Gulf Journal of Mathematics
Vol 6, Issue 2 (2018) 41-49

TRANSVERSAL DOMINATION IN GRAPHS

NAYAKA S. R.[1], ANWAR ALWARDI[2*] AND PUTTASWAMY[1]

Abstract. Let $G = (V, E)$ be a graph. A dominating set S of G which intersects every minimum dominating set in G is called a transversal dominating set. The minimum cardinality of a transversal dominating set is called the transversal domination number, denoted by $\gamma_{td}(G)$. In this paper, we begin to study this parameter. We calculate $\gamma_{td}(G)$ for some families of graphs. Further some bounds and relations with other domination parameters are obtained for $\gamma_{td}(G)$.

1. Introduction and preliminaries

Let $G = (V, E)$ be any graph. For any graph theoretic parameters not defined here, refer to [1, 2]. Recently, Ismail Sahul Hamid [4] defined and studied a domination parameter called as independent transversal domination. This parameter is a combination of two graph invariants namely independence and domination. Motivated by this, in this paper we begin to study the new domination parameter called transversal domination in graphs.

Let v be any vertex in G. Then, the open neighborhood of a vertex v is denoted by $N(v)$ and is defined by $N(v) = \{u \in V | uv \in E\}$, the set of all vertices adjacent to v. The closed neighborhood of v is denoted $N[v]$ and defined by $N[v] = N(v) \cup \{v\}$. For any subset S of G, the open and closed neighborhoods of S in G is defined by $N(S) = \bigcup_{v \in S} N(v)$ and $N(S) = \bigcup_{v \in S} N[v]$.

A set $S \subseteq V$ is called a dominating set of G if $N[S] = V(G)$. The minimum cardinality of a dominating set in G is called the domination number of G, denoted by $\gamma(G)$. A subset D of $V(G)$ is called an independent set in G if any two vertices in D are non-adjacent in G. In other words, the graph $\langle D \rangle$ is totally disconnected. The cardinality of the maximum independent set in G is called the independence number, denoted by $\beta(G)$.

An independent dominating set of G is a dominating set S such that S also an independent set in G. The independent domination number, denoted by $i(G)$ is the cardinality of the minimum independent dominating set in G. A dominating set S in G is called an independent transversal dominating set if it intersects every maximum independent set in G. The minimum cardinality of an independent

Date: Received: Jun 2, 2017; Accepted: Mar 16, 2018.
* Corresponding author.
2010 *Mathematics Subject Classification.* 05C69.
Key words and phrases. Dominating set, Independent Transversal Dominating set, Transversal Dominating set, Transversal Domination number.

transversal dominating set of G is called the independent transversal domination number in G, denoted by $\gamma_{it}(G)$. A complete tripartite graph $K_{1,1,2}$ is called a diamond graph.

Definition 1.1. The multi-star graph $K_m(a_1, a_2, \ldots, a_m)$ is a graph of order $a_1 + a_2 + \cdots + a_m + m$ formed by joining $a_1, a_2, \ldots, a_m$ end-edges to m vertices of K_m. For example, $K_2(a_1, a_2)$ is a double star.

Definition 1.2. The join $G = G_1 \vee G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

Definition 1.3. The corona product $G_1 \circ G_2$ is defined as the graph G obtained by taking one copy of G_1 of order n_1 and n_1 copies of G_2, and then joining the $i'th$ node of G_1 to every node in the $i'th$ copy of G_2.

Definition 1.4. A firefly graph $F_{s,t,n-2s-2t-1}$ $(s \geq 0, t \geq 0, n - 2s - 2t - 1 \geq 0)$ is a graph on n vertices having s triangles, t pendant paths of length 2 and $n - 2s - 2t - 1$ pendant edges sharing a common vertex.

We need the following theorem for our study:

Theorem 1.5. [3] *For any graph G with at least 2 vertices,* $3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1$.

2. Transversal Domination Number

Definition 2.1. A dominating set S which intersects every minimum dominating set in G is called a transversal dominating set. The minimum cardinality of a transversal dominating set in G is called the transversal domination number of G and is denoted by $\gamma_{td}(G)$. Any transversal dominating set S of G with $|S| = \gamma_{td}(G)$ is called a γ_{td}-set.

