ii) $\|\frac{\partial u_\epsilon}{\partial x_i}\|_{\mathcal{M}^{p_i^-\frac{q}{p}}(\Omega)}\leq C_8,\quad\forall i=1,...,N.$

Lemma 2.7. *For $i=1,...,N$, as $n\to+\infty$, we have*

$$a_i\left(x,\frac{\partial u_\epsilon}{\partial x_i}\right)\longrightarrow a_i\left(x,\frac{\partial u}{\partial x_i}\right)\ in\ L^1(\Omega)\ a.e.\ x\in\Omega. \tag{2.15}$$

Proposition 2.8. *There exists $u\in W_0^{1,\overrightarrow{p}(.)}(\Omega)\subset\mathcal{T}_0^{\overrightarrow{p}(.)}(\Omega)$ such that $u\in dom(\beta)$ a.e. in Ω and*

$$u_\epsilon\longrightarrow u\ in\ measure\ and\ a.e.\ in\ \ \Omega\ as\ \epsilon\longrightarrow 0. \tag{2.16}$$

As for $k>0$, T_k is continuous, then $T_k(u_\epsilon)\to T_k(u)$ a.e. in Ω. Finally, using Lemma 1.2 we deduce that for all $k>0$, $T_k(u)\in dom(\beta)$ a.e. in Ω. Since $T_k(u)\in dom(\beta)$, we get $u\in dom(\beta)$ *a.e.* in Ω and as $dom(\beta)$ is bounded, then $u\in W_0^{1,\overrightarrow{p}(.)}(\Omega)$.

Proposition 2.9. *Assume (1.3)-(1.8). If $u_\epsilon\in E$ is a weak solution of (2.5) then.*

(i) *For all $i=1,...,N$, $\frac{\partial u_\epsilon}{\partial x_i}$ converges in measure to the weak partial gradient of u.*

(ii) *For all* $i = 1, ..., N$ *and* $k > 0$, $a_i\left(x, \frac{\partial}{\partial x_i} T_k(u_\epsilon)\right)$ *converges to* $a_i\left(x, \frac{\partial}{\partial x_i} T_k(u)\right)$ *in* $L^1(\Omega)$ *strongly and in* $L^{p_i'(.)}(\Omega)$ *weakly.*

(iii) *For* $i = 1, ..., N$, $a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}$ *in* $L^1(\Omega)$ *and a.e.* $x \in \Omega$.

Proof. For the proof of (i) and (ii) we refer to [7] (see also [15, 18]).
(iii) The continuity of $a_i(x, \xi)$ with respect to $\xi \in \mathbb{R}$ gives us

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right) \text{ a.e. } x \in \Omega.$$

Therefore,

$$a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i} \quad \text{a.e. } x \in \Omega.$$

Setting $y_\epsilon = a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i}$ and $y = a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}$, for $i = 1, ..., N$, we have

$$\begin{cases} y_\epsilon \geq 0, y_\epsilon \to y \text{ a.e. in } \Omega, \; y \in L^1(\Omega), \\ \\ \displaystyle\int_\Omega y_\epsilon dx \to \int_\Omega y dx \end{cases}$$

and as $\displaystyle\int_\Omega |y_\epsilon - y| dx = 2\int_\Omega (y - y_\epsilon)^+ dx + \int_\Omega (y_\epsilon - y) dx$ and $(y - y_\epsilon)^+ \leq y$, it follows by using Lebesgue dominated convergence Theorem, that

$$\lim_{\epsilon \to 0} \int_\Omega |y_\epsilon - y| dx = 0,$$

which means that

$$a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial u_\epsilon}{\partial x_i} \longrightarrow a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i} \quad \text{strongly in } \; \mathrm{L}^1(\Omega).$$

□

We have the following lemma.

