

DECAY OF ENERGY FOR A CLASS OF p -LAPLACIAN WAVE EQUATION WITH NONLINEAR DAMPING AND SOURCE TERMS

KAMEL YAHIAOUI¹ AND SOUFIANE MOKEDDEM^{2*}

ABSTRACT. In this paper the energy decay of global solution for a class of p -Laplacian wave equation with a nonlinear dissipation and source term are discussed, which is based on multipliers technique and some integral inequalities.

1. INTRODUCTION

We consider the initial boundary value problem for the nonlinear wave equation of p -Laplacian type with a strong dissipation of the form

$$(P) \begin{cases} (|u_t|^{l-2}u_t)_t - \Delta_p u + |u_t|^{m-2}u_t + (\alpha|u|^{\mu-2} - |u|^{\nu-2})u = |u|^{r-2}u \text{ in } \Omega \times [0, +\infty[\\ u(x, t) = 0 \text{ on } \partial\Omega \times [0, +\infty[\\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ on } \Omega. \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla_x u|^{p-2}\nabla_x u)$ and $p, l, r, m \geq 2$ are real numbers, Ω is a bounded domain in \mathbf{R}^n with smooth boundary $\partial\Omega$ and the real numbers α, μ and ν satisfy appropriate conditions to be made precise in the sequel.

Global existence and decay rates of solutions for wave equation related to the problem (P) was investigated by many authors through various approaches and assumptions.

In the case where $l = 2$, Sango [14] considered the problem $u_{tt} - \Delta_p u - \Delta u_t + g(x, u) = f(x, u)$ in $\Omega \times [0, +\infty[$. By using Faedo-Galerkin approximation combining with the method of compactness and monotonicity (see [8] and [9]) he proved the global existence of solutions. Furthermore, under suitable conditions on $g(x, u)$ he established, both polynomial and exponential decay of energy.

Similar results concerning the decay properties of solution were obtained by Chen, Yao and Shao [3]. For this, they used a new method introduced by Martinez [11] based on new nonlinear integral inequality.

Date: Received: Sep 25, 2017; Accepted: Jan 23, 2018.

* Corresponding author.

2010 *Mathematics Subject Classification.* Primary 35B40, 35L70; Secondary 35A07.

Key words and phrases. p -Laplacian wave equation, Global solution, Nonlinear damping, Energy decay.

It is worth mentioning some other papers in connection with studies of the non-linear p -Laplacian wave equation type with damping term where several methods have been used to estimate the energy decay rate, e.g., [1], [16], [15], [10], [2], [15], [17], [18] and [12].

The main purpose of the present paper is to give an energy decay estimate of the solution of problem (P). Motivated by the ideas in [3], [18] and [12], we first investigate the global existence by constructing a stable set in $W_0^{1,p}(\Omega)$ (see [13]). Then we show the asymptotic behavior of solutions through the use of the integral inequality given by Komornik [5].

The content of this paper is organized as follows. In Section 2, some necessary preliminaries, lemmas and the main results are stated. The proof of asymptotic behavior of global solution is given in Section 3.

For simplicity of notation, hereafter we denote by $\|\cdot\|_p$ the Lebesgue space $L^p(\Omega)$ norm. Let l be a number with $2 \leq l \leq \infty$. We denote by $\|\cdot\|_l$ the L^l over Ω . In particular $\|\cdot\|_2$ denotes $L^2(\Omega)$ and (\cdot, \cdot) the inner product of $L^2(\Omega)$. We also write equivalent norm $\|\nabla \cdot\|_p$ instead of $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$. As usual, we write respectively $u(t)$ and $u_t(t)$ instead $u(x, t)$ and $u_t(t, x)$. Throughout this paper the functions considered are all real valued and C indicates positive constants, which may be differ in different places.

