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# ON NONEXISTENCE OF WARPED PRODUCT SUBMANIFOLDS OF LOCALLY CONFORMAL KAEHLER MANIFOLDS

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Abstract. The present note is devoted to the study of warped product submanifolds in locally conformal Kaehler manifolds. The second last section deals with the study of the existence of semi-slant submanifolds while in the last section we investigate the existence of doubly warped and doubly twisted product CR-submanifolds. Infact, we have generalized results of ([10], [11]) to locally conformal Kaehler manifolds.

## 1. Introduction

R. L. Bishop and B. O'Neill [2] gave the notion of warped product manifolds while constructing new examples of Riemannian manifolds with negative sectional curvatures. Warped product manifolds have applications in physics. For instance, they are widely used to provide setting to model space time near black holes or bodies with large gravitational force.

On the other hand, B. Y. Chen considered warped product CR-submanifolds of Kaehler manifolds and showed that there exist no warped product CR-submanifolds in the form $M_\perp \times_f M_T$, where $M_\perp$ is a totally real submanifold and $M_T$ is a holomorphic submanifold of a Kaehler manifold $M$. Then, he introduced CR-warped products which are warped product CR-submanifolds in the form $M_T \times_f M_\perp$ such that $M_T$ is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of $M$. Twisted product CR-submanifolds in Kaehler manifolds were also introduced by B. Y. Chen [3] and he showed that a twisted product CR-submanifold in the form $M_\perp \times_f M_T$ is a CR-product. Then, he considered twisted product CR- submanifolds in the form $M_T \times_f M_\perp$ and established a general sharp inequality for twisted product CR-submanifolds in Kaehler manifolds. B. Sahin [10] also studied doubly warped product and doubly twisted warped product submanifolds of a Kaehlerian manifold.

There is another generalization of complex (holomorphic) and totally real submanifolds known as slant submanifold, defined by Chen [5]. It is to be noted that a slant submanifold is called proper if it is neither holomorphic nor totally

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real submanifold and a proper CR-submanifold is never a slant submanifold. Papaghiuc [8] generalized notion of slant submanifolds to semi-slant submanifolds to obtain slant and CR-submanifolds as particular cases. In [11], B. Sahin also investigated semi-slant submanifolds in Kaehler manifolds which are warped products of the form $M_\theta \times_f M_T$ (respectively, $M_T \times_f M_\theta$), where $M_\theta$ is a proper slant submanifold and $M_T$ is a holomorphic submanifold of a Kaehler manifold.

In view of various applications of the warped product manifolds, the problem of existence (or non-existence) of warped product manifolds in various ambient manifolds assumes significance in general. Inspired by this, we devote the present paper to the study of warped product submanifolds in locally conformal Kaehler manifolds. The second last section deals with the study of the existence of semi-slant submanifolds while in the last section we investigate the existence of doubly warped and doubly twisted product CR-submanifolds. Infact, the purpose of this note is to generalize the results of ([10],[11]) to the case when abient manifold is locally conformal Kaehler manifold.

## 2. PRELIMINARIES

In this section, first we recall warped product manifold introduced by Bishop and O'Neill [2].

Let $(B, g_B)$ and $(F, g_F)$ be two Riemannian manifolds, $f : B \to (0, \infty)$ and let $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ be the projection maps given by $\pi(p, q) = p$ and $\sigma(p, q) = q$ for every $(p, q) \in B \times F$. Then, warped product manifold $M = B \times F$ is a manifold $B \times F$ with metric tensor

$$g(X, Y) = g_B(\pi_* X, \pi_* Y) + (f o \pi)^2 g_F(\sigma_* X, \sigma_* Y)$$

for every $X$ and $Y$ of $M$ and * denoting for tangent map. The function $f$ is called the warping function of the warped product manifold. In particular, if the warping function is constant, then manifold $M$ is said to be trivial. Now, let us recall the definition of l.c.K. manifold.