Example 2.2.
- Let G be a complete graph of order n. Then $\gamma_{td}(G) = n$.
- For $n \geq 3$, if $G \cong K_{1,n-1}$, then $\gamma_{td}(G) = 1$ and $\gamma_{td}(L(G)) = n - 1$.

Proposition 2.3. *Let $G \cong K_{m,n}$ be a complete bipartite graph. Then*

$$\gamma_{td}(K_{m,n}) = \begin{cases} 2 & \textit{if m=n=1;} \\ min\{m, n\} & \textit{otherwise.} \end{cases}$$

Proof. Let $G \cong K_{m,n}$ be a complete bipartite graph with the partite sets V_1 and V_2 where $|V_1| = m$ and $|V_2| = n$. Assume $m \leq n$. Let $S = V_1$. Then S is a dominating set in G intersecting every minimum dominating set in G. Hence $\gamma_{td}(G) \leq m$. For any two vertices u and v taken from V_1 and V_2 respectively, the set $\{u, v\}$ will be a minimum dominating set in G and so G contains mn minimum dominating sets. Hence each vertex in V_1 corresponds to n minimum dominating sets in G. Therefore $\gamma_{td}(G) \geq m$ and so $\gamma_{td}(G) = m$. i.e., $\gamma_{td}(G) = min\{m, n\}$. □

Proposition 2.4. *Let $G \cong K_{m_1,m_2,\ldots,m_r}$ be a multipartite graph with $m_1 \leq m_2, \ldots, \leq m_r$. Then*

$$\gamma_{td}(G) = \begin{cases} k & \text{if } m_i = 1 \text{ for } i \leq k; \\ m_1 + m_2 + \ldots + m_{r-1} & \text{otherwise.} \end{cases}$$

Proof. Let $G \cong K_{m_1,m_2,\ldots,m_r}$ be a multi-partite graph with n vertices. First, suppose $m_i = 1$ for $i \leq k$. Then $\gamma(G_i) = 1$, for each i, $1 \leq i \leq k$. Clearly, the set $S = \cup_{i=1}^{k} S_i$ of all $\gamma-$sets S_i of G_i will be the minimum transversal dominating set in G. Hence $\gamma_{td}(G) = k$.

Suppose $m_i \geq 2$ for all i and $V(G) = \{V_1, V_2, \ldots, V_m\}$ with $|V_i| = m_i$. Then, each vertex of V_i, for each i corresponds to a minimum dominating set in G. Let S be any transversal dominating set of G. Suppose $i \neq j$ and let u, v be the vertices of V_i and V_j respectively, then either u or v must be in S. For otherwise $\{u, v\}$ will be the minimum dominating set in G not intersected by S, a contradiction. Therefore $\gamma_{td}(G) \geq m_1 + m_2 + \cdots + m_{r-1}$. On the other hand $\cup_{i=1}^{r-1} V_i$ is a transversal dominating set in G of cardinality $m_1 + m_2 + \cdots + m_{r-1}$. Hence $\gamma_{td}(G) = m_1 + m_2 + \cdots + m_{r-1}$. □

Theorem 2.5. *For any path P_n of order $n \geq 2$, we have*

$$\gamma_{td}(P_n) = \begin{cases} \frac{n}{3}, & \text{if } n \equiv 0 \pmod 3; \\ 2 + \lceil \frac{n-3}{3} \rceil & \text{if } n \equiv 1 \text{ or } 2 \pmod 3. \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. We first observe that any minimum dominating set in P_n at least contains one of the vertex $v_i (1 \leq i \leq 2)$. We consider the following possible cases here:

Case 1: Suppose $n \equiv 0 \,(mod\, 3)$. Then $S = \{v_{3i-1} | 1 \leq i \leq \frac{n}{3}\}$ is a unique dominating set in G. Hence $\gamma_{td}(G) = \gamma(G) = \frac{n}{3}$.