Lemma 2.10. *For any* $h \in C_c^1(\mathbb{R})$ *and* $\xi \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$, *for any* $i = 1, ..., N$,

$$\frac{\partial}{\partial x_i}(h(u_\epsilon)\xi) \longrightarrow \frac{\partial}{\partial x_i}(h(u)\xi) \text{ strongly in } \; L^1(\Omega) \text{ as } \epsilon \to 0, \quad \text{for any } i = 1, ..., N.$$

Proof. For any $h \in C_c^1(\mathbb{R})$ and $\xi \in W_0^{1,\vec{p}(.)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned} \frac{\partial}{\partial x_i}[h(u_\epsilon)\varphi] = & \; \varphi h'(u_\epsilon)\frac{\partial u_\epsilon}{\partial x_i} + h(u_\epsilon)\frac{\partial \varphi}{\partial x_i} \\ = & \; \varphi h'(u_\epsilon)\frac{\partial T_l(u_\epsilon)}{\partial x_i} + h(u_\epsilon)\frac{\partial \varphi}{\partial x_i} \text{ for } l > 0 \text{ such that } \; supp(h) \subset (-l, +l). \end{aligned}$$

Since $|h(u_\epsilon)\frac{\partial\varphi}{\partial x_i}| \leq C(h)|\frac{\partial\varphi}{\partial x_i}| \in L^1(\Omega)$, using Lebesgue dominated convergence Theorem, we get

$$h(u_\epsilon)\frac{\partial\varphi}{\partial x_i} \to h(u)\frac{\partial\varphi}{\partial x_i} \text{ strongly in } L^1(\Omega) \text{ as } \epsilon \to 0.$$

Moreover, we have $|\varphi h'(u_\epsilon)\frac{\partial T_l(u_\epsilon)}{\partial x_i}| \leq C(h,||\varphi||_\infty)|\frac{\partial T_l(u_\epsilon)}{\partial x_i}| \longrightarrow C(h,||\varphi||_\infty)|\frac{\partial T_l(u)}{\partial x_i}|$ in $L^1(\Omega)$. Then, by using generalized convergence Theorem, we deduce that

$$\varphi h'(u_\epsilon)\frac{\partial T_l(u_\epsilon)}{\partial x_i} \to \varphi h'(u)\frac{\partial T_l(u)}{\partial x_i} \text{ strongly in } L^1(\Omega) \text{ as } \epsilon \to 0.$$

□

Now, we pass to the limit as $\epsilon \to 0$ in $\beta_\epsilon(u_\epsilon)$. Since, for any $k > 0$, $(h_k(u_\epsilon)z_\epsilon)_{\epsilon>0}$ is bounded in $L^1(\Omega)$, there exists $z_k \in \mathcal{M}_b(\Omega)$ such that

$$h_k(u_\epsilon)\beta_\epsilon(u_\epsilon) \rightharpoonup z_k \text{ in } \mathcal{M}_b(\Omega) \text{ as } \epsilon \to 0.$$

Moreover, for any $\xi \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$, we have

$$\int_\Omega \xi dz_k = \int_\Omega \xi h_k(u)d\mu - \int_\Omega \sum_{i=1}^N a_i(x,\frac{\partial u}{\partial x_i})\frac{\partial}{\partial x_i}[h_k(u)\xi]dx,$$

which implies that $z_k \in \mathcal{M}_b^{p_m(.)}(\Omega)$ and, for any $k \leq l$,

$$z_k = z_l \text{ on } [|T_k(u)| < k].$$

Let us consider the Radon measure defined by

$$\begin{cases} z = z_k, & \text{on } [|T_k(u)| < k] \text{ for } k \in \mathbb{N}^*, \\ z = 0 & \text{on } \displaystyle\bigcap_{k\in\mathbb{N}^*}[|T_k(u)| = k]. \end{cases} \tag{2.17}$$

For any $h \in \mathcal{C}_c(\mathbb{R})$, $h(u) \in L^\infty(\Omega, d|z|)$ and

$$\int_\Omega h(u)\xi dz = -\int_\Omega \sum_{i=1}^N a_i(x,\frac{\partial u}{\partial x_i})\frac{\partial}{\partial x_i}[h(u)\xi]dx + \int_\Omega h(u)\xi d\mu,$$

for any $\xi \in W_0^{1,\overrightarrow{p}(.)}(\Omega) \cap L^\infty(\Omega)$. Indeed, let $k_0 > 0$ be such that $supp(h) \subseteq [-k_0, k_0]$,

$$\begin{aligned}
\int_\Omega h(u)\xi dz &= \int_\Omega h(u)\xi dz_{k_0} \\
&= -\lim_{\epsilon\to 0}\int_\Omega \sum_{i=1}^N a_i(x,\frac{\partial u_\epsilon}{\partial x_i})\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\xi d\mu_\epsilon \\
&= -\lim_{\epsilon\to 0}\int_\Omega \sum_{i=1}^N a_i(x,\frac{\partial T_{k_0}(u_\epsilon)}{\partial x_i})\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx + \lim_{\epsilon\to 0}\int_\Omega h(u_\epsilon)\xi d\mu_\epsilon \\
&= -\int_\Omega \sum_{i=1}^N a_i(x,\frac{\partial u}{\partial x_i})\frac{\partial}{\partial x_i}[h(u)\xi]dx + \int_\Omega h(u)\xi d\mu.
\end{aligned}$$

Moreover, we have the following lemma (see [21], Lemma 4.7).