**Definition 2.1.** [6] *Let $(\widetilde{M}, J, g)$ be a Hermitian manifold of dimension $2m$. Let $\Omega$ be the Kaehler 2-form associated with $J$ and $g$ i.e. $\Omega(X, Y) = g(X, JY)$ for all $X, Y \in \chi(\widetilde{M})$. The manifold $\widetilde{M}$ is called locally conformal Kaehler (l.c.K.) manifold if there is a closed* 1-form $\omega$ *defined globally on $\widetilde{M}$ such that*

$$d\Omega = \omega \wedge \Omega.$$

The closed 1-form $\omega$ is called the Lee form of the l.c.K. manifold $\widetilde{M}$. Also $(\widetilde{M}, J, g)$ is globally conformal Kahler (g.c.K.) (respectively Kahler) if the Lee form $\omega$ is exact (respectively $\omega = 0$). Note that any simply connected l.c.K is g.c.K.

For a l.c.K. manifold $(\widetilde{M}, J, g)$ we define the Lee vector field $B = \omega^\sharp$ ,where $\sharp$ means the rising of the indices with respect to $g$, namely $g(X, B) = \omega(X)$; for all $X \in \chi(\widetilde{M})$. If $\widetilde{\nabla}$ denotes the Levi Civita connection of $(\widetilde{M}, J, g)$ then we have

$$(\widetilde{\nabla}_X J)Y = \frac{1}{2}\{\theta(Y)X - \omega(Y)JX - g(X, Y)A - \Omega(X, Y)B\} \tag{2.1}$$

for any $X, Y \in \chi(\widetilde{M})$. Here, $\theta = \omega o J$ and $A = -JB$ are the anti-Lee form and the anti-Lee vector field, respectively.

**Definition 2.2.** [1] *A Riemannian manifold $M$, isometrically immersed in a l.c.K. manifold $\widetilde{M}$ is called CR-submanifold if there exists on $M$ a differentiable holomorphic distribution $D$, i.e. $J_x D_x = D_x$ for all $x \in M$ whose orthogonal complement $D^\perp$ of $D$ in $T(M)$ is totally real distribution on $M$, i.e. $J_x D_x^\perp \subset T(M)_x^\perp$ for all $x \in M$. A CR-submanifold is called holomorphic submanifold if dim $D_x^\perp = 0$, totally real if dim $D_x = 0$ and proper if it is neither holomorphic nor totally real.*

The submanifold $M$ is called slant [5] if for all nonzero vector $X$ tangent to $M$ the angle $\theta(X)$ between $JX$ and $T(M)$ is a constant,that is, it does not depend on the choice of $x \in M$ and $X \in T(M)$. The submanifold $M$ is called semi-slant [8] if it is endowed with two orthogonal distributions $D$ and $D'$, where $D$ is invariant with respect to $J$ and $D'$ is slant, that is $\theta(X)$ between $JX$ and $D'_x$ is constant for $X \in D'_x$.

It is clear that holomorphic and totally real submanifolds are CR submanifolds (respectively, slant submanifolds) with $D^\perp = 0$ (resp. $\theta = 0$) and $D = 0$ (resp. $\theta = \frac{\pi}{2}$). It is also clear that CR-submanifolds and slant submanifolds are particular semi-slant submanifolds with $\theta = \frac{\pi}{2}$ and $D = 0$, respectively.

Let $M$ be a Riemannian manifold isometrically immersed in $\widetilde{M}$ and denote by the same symbol $g$ the Riemannian metric induced on $M$. Let $TM$ be the Lie algebra of vector fields in $M$ and $TM^\perp$, the set of all vector fields normal to $M$. Denote by $\nabla$ the Levi-Civita connection of $M$. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.2}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla_x^\perp N \tag{2.3}$$

for any $X, Y \in TM$ and any $N \in TM^\perp$, where $\nabla^\perp$ is the connection in the normal bundle $TM^\perp$, $h$ is the second fundamental form of $M$ and $A_N$ is the Weingarten endomorphism associated with $N$. The second fundamental form and the shape operator $A$ are related by

$$g(A_N X, Y) = g(h(X, Y), N) \tag{2.4}$$

We write

$$JX = TX + FX, \quad X \in T(M) \tag{2.5}$$

where, $TX$ is the tangential component of $JX$ and $FX$ is the normal component of $JX$. In a similar way for any vector field normal to $M$, We put

$$JN = BN + CN, \tag{2.6}$$

where,$BN$ and $CN$ are the tangential and the normal components of $JN$, respectively. It is easy to observe that $M$ is a slant submanifold of $M$ if and only if

$$T^2 = \lambda I \tag{2.7}$$

for some real number $\lambda \in [-1, 0]$[5], where $I$ denotes the identity transformation of the tangent bundle $TM$ of the submanifold $M$. Moreover, if $M$ is slant submanifold and $\theta$ is the slant angle of $M$, then $\lambda = -cos^2\theta$. Hence, for a slant submanifold, we have

$$g(TX, TY) = cos^2\theta g(X, Y) \tag{2.8}$$

$$g(FX, FY) = sin^2\theta g(X, Y) \tag{2.9}$$

for $X$, $Y$ tangent to $M$.