Case 2: Suppose $n \equiv 1 \, or \, 2 \,(mod\; 3)$. Since any minimum dominating set in G contains at least one vertex from the sequence $\{v_1, v_2\}$, it follows that $\gamma_{td}(G) = 2 + \gamma(P_{n-3})$. i.e., $\gamma_{td}(G) = 2 + \lceil \frac{n-3}{3} \rceil$. □

Theorem 2.6. *For any cycle C_n with $n \geq 3$, we have*

$$\gamma_{td}(C_n) = \begin{cases} 3, & \text{if } n = 3, 4; \\ 3 + \lceil \frac{n-5}{3} \rceil & \text{otherwise.} \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $\{v_1, v_2, v_3\}$ be any sequence of three consecutive vertices in C_n. If $n \in \{3, 4\}$, then $\{v_1, v_2, v_3\}$ itself a transversal dominating set in C_n, so we are done. Next, assume $n \geq 5$. Then any minimum dominating set in C_n at least contains one of the vertex $v_i (1 \leq i \leq 3)$. Therefore $\gamma_{td}(C_n) = 3 + \gamma(H)$ where H is the graph obtained by deleting the vertices v_1, v_2, v_3 of the sequence and its neighbors. Clearly $H \cong P_5$. Therefore, $\gamma_{td}(C_n) = 3 + \gamma(P_{n-5}) = 3 + \lceil \frac{n-5}{3} \rceil$. □

Proposition 2.7. *Let $G = K_m(a_1, a_2, \ldots, a_m)$ be a multi-star graph. Then $\gamma_{td}(G) = m$.*

Proof. Let $G = K_m(a_1, a_2, \ldots, a_m)$ be a multi-star graph as shown in the Figure 1. Then G contains a unique minimum dominating set consisting of all support vertices of G. Hence $\gamma_{td}(G) = \gamma(G) = m$.

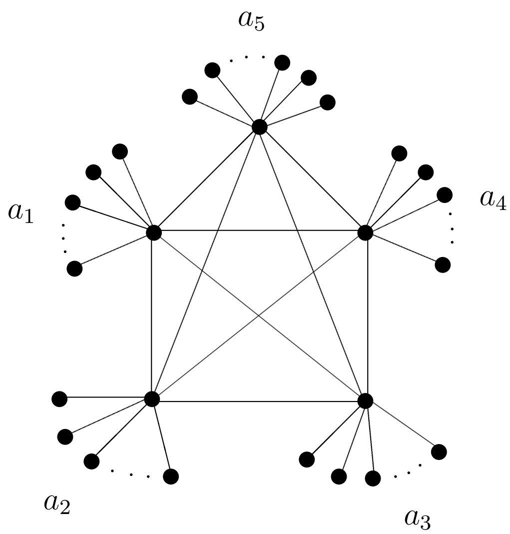

FIGURE 1. Multi star $K_m(a_1, a_2, \ldots, a_m)$.

Proposition 2.8. *Let G be a firefly graph with t pendant paths. Then $\gamma_{td}(G) = t + 1$.*

Proof. Let G be a firefly graph and let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let v_n be the vertex common to the triangles, pendent edges and pendent paths in G. Then, the vertex v_n dominates the triangles and the pendent edges in G. If $t = 0$, then $S = \{v_n\}$ is a unique dominating set in G. Hence $\gamma_{td}(G) = \gamma(G) = 1$.
Assume $t \geq 1$. Then G contains two minimum dominating sets having the vertex v_n in common. Hence the minimum dominating set in G itself the $\gamma_{td}-$set of G. Therefore $\gamma_{td}(G) = \gamma(G) = t + 1$. □

Proposition 2.9. *Let G_1 and G_2 be any two graphs with n_1 and n_2 vertices respectively. Then $\gamma_{td}(G_1 \vee G_2) = 1$ if and only if either $n_1 = 1$ or $n_2 = 1$.*

Proof. Suppose $\gamma_{td}(G_1 \vee G_2) = 1$. On contrary assume that $n_1, n_2 \geq 2$. Then by the definition of join of two graphs each vertex of G_1 is adjacent to every vertex in G_2 and vice versa. Hence $\gamma(G_1 \vee G_2) = 2$ and every vertex in G_1 correspond to a distinct minimum dominating set in $G_1 \vee G_2$. Hence $\gamma_{td}(G_1 \vee G_2) \geq min\{n_1, n_2\}$. By our assumption, we have $\gamma_{td}(G_1 \vee G_2) \geq 2$, a contradiction. This contradiction establishes the result. Converse is obvious. □