2. PRELIMINARIES AND MAIN RESULT

We begin by introducing some preliminary definitions which will be used throughout this work. We first define the functional:

$$J(u) = \frac{1}{p} \|\nabla_x u\|_p^p + \frac{1}{r} \int_{\Omega} H(x, u) dx - \frac{1}{r} \|u\|_r^r,$$

for $u \in W_0^{1,p}(\Omega)$. Where $H(x, u) = \int_0^u h(x, s) ds$ and $h(x, u) = (\alpha|u|^{\mu-2} - |u|^{\nu-2})u$. Then, we are able to define the stable set \mathcal{S} for the problem (P) as,

$$\mathcal{S} \equiv \{u \in W_0^{1,p}(\Omega), K(u) > 0\} \cup \{0\},$$

where

$$K(u) = \|\nabla u\|_p^p + \int_{\Omega} H(x, u) dx - \|u\|_r^r.$$

We also define the energy function associated to the solution of the problem (P) as follows

$$E(t) = \frac{l-1}{l} \|u_t\|_l^l + \frac{1}{p} \|\nabla u\|_p^p + \int_{\Omega} H(x, u) dx - \frac{1}{r} \|u\|_r^r, \quad (2.1)$$

for $u \in W_0^{1,p}(\Omega)$ and $t \geq 0$.

Now we recall the following local existence theorem which can be established by using Faedo-Galerkin method. The proof closely follows the argument presented in [3], [8] and [15].

Theorem 2.1. *Assume $2 \leq m \leq p$ and let $2 < p < r < \frac{np}{n-p}$ if $n > p$ and $2 < p < r < \infty$ if $n \leq p$. Then, for $(u_0, u_1) \in W_0^{1,p}(\Omega) \times L^l(\Omega)$, the problem (P) has a unique local solution $u(t)$ such that*

$$\begin{aligned} u &\in L^\infty([0, T]; W_0^{1,p}(\Omega)), \\ u_t &\in L^\infty([0, T]; L^l(\Omega)) \cap L^m([0, T]; L^m(\Omega)). \end{aligned}$$

for any $T > 0$.

We first prepare the following useful lemmas before stating the global existence and energy estimate theorem. From now on, we denote the life span of the solution $u(t)$ of the problem (P) by T_{max} .

Lemma 2.2. *Let $u(t, x)$ be a solution to the problem (P) on $[0, \infty)$. Then $E(t)$ is a nonincreasing function, that is,*

$$\frac{d}{dt}E(t) = -\|u_t\|_m^m \leq 0.$$

for all $t \in [0, \infty)$.

This Lemma can be easily proved by multiplying the both sides of the first equation of (P) by u_t , integrating over Ω and then using integration by parts.

Lemma 2.3 (Sobolev-Poincaré's inequality). *Let r be a number with $2 \leq r < +\infty$ ($n \leq p$) or $2 \leq r \leq np/(n-p)$ ($n \geq p+1$). Then there is a constant $c_* = c_*(\Omega, r)$ such that*

$$\|u\|_r \leq c_* \|\nabla u\|_p \quad \text{for } u \in W_0^{1,p}(\Omega).$$

Lemma 2.4. *Assume that the hypotheses in Theorem 2.1 hold, then*

$$\frac{r-p}{rp} \|\nabla_x u\|_p^p \leq E(t), \tag{2.2}$$

for $u \in \mathcal{S}$.

Proof. The definition of $K(u)$ and $J(u)$ assume that

$$K(u) + \frac{r-p}{p} \|\nabla_x u\|_p^p = rJ(u).$$

Since $u \in \mathcal{S}$, so we have $K(u) \geq 0$. Hence we deduce from (2.1) that

$$\frac{r-p}{rp} \|\nabla_x u\|_p^p \leq J(u) \leq E(t).$$

□

Lemma 2.5 ([18]). *Let $u(t)$ be a solution to problem (P) on $[0, T_{max})$. Suppose that $2 \leq p < r \leq \frac{np}{n-p}$, $n > p$ and $2 < p < r < +\infty$, $n \leq p$. If $u_0 \in \mathcal{S}$ and $u_1 \in L^l(\Omega)$ satisfy*

$$\mathcal{M} = C^r \left(\frac{r-p}{rp} E(0) \right)^{\frac{r-p}{p}} < 1, \tag{2.3}$$

then $u(t) \in \mathcal{S}$, for each $t \in [0, T_{max})$.

In order to solve the energy decay of (P) , we use the following lemma.