## 3. Warped Product Semi-Slant Submanifolds

In the present section, we investigate semi-slant submanifolds which are warped products of the form $M_\theta \times_f M_T$ (respectively, $M_T \times_f M_\theta$), where $M_\theta$ is a proper slant submanifold and $M_T$ is a holomorphic submanifold of $\widetilde{M}$. First, we recall that if $X$ and $Y$ be vector fields on $B$ and $V$, $W$ vector fields on $F$, then from Lemma 7.3 of [2], we have

$$\nabla_X V = \nabla_V X = (\frac{Xf}{f})V \tag{3.1}$$

Now, we prove our results.

**Theorem 3.1.** *Let $\widetilde{M}$ be a locally conformal Kaehler manifold. Then there do not exist warped product submanifolds $M = M_\theta \times_f M_T$ in $\widetilde{M}$ such that $M_\theta$ is a proper slant submanifold and $M_T$ is a holomorphic submanifold of $\widetilde{M}$.*

**Proof.** Let us suppose that M be a warped product semi-slant submanifold of a l.c.K. manifold $\widetilde{M}$. Then in the light of (2.3) we have

$$-g(A_{FV}JX, JY) = g(\widetilde{\nabla}_{JX}FV, JY)$$

for $X, Y \in \Gamma(TM_T)$ and $V \in \Gamma(TM_\theta)$. Thus, taking into account that $\widetilde{\nabla}$ is the Levi-civita connection. We obtain

$$g(A_{FV}JX, JY) = g(\widetilde{\nabla}_{JX}JY, FV).$$

In view of (2.5), we have

$$\begin{aligned} g(A_{FV}JX, JY) &= g(\widetilde{\nabla}_{JX}JY, JV - TV) \\ &= g(\widetilde{\nabla}_{JX}JY, JV) - g(\widetilde{\nabla}_{JX}JY, TV) \end{aligned}$$

which on using (2.1) and (3.1) reduces to

$$\begin{aligned} g(A_{FV}JX, JY) &= g(\widetilde{\nabla}_{JX}Y, V) + g(\widetilde{\nabla}_{JX}TV, JY) \\ &= g(\nabla_{JX}Y, V) - g(\nabla_{JX}TV, JY) \\ &= -V(lnf)g(JX, Y) + \frac{1}{2}g\{g(JX, Y)V \\ &\quad -g(X, Y)JV, B\} + TV(lnf)g(X, Y) \end{aligned} \tag{3.2}$$

for $X, Y \in \Gamma(TM_T)$ and $V \in \Gamma(TM_\theta)$.

On the other hand, by definition of semi-slant submanifolds, in the light of (3.1) we have

$$g(\nabla_{JX}V, X) = V(lnf)g(JX, X) = 0$$

for $X \in \Gamma(TM_T)$ and $V \in \Gamma(TM_\theta)$. Thus, using the fact that $\nabla$ is the Levi-Civita connection, we obtain

$$g(\nabla_{JX}X, V) = 0.$$

In view of Gauss formula and (2.1), we get

$$g(\widetilde{\nabla}_{JX}JX, JV) = 0$$
$$\Rightarrow g(\nabla_{JX}TV, JX) = g(h(JX, JX), FV)$$

where we have used (2.5) and the fact that $\nabla$ is the Levi-Civita connection. In the light of (3.1), from above equation, we have

$$TV(lnf)g(JX, JX) = g(h(JX, JX), FV).$$

Thus, by polarization identity, we get

$$TV(lnf)g(X, Y) = g(h(JX, JY), FV) \tag{3.3}$$

for $X, Y \in \Gamma(TM_T)$ and $V \in \Gamma(TM_\theta)$.