Corollary 2.10. *If W_n is a wheel on n vertices, then*

$$\gamma_{td}(W_n) = \begin{cases} 4, & \text{if } n = 4; \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let W_n be a wheel with n vertices. Then $W_n = K_1 \vee C_{n-1}$ where K_1 is a complete graph with one vertex and C_{n-1} is a cycle with $n - 1$ vertices. For $n = 4$, W_4 is a complete graph and so $\gamma_{td}(W_n) = 4$. For $n \neq 4$, we have from Proposition 1.5 that $\gamma_{td}(W_n) = 1$. □

Theorem 2.11. *Let G be any graph having k vertex disjoint minimum dominating sets. Then*

$$\gamma_{td}(G) \leq \gamma(G) + k - 1.$$

Proof. Let G be any graph and $S_1, S_2, \ldots, S_k$ be the vertex disjoint minimum dominating sets in G. Suppose $S = S_1 \cup \{v_i | v_i \in S_i, 2 \leq i \leq k\}$ obtained by attaching $k-1$ vertices to S_1. Then S clearly a dominating set in G intersecting every minimum dominating set in G. Hence $\gamma_{td}(G) \leq |S| = \gamma(G) + k - 1$. □

The bound given in Theorem 2.11 is sharp. For instance, the complete graph K_n contains n vertex disjoint minimum dominating sets each of cardinality one. Then $\gamma(G) + n - 1 = n = \gamma_{td}(G)$. But strict inequality holds for multi-partite graphs. For example, consider $G \cong K_{m,n}$ with $m \leq n$. Then G contains m vertex disjoint $\gamma-$sets each of cardinality 2. So $\gamma(G) + m - 1 = m + 1 > m = \gamma_{td}(G)$.

Corollary 2.12. *Let G be any graph. Then $\gamma_{td}(G) - \gamma(G) + 1$ gives the minimum number of disjoint $\gamma-$sets in G.*

Corollary 2.13. *Let G be any graph. Then $\gamma_{td}(G) = \gamma(G)$ if G contains a unique minimum dominating set.*

Corollary 2.14. *Let G be a Cycle with $n \geq 3$ vertices. Then $\gamma_{td}(G) = \gamma + 1$ or $\gamma + 2$.*

Proposition 2.15. *Let G be any graph of order n. Then for a non-complete graph H,*

$$\gamma_{td}(G \circ H) = \begin{cases} n, & \text{if } \gamma(H) \geq 2; \\ n+1 & \text{otherwise.} \end{cases}$$

Proof. Let G be any graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Now, for any graph H, if $\gamma(H) = 2$, then the graph corona $G \circ H$ contains a unique minimum dominating set namely the vertex set of G. Hence $\gamma_{td}(G) = \gamma(G) = n$. Next, Suppose $\gamma(H) = 1$, then $G \circ H$ contains two vertex distinct minimum dominating sets. For if $\{u\}$ is a $\gamma-$set of H then $V(G) \cup \{u\}$ will be a $\gamma_{td}-$set in $G \circ H$. Therefore $\gamma_{td}(G \circ H) = n + 1$. □

Corollary 2.16. *Let G be a sunlet graph. Then $\gamma_{td}(G) = n + 1$.*

Theorem 2.17. *If G is a disconnected graph with components $G_1, G_2, \ldots, G_m$ then $\gamma_{td}(G) = min_{1 \leq i \leq m}\{\gamma_{td}(G_i) + \Sigma_{j=1, j \neq i}^{m} \gamma(G_j)\}$.*

Proof. We prove this result by using mathematical induction. The result is trivially true for $m = 1$. Suppose that $m = 2$. Then $G = G_1 \cup G_2$. Let D_1, D_2 be the $\gamma_{td}-$sets of G_1 and G_2 respectively. Then, clearly $D_1 \cup S_2$ and $D_2 \cup S_1$ are transversal dominating sets in G, where S_1 and S_2 are $\gamma-$sets of G_1 and G_2 respectively. Therefore $\gamma_{td}(G) \leq min\{\gamma_{td}(G_1) + \gamma(G_2), \gamma_{td}(G_2) + \gamma(G_1)\}$.