Lemma 2.11. *The Radon-Nikodym decomposition of the measure z given by* (2.17) *with respect to* $\mathcal{L}^N$,

$$z = w\mathcal{L}^N + \nu \quad \textit{with } \nu \perp \mathcal{L}^N$$

satisfies the following properties

$$\begin{cases} w \in \beta(u)\mathcal{L}^N - a.e. \text{ in } \Omega, \; w \in L^1(\Omega), \; \nu \in \mathcal{M}_b^{p_m(\cdot)}(\Omega), \\ \nu^+ \text{ is concentrated on } [u = M], \\ \nu^- \text{ is concentrated on } [u = m]. \end{cases}$$

To end the proof of the Theorem 2.1, we consider $\xi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $h \in C_c^1(\mathbb{R})$. Then, we take $h(u_\epsilon)\xi$ as a test function in (2.6) to get

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx + \int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx = \int_\Omega f_\epsilon h(u_\epsilon)\xi dx + \int_\Omega F.\nabla[h(u_\epsilon)\xi]dx. \tag{2.18}$$

Since $F \in (L^{p'_m(\cdot)}(\Omega))^N$ and $\nabla[h(u_\epsilon)\xi] \rightharpoonup \nabla[h(u)\xi]$ in $(L^{p_m(\cdot)}(\Omega))^N$ as $\epsilon \to 0$, we have

$$\lim_{\epsilon\to 0} \int_\Omega F.\nabla[h(u_\epsilon)\xi]dx = \int_\Omega F.\nabla[h(u)\xi]dx$$

and, using Lebesgue dominated convergence Theorem we obtain

$$\lim_{\epsilon\to 0} \int_\Omega f_\epsilon h(u_\epsilon)\xi dx = \int_\Omega f h(u)\xi dx.$$

We deduce that

$$\lim_{\epsilon\to 0}\left(\int_\Omega f_\epsilon h(u_\epsilon)\xi dx + \int_\Omega F.\nabla[h(u_\epsilon)\xi]dx\right) = \int_\Omega h(u)\xi d\mu.$$

The first term of (2.18) can be written as

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial T_{l_0+1}(u_\epsilon)}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx = \sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx,$$

for $l_0 > 0$ so that, by lemmas 2.7 and 2.10, we have

$$\begin{aligned} \lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u_\epsilon}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx &= \lim_{\epsilon\to 0} \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial T_{l_0+1}(u_\epsilon)}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u_\epsilon)\xi]dx \\ &= \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial T_{l_0+1}(u)}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u)\xi]dx \\ &= \int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}[h(u)\xi]dx. \end{aligned}$$

From the convergence result of lemmas 2.7 and 2.10, and using (2.18), we get

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\Omega \beta_\epsilon(u_\epsilon)h(u_\epsilon)\xi dx &= \int_\Omega h(u)\xi d\mu - \sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h(u)\xi]dx \\ &= \int_\Omega h(u)\xi dz \\ &= \int_\Omega h(u)\xi w dx + \int_\Omega h(u)\xi d\nu.\end{aligned}$$

Passing to the limit in (2.18) as $\epsilon \to 0$, we get

$$\sum_{i=1}^N \int_\Omega a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\frac{\partial}{\partial x_i}[h(u)\xi]dx + \int_\Omega wh(u)\xi dx + \int_\Omega h(u)\xi d\nu = \int_\Omega h(u)\xi d\mu. \tag{2.19}$$

Since (2.19) holds for any $h \in C_c^1(\mathbb{R})$, we can take $h = h_{l_0}$ with $[m, M] \subset [-l_0, +l_0]$ such that (2.1) holds.