Therefore, using (2.4) we obtain

$$g(A_{FV}JX, JY) = g(h(JX, JY), FV). \tag{3.4}$$

Thus, (3.2),(3.3) and (3.4) imply

$$V(lnf)g(JX, Y) = 0$$

for $X, Y \in \Gamma(TM_T)$ and $V \in \Gamma(TM_\theta)$. Since, $M_T \neq \{0\}$ is a Riemannian and invariant, we obtain

$$V(lnf) = 0,$$

which shows that $f$ is constant. □

Theorem 3.1 tells us that there do not exist warped product semi-slant submanifolds in the form $M_\theta \times_f M_T$ in l.c.K. manifolds such that $M_\theta$ is a proper slant submanifold and $M_T$ is a holomorphic submanifold of $\widetilde{M}$ . Next, we prove

**Theorem 3.2.** *Let $\widetilde{M}$ be a locally conformal Kaehler manifold. Then there do not exist warped product submanifolds $M = M_T \times_f M_\theta$ in $\widetilde{M}$ such that $M_T$ is a holomorphic submanifold and $M_\theta$ is a proper slant submanifold of $\widetilde{M}$.*

**Proof.** From Gauss formula, we have

$$g(h(TZ, X), FW) = g(\widetilde{\nabla}_X TZ, FW)$$

for $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$. As $\widetilde{\nabla}$ is the Levi-Civita connection, using (2.5) we have

$$\begin{aligned} g(h(TZ, X), FW) &= -g(TZ, \widetilde{\nabla}_X JW) + g(TZ, \widetilde{\nabla}_X TW) \\ &= g(JTZ, \widetilde{\nabla}_X W) + g(TZ, \nabla_X TW) \end{aligned}$$

where we have used (2.1) and (2.2). In view of (2.5) and (2.2), we get

$$g(h(TZ,X),FW) = g(T^2Z,\widetilde{\nabla}_X W) + g(FTZ,h(X,W)) + g(TZ,\nabla_X TW)$$

which on applying (2.7) and (3.1) reduces to

$$\begin{aligned} g(h(TZ,X),FW) = & - cos^2\theta X(lnf)g(Z,W) + g(FTZ,h(X,W)) \\ & + X(lnf)g(TZ,TW). \end{aligned}$$

In the light of (2.8), we obtain

$$g(h(TZ,X),FW) = g(FTZ,h(X,W)).$$

Thus, for $Z = W$, we get

$$g(h(TZ,X),FZ) = g(FTZ,h(X,Z)) \tag{3.5}$$

for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$.

On the other hand, from (3.1) we have $g(\nabla_{TZ}X,Z) = X(lnf)g(TZ,Z) = 0$ due to $g(TZ,Z) = 0$ for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$. In the light of orthogonality of $TM_T$ and $TM_\theta$ and using (2.1), we obtain

$$g(\widetilde{\nabla}_{TZ}JZ,JX) = 0 \quad \forall X \in \Gamma(TM_T), Z \in \Gamma(TM_\theta).$$

In view of (2.5), we have

$$\begin{aligned} g(\widetilde{\nabla}_{TZ}TZ,JX) + g(\widetilde{\nabla}_{TZ}FZ,JX) = 0 \\ \Rightarrow -g(\nabla_{TZ}JX,TZ) - g(JX,A_{FZ}TZ) = 0 \end{aligned}$$

where we have used (2.2) and (2.3). Further, using (2.4), we get

$$g(\nabla_{TZ}JX,TZ) + g(h(TZ,JX),FZ) = 0$$

which on applying (3.1) reduces to

$$JX(lnf)g(TZ,TZ) + g(h(TZ,JX),FZ) = 0.$$

Taking account of (2.8), we have

$$JX(lnf)cos^2\theta g(Z,Z) = -g(h(TZ,JX),FZ) = 0.$$

Replacing $X$ by $JX$

$$X(lnf)cos^2\theta g(Z,Z) = -g(h(TZ,X),FZ) = 0. \tag{3.6}$$

and replacing $Z$ by $TZ$, we have

$$X(lnf)cos^2\theta g(TZ,TZ) = -g(h(T^2Z,X),FTZ)$$

which on using (2.7) and (2.8), gives

$$X(lnf)cos^2\theta g(Z,Z) = g(h(Z,X),FTZ). \tag{3.7}$$

Therefore, from (3.5), (3.6) and (3.7) we have $X(lnf)cos^2\theta g(Z,Z) = 0$ for $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$. As $M_\theta$ is a proper slant and $Z$ is non-null, we get $X(lnf) = 0$. □