Let S be any transversal dominating set in G. Then S must intersect the vertex set of both G_1 and G_2 and S contains the dominating set of G_1 and G_2. Further, the set S must contain the transversal dominating set of G_1 or G_2. If not, there are minimum dominating sets S_1 and S_2 of G_1 and G_2 respectively such that $S_1 \cup S_2$ will be a minimum dominating set of G not intersected by S, a contradiction. Thus $|S| \geq min\{\gamma_{td}(G_1) + \gamma(G_2), \gamma_{td}(G_2) + \gamma(G_1)\}$. Hence the result is true for $m = 2$.

Suppose $m > 2$ and assume the result for $m = k - 1$. Suppose G is a graph with components $G_1, G_2, ..., G_{k-1}, G_k$. Let G' be the graph whose components are $G_1, G_2, ..., G_{k-1}$. By induction hypothesis we have $\gamma_{td}(G') = min_{1\leq i\leq k-1}\{\gamma_{td}(G_i) + \Sigma_{j=1,j\neq i}^{k-1}\gamma(G_j)\}$. Now, we have $G = G' \cup G_m$. That is, G is a graph with two components G' and G_m. Hence from the case $m = 2$, we obtain that $\gamma_{td}(G) = min_{1\leq i\leq k}\{\gamma_{td}(G_i)+\Sigma_{j=1,j\neq i}^{m}\gamma(G_j)\}$. Therefore the result is true for $m = k$ and hence true for any positive integer m. □

Corollary 2.18. *Let $G = \cup_{i=1}^{m} G_i$. Then $\gamma_{td}(G) = m$ if and only if each component G_i contains a unique $\gamma-$set of size one.*

Proof. Let G be any disconnected graph and let $G_1, G_2, ..., G_m$ be the components of G. Then $|w(G)| = m$. First, we assume that $\gamma(G_i) = 1$, for each $i\,(1 \leq i \leq n)$, then from the above theorem, it follows that $\gamma_{td}(G) = \Sigma_{i=1}^{m-1}\gamma(G_j) = m$.

Conversely, assume $\gamma_{td}(G) = m$ and let S be the $\gamma_{td}-$set in G. Suppose that $\gamma(G_i) \geq 2$ for some i. Since S has to intersect every component $G_j(1 \leq i \leq n,\ j \neq i)$ of G, it contains at least $m - 1$ vertices, taking one from each of the remaining components. Hence, $|S| \geq m + 1$, a contradiction. This contradiction establishes the result. □

Corollary 2.19. *Let G be any disconnected graph having an isolated vertex. Suppose $\gamma(G_i) = 1$ for each component G_i of G then $\gamma(G) = \gamma_{td}(G) = i(G)$.*

Proof. Let G be a disconnected graph with an isloated vertex v. Then every $\gamma-$set in G contains the vertex v. Hence any $\gamma-$set in G will be a transversal dominating set in G proving that $\gamma_{td}(G) = \gamma(G)$. Further we have $\gamma(G_i) = 1$ for each i, so for any $\gamma-$set S in G, the induced graph $\langle S\rangle$ will be totally disconnected. Hence $\gamma(G) = i(G)$. □

Corollary 2.20. *Let G be any graph having an isolated vertex. Then*

$$\gamma_{td}(G) = \gamma_{it}(G) = \gamma(G).$$

Proof. Suppose G contains an isolated vertex. Then $\gamma(G) = \gamma_{td}(G)$. Since an isolated vertex lies in every maximum independent set in G, any $\gamma-$set S intersects every maximum independent set in G. Hence S is a transversal dominating set in G, proving that $\gamma_{it}(G) = \gamma(G)$. □

Converse of Corollary 2.19 and 2.20 need not be true. For example, consider a star $G \cong K_{1,n-1}$. Then G^+ is the graph obtained by attaching K_1 to each vertex of G. Clearly G contains no isolated vertex but $\gamma(G) = \gamma_{td}(G) = i(G) = \gamma_{it}(G)$.