Lemma 2.6 ([5]). *Let $E : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be a non-increasing function such that there are nonnegative constants q and $\omega > 0$ with*

$$\int_S^{+\infty} E(t)^{q+1} dt \leq \omega E(0)^q E(S), \quad 0 \leq S < +\infty, \quad (2.4)$$

then we have

$$E(t) \leq E(0) \left(\frac{\omega + qt}{\omega + q\omega} \right)^{-1/q} \quad \forall t \geq 0, \quad \text{if } q > 0,$$

and

$$E(t) \leq E(0) e^{1 - \frac{t}{\omega}} \quad \forall t \geq 0, \quad \text{if } q = 0.$$

This lemma is proved in [4] and [5] (see also [7] and [6]).

Here is our main result.

Theorem 2.7. *Let $u(t, x)$ be a local solution of problem (P) on $[0, T_{\max})$ with initial data $u_0 \in \mathcal{S}$, $u_1 \in L^l(\Omega)$ and sufficiently small initial energy $E(0)$. If the hypotheses in Theorem 2.1 are valid and $2 < m < \frac{np}{n-p}$, $n > p$ and $2 < m < \infty$, $n \leq p$, and assume that $2 \leq m \leq p$, then $T_{\max} = \infty$. Assume further that $p \leq \nu \leq 2p$ and $\nu < \mu < \frac{np}{n-p}$. Then, there exists $C_0 = C(u_0, u_1)$ and $\alpha' > 0$ with $\alpha > \alpha'$ such that the global solution of the problem (P) has the following energy decay property*

$$E(t) \leq C_0 t^{-\frac{p}{p-2}}, \quad \forall t \geq 0.$$

3. PROOF OF MAIN RESULT

Since $E(t)$ is a nonincreasing function on t , we have from (2.2) that

$$\frac{l-1}{l} \|u_t\|_l^l + \frac{r-p}{rp} \|\nabla u\|_p^p \leq \frac{l-1}{l} \|u_t\|_l^l + J(u) = E(t) \leq E(0).$$

Hence, we get

$$\|u_t\|_l^l + \|\nabla u\|_p^p \leq \max\left(\frac{l}{l-1}, \frac{rp}{r-p}\right) E(0) < +\infty.$$

The above inequality and the continuation principle lead to the global existence of the solution, i.e., $T_{\max} = \infty$.

Now we begin the estimates of energy associated to the solution of the problem (P) . From now on, C indicates positive constants, which may be differ in different places.

Let us remark that multiplying the both sides of first equation of (P) by $E(t)^q u$ and integrating over $\Omega \times [S, T]$, where $q = (p-2)/p > 0$ and $0 \leq S \leq T \leq \infty$, we obtain

$$0 = \int_S^T \int_{\Omega} E(t)^q u [(|u_t|^{l-2} u_t)_t - \operatorname{div}(|\nabla_x u|^{p-2} \nabla_x u) + |u_t|^{m-2} u_t + h(x, u) - |u|^{r-2} u] dx dt.$$

In addition

$$\begin{aligned}
& \int_S^T \int_\Omega E(t)^q u (|u_t|^{l-2} u_t)_t dx dt \\
&= \left[E(t)^q \int_\Omega u (|u_t|^{l-2} u_t) dx \right]_S^T - \int_S^T \int_\Omega E(t)^q |u_t|^l dx dt \\
&\quad - q \int_S^T \int_\Omega E(t)^{q-1} E'(t) u |u_t|^{l-2} u_t dx dt,
\end{aligned}$$

Hence from the definition of energy and a simple argument we can obtain

$$\begin{aligned}
& p \int_S^T E(t)^{q+1} dt \\
&= \left(\frac{p(l-1)}{l} + 1 \right) \int_S^T E(t)^q \|u_t\|_l^l dt - \int_S^T \int_\Omega E(t)^q u |u_t|^{m-2} u_t dx dt \\
&\quad - \left[E(t)^q \int_\Omega u |u_t|^{l-2} u_t dx \right]_S^T + q \int_S^T \int_\Omega E(t)^{q-1} E'(t) u |u_t|^{l-2} u_t dx dt \\
&\quad + \left(1 - \frac{p}{r} \right) \int_S^T E(t)^q \int_\Omega \|u\|_r^r dt + \int_S^T E^q(t) \int_\Omega (p H(u) - u h(x, u)) dx dt.
\end{aligned} \tag{3.1}$$