## 4. Doubly warped and doubly twisted product CR-submanifolds

In the present section, we consider CR-submanifolds which are doubly warped or doubly twisted products in the form ${}_fM_T \times_b M_\perp$, where $M_T$ is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of $\widetilde{M}$ (locally conformal Kaehler manifolds or locally conformal quaternion Kaehler manifolds) and investigate their existence in $\widetilde{M}$. In general, doubly warped and doubly twisted product manifolds can be considered as generalization of warped products. Suppose that $(B, g_B)$ and $(F, g_F)$ be Riemannian manifolds of dimensions $m$ and $n$, respectively and further suppose that $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ be the canonical projections. Let $b : B \times F \to (0, \infty)$, $f : B \times F \to (0, \infty)$ be smooth functions. Then the doubly twisted product ([7],[9]) of $(B, g_B)$ and $(F, g_F)$ with twisting functions $b$ and $f$ is defined to be the product manifold $M = B \times F$ with metric tensor

$$g = f^2 g_B \oplus b^2 g_F$$

Denoting this kind of manifolds by ${}_fB \times_b F$ and by $F(B)$ denoting the algebra of smooth functions on $B$ and by $\Gamma(E)$ the $F(B)$ module of smooth sections of a vector bundle $E$ (same notation for any other bundle) over $B$. If $X \in \Gamma(TB)$ and $V \in \Gamma(TF)$, then from Proposition 1 of [7], we have

$$\nabla_X V = V(lnf)X + X(lnb)V \tag{4.1}$$

where $\nabla$ denotes the Levi-Civita connection of the doubly twisted product ${}_fB \times_b F$ of $(B, g_B)$ and $(F, g_F)$. In particular, if $f = 1$, then $B \times_b F$ is called the twisted product of $(B, g_B)$ and $(F, g_F)$ with twisting function $b$. We note that the notion of twisted products was introduced in [4].

If $M = B \times_b F$ is a twisted product manifold, then (3.1) becomes

$$\nabla_X V = X(lnb)V \tag{4.2}$$

Moreover, if $b$ only depends on the points of $B$, then $B \times_b F$ is called warped product of $(B, g_B)$ and $(F, g_F)$ with warping function $b$. In this case, for $X \in \Gamma(TB)$ and $V \in \Gamma(TF)$, from Lemma 7.3 of [2], we have

$$\nabla_X V = X(lnb)V \tag{4.3}$$

where $b$ depends on the points of $B$ and $X \in \Gamma(TB)$, $V \in \Gamma(TF)$. ${}_fB \times_b F$ is called the doubly warped product of Riemannian manifolds $(B, g_B)$ and $(F, g_F)$ with warping functions $b$ and $f$ if only depend on the points of $B$ and $F$ , respectively.

We prove the following Theorem

**Theorem 4.1.** *Let $\widetilde{M}$ be a locally conformal Kaehler manifold. Then there do not exist doubly warped product CR-submanifolds which are not (singly) warped product CR-submanifolds in the form ${}_fM_T \times_b M_\perp$ such that $M_T$ is a holomorphic submanifold and $M_\perp$ is a totally real submanifold of $\widetilde{M}$.*

**Proof.** For a doubly warped product CR-submanifold M of a l.c.K manifold $\widetilde{M}$ from equation (2.1), we observe that

$$\widetilde{\nabla}_X JY - J\widetilde{\nabla}_X Y = \frac{1}{2}\{\theta(Y)X - \omega(Y)JX - g(X,Y)A - \Omega(X,Y)B\}$$

for $X, Y, B \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. Using (2.2), we have

$$g(\nabla_X JY, V) + g(h(X,Y), JV) = 0 \tag{4.4}$$

But (4.1) gives the following

$$g(\nabla_X V, JY) = V(lnf)g(X, JY) \tag{4.5}$$

Taking into account that $D$ and $D^{\perp}$ are orthogonal, we obtain

$$g(\nabla_X JY, V) = -g(JY, \nabla_X V).$$

Hence, from (4.4) and (4.5), we have

$$g(h(X,Y), JV) - V(lnf)g(X, JY) = 0.$$

For $X = JX$ and $Y = JY$ above equation reduces to

$$g(h(JX, JY), JV) + V(lnf)g(JX, Y) = 0 \tag{4.6}$$

for $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$.