Theorem 2.21. *Let $S_1,\ S_2, \ldots, S_k$ be the $\gamma-$sets of G. Then $\gamma_{td}(G) = \gamma(G)$ if and only if for some index $i\ \exists\ v \in S_i$ such that $N(v) \cap S_j \neq \phi$.*

Proof. Let G be any graph. Assume $\gamma_{td}(G) = \gamma(G)$. Suppose for all i and for all vertices v in S_i assume $N(v) \cap S_j = \phi$ for $j \neq i(1 \leq j \leq k)$. Let S be any $\gamma_{td}-$set in G. Then S is also a $\gamma-$set in G and hence either $S = S_i$ or $S = S_j$ where $j \neq i$. Clearly $S \neq S_i$, since $S_i \cap S_j = \phi$ for all $j \neq i$. But, if $S = S_j$ then also S fails to intersect the $\gamma-$set S_i of G, which is not possible. Hence $\gamma_{td}(G) \neq \gamma(G)$.

Conversely, Suppose there exists an index i and a vertex v in S_i such that $N(v) \cap S_j \neq \phi$. Let u be the neighbor vertex of v in S_j. Then the set $S' = (S - \{u\}) \cup \{v\}$ is transversal dominating set in G of cardinality γ. Therefore $\gamma_{td}(G) = \gamma(G)$. □

3. Bounds for γ_{td}

Proposition 3.1. *Let G be any connected graph of order n. Then $1 \leq \gamma_{td}(G) \leq n$. Further for $n \geq 2$, $\gamma_{td}(G) = n$ if and only if G is a complete graph.*

Proof. The inequalities are trivial. Now, suppose $n \geq 2$ and $\gamma_{td}(G) = n$. If $\gamma(G) \geq 2$, then for any vertex v in G, the set $V - \{v\}$ will be a minimum transversal dominating set of G and hence $\gamma_{td}(G) \leq n-1$, which is a contradiction. Thus $\gamma(G) = 1$ so that $G = K_n$. □

Proposition 3.2. *Let G be a connected graph with n vertices. Then $\gamma_{td}(G) = n - 1$ if and only if G is a path P_4 or a cycle C_4 or a diamond graph.*

Proof. Suppose $\gamma_{td}(G) = n - 1$. Choose an edge $e = xy$ in G and fix it. Since G connected, any vertex in G is adjacent either to x or y. If all the vertices of G are adjacent x or y alone, then G will be a star and $\gamma_{td}(G) = 1$, which is not possible. Clearly $n \geq 4$. Assume $n = 4$. Let u, v be the vertices of G adjacent to x and y respectively. First, suppose u and v are adjacent then $G \cong C_4$. If they are not adjacent then $G \cong P_4$. Suppose if u and v are not adjacent but they are adjacent to both x and y, then G is a diamond graph. Suppose $n > 4$. Then the vertices u and v must be adjacent to both x and y. Let w be any other vertex in G. Then w must be adjacent to both x and y. If w is adjacent to x or y alone then $\gamma_{td}(G) = 1$. For if w is adjacent to u or v, then $\gamma_{td}(G) \leq n - 2$, a contradiction. This contradiction shows that G is either a path P_4 or a cycle C_4 or a diamond graph. Converse is obvious. □

Proposition 3.3. *Let G be any connected graph with n vertices. Then*

$$\gamma(G) \leq \gamma_{td}(G) \leq \gamma_{it}(G) \leq n.$$

Proof. Let G be any graph with n vertices. The first inequality is trivial. Let S be any $\gamma_{it}-$set of G. Then S intersects every maximum independent set in G. Hence S intersects every minimal dominating set in G. Since any minimum dominating set is also a minimal dominating set, it follows that S intersects every minimum dominating set in G. Therefore S is a transversal dominating set of G and so we obtain that $\gamma_{td}(G) \leq \gamma_{it}(G)$. □

Theorem 3.4. *Let k be any positive integer. Then there is a graph G of order $2k$ such that $\gamma(G) = k$, $\gamma_{td}(G) = k + 1$.*

Proof. Let H be any connected graph with k vertices and let $V(H) = \{v_1, v_2, \ldots, v_k\}$. Now, for each vertex v_i of H attach a pendant edge to obtain the graph $G = H^+$. Then clearly G is a graph of order $2k$ having 2 minimum vertex disjoint dominating sets. Therefore $\gamma_{td}(G) = \gamma(G) + 1$. Since $V(H)$ is the $\gamma-$set of G, we have $\gamma(G) = k$ and $\gamma_{td}(G) = k + 1$. □