Now we must estimate every terms of the right-hand side of (3.1) to arrive at a similar inequality as (2.4). From (2.1) we see that

$$\int_S^T E(t)^q \|u_t\|_l^l dt \leq C \int_S^T E(t)^{q+1} dt. \tag{3.2}$$

Employing Hölder's inequality, Lemma 2.3 with the nonincreasingness of the energy $E(t)$, we obtain that

$$\begin{aligned}
& \left[E(t)^q \int_\Omega u |u_t|^{l-2} u_t dx \right]_S^T \\
&\leq C \left[E(t)^q \|u\|_2 \|u_t\|_{2(l-1)}^{l-1} \right]_S^T \\
&\leq C \left[E(t)^q \|\nabla u\|_p |\Omega|^{\frac{2-l}{2l}} \|u_t\|_l^{l-1} \right]_S^T \\
&\leq C \left[E(t)^q E(t)^{\frac{1}{p}} E(t)^{\frac{l-1}{l}} \right]_S^T \\
&\leq C E(S)^{q+\frac{1}{p}+\frac{l-1}{l}},
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
& \left| \int_S^T \int_{\Omega} E(t)^{q-1} E'(t) u u_t^{l-1} dx dt \right| \\
& \leq \int_S^T |E'(t)| E(S)^{q-1+\frac{1}{p}+\frac{l-1}{l}} dt \\
& \leq - \int_S^T E'(t) E(S)^{q-1+\frac{1}{p}+\frac{l-1}{l}} dt \\
& \leq C E(S)^{q+\frac{1}{p}+\frac{l-1}{l}},
\end{aligned} \tag{3.4}$$

Exploiting, the Sobolev-Poincaré inequality, (2.2) and (2.3) we get

$$\begin{aligned}
& \left(1 - \frac{p}{r}\right) \int_S^T E(t)^q \|u\|_r^r dt \\
& \leq \left(1 - \frac{p}{r}\right) \int_S^T E(t)^q C^r \|\nabla u\|_p^r dt \\
& = \left(1 - \frac{p}{r}\right) \int_S^T E(t)^q C^r \|\nabla u\|_p^{r-p} \|\nabla u\|_p^p dt \\
& \leq \left(1 - \frac{p}{r}\right) \int_S^T E(t)^q C^r \left(\frac{rp}{r-p} E(0)\right)^{\frac{r-p}{p}} \frac{rp}{r-p} E(t) dt \\
& = M_p \int_S^T E(t)^{q+1} dt,
\end{aligned} \tag{3.5}$$

Employing Hölder's inequality and Young's inequality we get the estimate

$$\begin{aligned}
& \left| - \int_S^T E(t)^q \int_{\Omega} u |u_t|^{m-2} u_t dx dt \right| \\
& \leq C \int_S^T E(t)^q \|u_t\|_m^{m-1} \|u\|_m dt \\
& \leq C \int_S^T E(t)^q (C(\epsilon) \|u_t\|_m^m + \epsilon \|u\|_m^m) dt \\
& = C \int_S^T E(t)^q \|u_t\|_m^m dt + C \int_S^T E(t)^q \|u\|_m^m dt \\
& = C \int_S^T E(t)^q (-E'(t)) dt + C \int_S^T E(t)^q \|u\|_m^m dt \\
& \leq C E(S)^{q+1} + C \int_S^T E(t)^q \|u\|_m^m dt,
\end{aligned}$$

where the fact that $E(t)$ in nonincreasing is used.

On the other hand, from the convexity of the function $\frac{u^y}{y}$ in y for $u \geq 0$ and $y > 0$, we can write

$$\frac{u^{C_1 y_1 + C_2 y_2}}{C_1 y_1 + C_2 y_2} \leq C_1 \frac{u^{y_1}}{y_1} + C_2 \frac{u^{y_2}}{y_2},$$

thus

$$\frac{\|u\|_m^m}{m} \leq 2C_1 \frac{\|u\|_p^2}{p} + C_2 \frac{\|u\|_r^r}{r},$$

where, $C_1 = \frac{4(r-m)}{p(r-2)}$ and $C_2 = \frac{m-2}{r-2}$ are positive constants.