Now, in the light of equation (4.1) we get

$$g(\nabla_{JX} V, X) = V(lnf)g(JX, X) = 0$$

for $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. Equation (2.2) gives $g(\widetilde{\nabla}_{JX} V, X) = 0$ and so by the orthogonality of $D$ and $D^{\perp}$ we obtain

$$g(\widetilde{\nabla}_{JX} X, V) = 0.$$

In view of equation (2.1), above equation reduces to $g(\widetilde{\nabla}_{JX} JX, JV) = 0$.
From gauss formula, we have

$$g(h(JX, JX), JV) = 0$$

Substituting $X$ by $X + Y$ in above equation and using symmetry of $h$, we have

$$g(h(JX, JY), JV) = 0 \tag{4.7}$$

for $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. Equations (4.6) and (4.7) give the following

$$V(lnf)g(JX, Y) = 0.$$

For $Y = JX$, we get $V(lnf)g(JX, JX) = 0$ and so we drive $V(lnf) = 0$ due to $g(Y,Y) \neq 0$. Hence, $V(lnf) = 0$ implies that $f$ is constant. Thus, $M$ is a warped product CR-submanifold in the form $M_T \times_b M_{\perp}$ which is called CR-warped product (see [4]). This proves the result. □

The rest part of this section deals with the investigation of existence of doubly twisted product CR-submanifolds in l.c.K. manifolds.

**Theorem 4.2.** *Let $\widetilde{M}$ be a locally conformal Kaehler manifold. Then there does not exist doubly twisted product CR-submanifolds of $\widetilde{M}$ which are not (singly) twisted product CR-submanifolds in the form ${}_fM_T \times_b M_{\perp}$ such that $M_T$ is a holomorphic submanifold and $M_{\perp}$ is a totally real submanifold of $\widetilde{M}$.*

**Proof.** Let us suppose that $M$ be a doubly twisted product CR-submanifold of a l.c.K. manifold $\widetilde{M}$. Then from (2.1) we have

$$\widetilde{\nabla}_X JY - J\widetilde{\nabla}_X Y = \frac{1}{2}\{\theta(Y)X - \omega(Y)JX - g(X,Y)A - \Omega(X,Y)B\}$$

for $X, Y, B \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. Using (2.2) we have

$$g(\nabla_X Y, V) = g(h(X, JY), JV) \tag{4.8}$$

From equation (4.1), we get

$$g(\nabla_X V, Y) = V(lnf)g(X,Y)$$

Taking into account that $D$ and $D^{\perp}$ are orthogonal, we obtain

$$-g(V, \nabla_X Y) = V(lnf)g(X,Y) \tag{4.9}$$

So, from (4.8) and (4.9), we have

$$-g(h(X, JY), JV) = V(lnf)g(X,Y) \tag{4.10}$$

for $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. But, by the use of Gauss formula we have, $g(h(X, JY), JV) = g(\widetilde{\nabla}_{JY} X, JV)$ which upon applying equation (2.1) reduces to $g(h(X, JY), JV) = -g(\widetilde{\nabla}_{JY} JX, V)$. In the light of Gauss formula and taking into account the orthogonality of $D$ and $D^{\perp}$ , above equation reduces to

$$g(h(X, JY), JV) = g(\nabla_{JY} V, JX)$$

In view of equation (4.1), above equation gives

$$g(h(X, JY), JV) = V(lnf)g(JX, JY) \tag{4.11}$$

for $X, Y \in \Gamma(D)$ and $V \in \Gamma(D^{\perp})$. From equations (4.10) and (4.11), we have $V(lnf)g(X,Y) = 0$. Since $D$ is Riemannian, we get $V(lnf) = 0$. This implies that $f$ only depends on the point of $M_T$. Thus, we can write

$$g = \overline{g}_{M_T} \oplus b^2 g_{M_\perp}$$

where $\overline{g}_{M_T} = f^2 g_{M_T}$.

Thus, it follows that $M$ is a twisted product CR-submanifold in the form $M_T \times_b M_\perp$ ([3] for twisted product CR-submanifolds). Hence, we conclude that there are no doubly twisted product CR-submanifolds in l.c.K. manifold, other than twisted product CR-submanifolds. □

**Applications.** The notion of warped product manifolds was investigated as a need of constructing new examples of Riemannian manifolds with negative sectional curvatures [2]. Warped product manifolds have applications in physics and relativity. In fact, warped product turned out to be the standard space time model of the universe as they are widely used to provide setting to model space time near black holes or bodies with large gravitational force.

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