Lemma 2.12. *Let $\eta \in W_0^{1,p_m(.)}(\Omega)$, Z in $\mathcal{M}_b^{p_m(.)}(\Omega)$ and $\lambda \in \mathbb{R}$ be such that*

$$\begin{cases} \eta \leq \lambda \text{ a.e. in } \Omega \text{ (resp. } \eta \geq \lambda) \\ Z = -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{2.20}$$

Then

$$\int_{[\eta=\lambda]} \xi dZ \geq 0, \tag{2.21}$$

(resp.)

$$\int_{[\eta=\lambda]} \xi dZ \leq 0, \tag{2.22}$$

for any $\xi \in C_c^1(\Omega)$, $\xi \geq 0$.

Proof. For the sake of completeness, let us give the arguments. For $n \geq 1$, let $\varphi_n(r) = \inf(1, (nr + 1 - n\lambda)^+)$. Note that $\varphi_n(r)$ converges to $\chi_{[\lambda,\infty)}(r)$ for every $r \in \mathbb{R}$, so $\varphi_n(\eta(x))$ converges to $\chi_{[\lambda,\infty)}(\eta(x))$ at every x where $\eta(x)$ is defined. As η is defined quasi everywhere and $\chi_{[\lambda,\infty)} \circ \eta = \chi_{\{x\in\Omega:\eta(x)=\lambda\}}$, then the convergence of $\varphi_n(\eta)$ to $\chi_{[\lambda,\infty)}(\eta)$ is quasi everywhere.
Therefore, since Z is diffuse, then $\varphi_n(\eta)$ converges to $\chi_{\{x\in\Omega:\eta(x)=\lambda\}}$, $Z-$a.e. in Ω.

Next, we use the Lebesgue dominated convergence theorem and (1.6) to get

$$\begin{aligned}\int_{[\eta=\lambda]} \xi dZ &= \lim_{n\to+\infty} \int_\Omega \xi\varphi_n(\eta)dZ \\ &= \lim_{n\to+\infty} \int_\Omega \sum_{i=1}^N a_i\Big(x, \frac{\partial\eta}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(\xi\varphi_n(\eta))dx \\ &\geq \lim_{n\to+\infty} \int_\Omega \sum_{i=1}^N a_i\Big(x, \frac{\partial\eta}{\partial x_i}\Big)\varphi_n(\eta)\frac{\partial}{\partial x_i}(\xi)dx \\ &\geq -\max_i ||\frac{\partial\xi}{\partial x_i}||_\infty \lim_{n\to+\infty} \sum_{i=1}^N \int_{\{x\in\Omega:\lambda-\frac{1}{n}\leq\eta(x)\leq\lambda\}} |a_i(x, \frac{\partial\eta}{\partial x_i})|dx \\ &\geq -\max_i ||\frac{\partial\xi}{\partial x_i}||_\infty \sum_{i=1}^N \int_\Omega |a_i(x,0)|dx \\ &= 0,\end{aligned}$$

since $a_i(x,0) = 0$ for any $i = 1, ..., N$ and for a.e. $x \in \Omega$. Indeed, for $x \in \Omega$ fixed, denote $z_i = a_i(x,0) \in \mathbb{R}$.
By the continuity of $a_i(x,.)$, we have $\lim_{\xi\to 0} a_i(x,\xi) = z_i$. Suppose now that $z_i \neq 0$ and choose $\xi_{0,i} = -sz_i$ with $s > 0$ used to tend toward 0; then $a_i(x;\xi_{0,i}).\xi_{0,i} = -s(z_i + \epsilon(s)).z_i \leq -s|z_i|^2 + s|z_i||\epsilon(s)|$, where $\lim_{s\to 0} |\epsilon(s)| = 0$. Therefore, for s sufficiently small, $-s|z_i|^2 + s|z_i||\epsilon(s)| < 0$, which is a contradiction by assumption (1.6). Thus, $z_i = 0$ for any $i = 1, ..., N$.
Finally, if $\eta \geq \lambda$, we do the same calculus with $\tilde{\eta} = -\eta$, $\tilde{\lambda} = -\lambda$ and $\tilde{a}_i(x,\eta) = -a_i(x,-\eta)$ to get the result.
Since

$$\nu = \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}) - w\mathcal{L}^N + \mu,$$

we have

$$\mu - \nu - w\mathcal{L}^N = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \frac{\partial u}{\partial x_i}).$$