The idea of the above proof is the following. For any connected graph H, the graph $G \cong H \circ K_1$ is a graph of order $2|H|$ and having the minimum transversal domination number $2|H|+1$. More generally, we may state the above result as follows:

Proposition 3.5. *Let G be any graph. Then $\gamma(G) \leq \gamma_{td}(G) \leq \gamma(G) + \delta(G)$.*

Proof. Let G be any graph. The first inequality is trivial. Let S be any tansversal dominating set in G. Then, by definition S intersects every minimum dominating set in G. But, in a graph G, there will be $\delta(G)+1$ number of distinct dominating sets. So, adjoining $\delta(G)$ number of vertices to the minimum dominating set of S, the resulting set intersects each minimum dominating sets at the vertex adjoined to it. Therefore $\gamma_{td}(G) \leq \gamma(G) + \delta(G)$. □

Corollary 3.6. *Let G be any Forest. Then $\gamma_{td}(G)$ is either $\gamma(G)$ or $\gamma(G)+1$.*

Theorem 3.7. *Let G be any graph with at least two vertices. Then*

$$4 \leq \gamma_{td}(G) + \gamma_{td}(\overline{G}) \leq 2n.$$

Proof. Let G be any graph having at least two vertices. Then, we have $\gamma_{td}(G) \leq n$ and $\gamma_{td}(\overline{(G)}) \leq n$. Therefore $\gamma_{td}(G) + \gamma_{td}(\overline{(G)}) \leq 2n$. Similarly, we have $\gamma_{td}(G) \geq 2$ and $\gamma_{td}(\overline{(G)}) \geq 2$ and so $\gamma_{td}(G) + \gamma_{td}(\overline{G}) \geq 4$. This proves that, $4 \leq \gamma_{td}(G) + \gamma_{td}(\overline{G}) \leq 2n$. □

The bounds given in the above Theorem are sharp. Clearly, the lower bound is trivial. For instance, we may find equality for if G is the Path graph P_2. The upper bound attains for if G is the complete graph K_n.

Proposition 3.8. *Let G be any graph having isolate vertex such that $\Delta(G) = n-1$. Then $3 \leq \gamma_{pe}(G) + \gamma_{pe}(\overline{G}) \leq n+1$.*

Proof. Let G be any graph having isolate vertex such that $\Delta(G) = n-1$. Then both G and $\overline{G}$ contains an isolated vertex. Hence $\gamma_{td}(G) = \gamma(G)$ and $\gamma_{td}(\overline{(G)}) = \gamma\overline{(G)}$. Therefore, from Theorem 3.7, we obtain that $3 \leq \gamma_{pe}(G) + \gamma_{pe}(\overline{G}) \leq n+1$. □

Proposition 3.9. *Let G be any graph of diameter 2. Then $\gamma_{td}(G) \leq \delta(G)+1$.*

Proof. Let G be any graph of diameter 2 and let v be any vertex of G such that $degv = \delta(G)$. Since $diamG = 2$, any vertex in G adjacent to a vertex in $N[v]$. From which it follows that every dominating set in G contains a vertex from $N[v]$. Therefore $N[v]$ minimum transversal dominating set so that $\gamma_{td}(G) \leq \delta(G)+1$. □

References

1. J. A. Bondy, U. S. R. Murty, *Graph theory with application*, Elsevier science Publishing Co, Sixth printing. 1984.
2. T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker New york, 1998).
3. T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Domination in Graphs: Advanced Topics*, (Marcel Dekker, New York, 1998).

4. Ismail Sahul Hamid, *Independent Transversal Domination in Graphs*, Discussiones Mathematicae Graph Theory **32** (2012) 5-17.

[1] Department of Mathematics, P. E. S. College of Engineering, Mandya
Email address: nayaka.abhi11@gmail.com, prof.puttaswamy@gmail.com

[2] Department of Mathematics, College of Education, Yafea, University of Aden, Aden, Yemen.
Email address: a_wardi@hotmail.com