Furthermore, we see that

$$\begin{aligned} \|u\|_m^m &\leq C_1 \|u\|_2^p + C_2 \|u\|_r^r \\ &\leq C(\|\nabla u\|_p^p + \|u\|_r^r) \\ &\leq CE(t), \end{aligned}$$

which implies that

$$\begin{aligned} &\left| - \int_S^T E(t)^q \int_{\Omega} u |u_t|^{m-2} u_t \, dx \, dt \right| \\ &\leq CE(S)^{q+1} + C \int_S^T E(t)^{q+1} \, dt. \end{aligned} \tag{3.6}$$

The last term of the right hand side of the first equation of (P) can be estimated as follows. From Sobolev-Poincaré inequality, there exists $\alpha' > 0$ such that

$$\alpha' \|u\|_p^p \leq \|\nabla u\|_p^p, \quad \forall u \in W_0^{1,p}(\Omega)$$

As $h(x, u) = (\alpha|u|^{\mu-2} - |u|^{\nu-2})u$, with $H(x, u) = \int_0^u h(x, s)ds$ we have,

$$H(u) = \alpha \frac{|u|^\mu}{\mu} - \frac{|u|^\nu}{\nu}. \tag{3.7}$$

We also notice that there exists $\alpha' > 0$ where $\alpha > \alpha'$ so that,

$$\frac{\alpha'}{2p} |u|^p + H(u) \geq \frac{\alpha}{2\mu} |u|^\mu, \quad \forall u \in \mathbf{R}. \tag{3.8}$$

Thus, taking into account (3.7) and (3.8), we have

$$\begin{aligned} &\int_S^T E^q(t) \int_{\Omega} (pH(u) - u h(x, u)) \, dx \, dt \\ &= \int_S^T E^q(t) \int_{\Omega} \left(\frac{\nu-p}{\nu} |u|^\nu - \frac{\alpha(\mu-p)}{\mu} |u|^\mu \right) \, dx \, dt \\ &\leq \int_S^T E^q(t) \int_{\Omega} \frac{\nu-p}{\nu} |u|^\nu \, dx \, dt \\ &= \int_S^T E^q(t) \int_{\Omega} (\nu-p) \left(\frac{\alpha}{\mu} |u|^\mu - H(u) \right) \, dx \, dt \\ &\leq \int_S^T E^q(t) \int_{\Omega} (\nu-p) \left(\frac{\alpha'}{p} |u|^p + 2H(u) - H(u) \right) \, dx \, dt \\ &\leq \int_S^T E^q(t) \int_{\Omega} (\nu-p) \left(\frac{\alpha'}{p} |u|^p + H(u) \right) \, dx \, dt \end{aligned}$$

Using the definition of the energy $E(t)$ we see that

$$\int_{\Omega} \left(\frac{\alpha'}{p} |u|^p + H(u) \right) dx \leq CE(t)$$

Consequently,

$$\begin{aligned} \int_S^T E^q(t) \int_{\Omega} (pH(u) - uh(u)) dx dt \\ \leq C(\nu - p) \int_S^T E^q E(t) dt \\ \leq C(\nu - p) \int_S^T E^{q+1}(t) dt. \end{aligned} \quad (3.9)$$

Then substituting the estimates (3.2), (3.3), (3.4), (3.5), (3.6) and (3.9) into (3.1), we get

$$\begin{aligned} \int_S^T E(t)^{q+1} \\ \leq C \left(E(S)^{q+\frac{1}{p}+\frac{l-1}{l}} + E(S)^{q+1} \right) \\ = CE(S) \left(E(S)^{q+\frac{1}{p}+\frac{l-1}{l}-1} + E(S)^q \right) \\ \leq CE(S) E(0)^q \left(E(0)^{\frac{1}{p}-\frac{1}{l}} + 1 \right) \\ = \omega E(S) E(0)^q, \end{aligned}$$

where $\omega = C \left(E(0)^{\frac{1}{p}-\frac{1}{l}} + 1 \right)$.

Letting $T \rightarrow +\infty$, this yield the following estimate,

$$\int_S^{+\infty} E(t)^{q+1} \leq \omega E(0)^q E(S), \quad \forall S \geq 0,$$

and we conclude from Lemma 2.6 that

$$E(t) \leq E(0) \left(\frac{\omega + qt}{\omega + q\omega} \right)^{-1/q} \leq E(0) \left(\frac{1 + \frac{q}{\omega}t}{1 + q} \right)^{-1/q} \leq C t^{-p/(p-2)}.$$

This completes the proof of Theorem 2.7.