By the Lemma 2.12, for any $\xi \in C_c^1(\Omega), \xi \geq 0$, we have

$$\int_{[u=M]} \xi d\nu^+ \leq \int_{[u=M]} \xi d\mu - \int_{[u=M]} \xi w dx$$

and

$$\int_{[u=m]} \xi d\nu^- \leq -\int_{[u=m]} \xi d\mu + \int_{[u=m]} \xi w dx.$$

The first inequality implies that

$$\int_\Omega \xi d\nu^+ \leq \int_\Omega \xi d\mu\lfloor[u=M] - \int_\Omega \xi w \chi_{[u=M]} dx.$$

Consequently (2.2) holds. Similarly we get (2.3). □

Proof of Theorem 2.2. If (u,w) is a solution of (1.1) in the sense of Theorem 2.1, for any $\xi\in W_0^{1,\overrightarrow{p}(.)}(\Omega)\cap L^\infty(\Omega)$ with $\xi\in dom(\beta)$ and for any for $k>0$, the function $T_k(u-\xi)$ belongs to $W_0^{1,\overrightarrow{p}(.)}(\Omega)\cap L^\infty(\Omega)$ and then, it can be used as a test function in (2.1) to get

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-\xi)dx+\int_\Omega wT_k(u-\xi)dx+\int_\Omega T_k(u-\xi)d\nu=\int_\Omega T_k(u-\xi)d\mu,\ \text{for any } k>0. \tag{2.23}$$

We split the third term in (2.23) as follow.

$$\begin{aligned}\int_\Omega T_k(u-\xi)d\nu &= \int_{[u=M]}T_k(u-\xi)d\nu^+ - \int_{[u=m]}T_k(u-\xi)d\nu^-\\ &= \int_{[u=M]}T_k(M-\xi)d\nu^+ - \int_{[u=M]}T_k(m-\xi)d\nu^-\\ &\geq 0.\end{aligned}$$

Then, from (2.23) we have (2.4).

We now prove the uniqueness of the solution.
Suppose that $(u_1,w_1),\ (u_2,w_2)$ are two solutions of (1.1). For u_1 we choose $\xi=u_2$ as a test function in (2.4) to get

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_1}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u_1-u_2)dx+\int_\Omega w_1T_k(u_1-u_2)dx\leq\int_\Omega T_k(u_1-u_2)d\mu.$$

Similarly, for u_2 by taking $\xi=u_1$ as a test function in (2.4), we get

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u_2}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u_2-u_1)dx+\int_\Omega w_2T_k(u_2-u_1)dx\leq\int_\Omega T_k(u_2-u_1)d\mu.$$

Adding these two last inequalities yields

$$\int_\Omega\sum_{i=1}^{N}\left(a_i(x,\frac{\partial u_1}{\partial x_i})-a_i(x,\frac{\partial u_2}{\partial x_i})\right)\frac{\partial}{\partial x_i}T_k(u_1-u_2)dx+\int_\Omega(w_1-w_2)T_k(u_1-u_2)dx\leq 0. \tag{2.24}$$

Since $a_i(x,.)$ and ∂j are monotones, for any $k>0$, it follows from (2.24) that

$$\int_\Omega\sum_{i=1}^{N}\left(a_i(x,\frac{\partial u_1}{\partial x_i})-a_i(x,\frac{\partial u_2}{\partial x_i})\right)\frac{\partial}{\partial x_i}T_k(u_1-u_2)dx=0. \tag{2.25}$$

From (2.25), it follows that there exists a constant c such that $u_1-u_2=c$ a.e. in Ω.
Using the fact that $u_1=u_2=0$ on $\partial\Omega$ we get $c=0$. Thus, $u_1=u_2$ a.e. in Ω. At last, let us see that $w_1=w_2$ a.e. in Ω and $\nu_1=\nu_2$. Indeed, for any $\varphi\in\mathcal{D}(\Omega)$, taking φ as a test function in (2.1) for the solutions (u_1,w_1) and (u_1,w_2) after substraction of these equalities, we get

$$\int_\Omega(w_1-w_2)\varphi dx+\int_\Omega\varphi d(\nu_1-\nu_2)=0.$$

Hence

$$\int_\Omega w_1 \varphi dx + \int_\Omega \varphi d\nu_1 = \int_\Omega w_2 \varphi dx + \int_\Omega \varphi d\nu_2.$$

Therefore

$$w_1 \mathcal{L}^N + \nu_1 = w_2 \mathcal{L}^N + \nu_2.$$

Since the Radon-Nikodym decomposition of a measure is unique, we get $w_1 = w_2$ a.e. in Ω and $\nu_1 = \nu_2$. □

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