REFERENCES

- [1] D. Ang and A. Dinh, *Strong solutions of quasilinear wave equation with nonlinear damping*, Siam Journal on Mathematical Analysis. **19** (1988), 289 – 299.
- [2] A. Benaissa and S. Mimouni, *Energy decay of solutions of wave equation of p -Laplacian type with a weakly nonlinear dissipation*, JIPAM. J. Inequal. Pure Appl. Math. **7** (2006)-1, Article 15, 8 pp.
- [3] C. Chen, H. Yao and L. Shao, *Global Existence, Uniqueness, and Asymptotic behavior of solution for p -Laplacian Type wave equation*, J. Inequal. Appl. 2010, Art. ID 216760, 15 pp.
- [4] A. Haraux, *Semi-groupes linéaires et équations d'évolution linéaires périodiques (in French)*, Preprint No. 78011, Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, Paris, 1978.

- [5] V. Komornik, *Exact Controllability and Stabilization*. RAM Research Appl. Math. Paris: Masson, Chichester: John Wiley 1994.
- [6] V. Komornik, *Decay estimates for the wave equation with internal damping*. In: Control and Estimation of Distributed Parameter Systems: Nonlinear Phenomena, (Proceedings Vorau (Austria) 1993; eds.: W. Desch et al.). Intern. Ser. Numer. Math. 118. Basel: Birkhäuser 1994, pp. 253 – 266.
- [7] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Diff. Equations. **50**(1983), 163 – 182.
- [8] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod-Gauthier Villars. Paris, France, 1969.
- [9] J. L. Lions and W. A. Strauss, *Some non-linear evolution equations*, Bull. Soc. Math. France. **93**(1965), 43 – 96.
- [10] T. F. Ma and J. A. Soriano, *On weak solutions for an evolution equation with exponential nonlinearities*, Nonlinear Anal. **37**(1999)(8), 1029 – 1038.
- [11] P. Martinez, *A new method to decay rate estimates for dissipative systems*, ESAIM Control Optim. Calc. Var. **4**(1999), 419 – 444.
- [12] S. Mokeddem and Kh. B. Mansour, *Asymptotic behaviour of solutions for p -Laplacian wave equation with m -Laplacian dissipation*, Z. Anal. Anwend. vol. **33**, no. 3, pp. 259 – 269, 2014.
- [13] D. H. Sattinger, *On global solution of nonlinear hyperbolic equations*. Arch. Ration. Mech. Anal. **30** (1968), 148 – 172.
- [14] M. Sango, *On a nonlinear hyperbolic equation with anisotropy: global existence and decay of solution*, Nonlinear Analysis: Theory, Methods and Applications. vol. **70**, no. 7, pp. 2816 – 2823, 2009.
- [15] Z. Yang, *Existence and asymptotic behaviour of solutions for a class of quasilinear evolution equations with nonlinear damping and source terms*, Mathematical Methods in the Applied Sciences. **25** (2002), 795 – 814.
- [16] Y. J. Ye, *On the decay of solutions for some nonlinear dissipative hyperbolic equations*, Acta Math. Appl. Sin. Engl. Ser. **20** (2004), 93 – 100.
- [17] Y. J. Ye, *Exponential decay of energy for some nonlinear hyperbolic equations with strong dissipation*, Adv. Difference Equ. 2010, Art. ID 357404, 12 pp.
- [18] Y. J. Ye, *Global existence and asymptotic behavior of solutions for some nonlinear hyperbolic equation*, J. Inequal. Appl. 2010, Art. ID 895121, 10 pp.

¹ LABORATORY OF BIOMATHEMATICS, DJILLALI LIABES UNIVERSITY, P. B. 89, SIDI BEL ABBES 22000, ALGERIA.

Email address: yahiaoui_kamel@yahoo.fr

² LABORATORY OF BIOMATHEMATICS, DJILLALI LIABES UNIVERSITY, P. B. 89, SIDI BEL ABBES 22000, ALGERIA

Email address: s_mokeddem@yahoo.com