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A NEW GENERALIZATION OF $(\in, \in\vee q)$-FUZZY SUBRINGS AND IDEALS

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Abstract. Jun et al. [7] introduced more general form, so called δ-quasi-coincident with a fuzzy set, of "quasi-coincident with" relation (q) of a fuzzy point with a fuzzy set. Using this notion, the concepts of $(\in, \in\vee q_0^\delta)$-fuzzy subrings/ideals, $(\in, \in\vee q_0^\delta)$-fuzzy radicals and $(\in, \in\vee q_0^\delta)$-fuzzy coset of a fuzzy set determined by an element of a ring are introduced, and related properties are investigated.

1. Introduction

Since the inception of the notion of a fuzzy set in 1965 which laid the foundations of fuzzy set theory, the literature on fuzzy set theory and its applications has been growing rapidly amounting by now to several papers. Murali [9] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [11], played a vital role to generate some different types of fuzzy algebraic structures, for example, semigroup theory (see [8]), group theory (see [1, 2, 3, 12]), ring theory (see [5]), near-ring theory (see [6, 10]) and so on. In 1996, Bhakat and Das [4] defined $(\in, \in\vee q)$-fuzzy subrings and ideals of a ring. Also, they introduced the concepts of $(\in, \in\vee q)$-fuzzy semiprime, prime, semiprimary, primary and maximal ideals, and obtained characterization of such fuzzy ideals. Bhakat and Das [5] discussed $(\in, \in\vee q)$-fuzzy cosets determined by $(\in, \in\vee q)$-fuzzy ideals and $(\in, \in\vee q)$-fuzzy radicals of $(\in, \in\vee q)$-fuzzy ideals. It is now natural to investigate more general form of "quasi-coincident with" relation (q), and Jun et al. [7] introduced the concept of "δ-quasi-coincident with" relation (q_0^δ), and apply it to fuzzy subgroups.

In this paper, we apply this new notion to rings, and generalize the contents in [5]. We introduce the notion of $(\in, \in\vee q_0^\delta)$-fuzzy subrings and ideals, which is a generalization of $(\in, \in\vee q)$-fuzzy subrings and ideals, and investigate related properties. We discuss relations between an $(\in, \in\vee q)$-fuzzy subring/ideal and an $(\in, \in\vee q_0^\delta)$-fuzzy subring/ideal. We consider characterizations of an $(\in, \in\vee q_0^\delta)$-fuzzy subring and ideal. We introduce $(\in, \in\vee q_0^\delta)$-fuzzy radicals and investigate

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its properties. We also introduce the notion of $(\in, \in\vee q_0^\delta)$-fuzzy coset of a fuzzy set determined by an element of a ring. We show that for any $(\in, \in\vee q_0^\delta)$-fuzzy ideal of a ring R, the set of all $(\in, \in\vee q_0^\delta)$-fuzzy cosets of λ in R is a ring under operations $\oplus$ and $\odot$. We induce a homomorphism between a given ring and a new ring, and investigate related properties.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.

Definition 2.1. ([11])A fuzzy set λ in a set X of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & if\ \ y = x, \\ 0 & if\ \ y \neq x, \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t.

Definition 2.2. ([11]) For a fuzzy point x_t and a fuzzy set λ in a set X, we say that

i) $x_t \in \lambda$ (resp. $x_t\, q\, \lambda$) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). In this case, x_t is said to *belong to* (resp. *be quasi-coincident with*) a fuzzy set λ.

ii) $x_t \in\vee\, q\, \lambda$ (resp. $x_t \in\wedge\, q\, \lambda$) if $x_t \in \lambda$ or $x_t\, q\, \lambda$ (resp. $x_t \in \lambda$ and $x_t\, q\, \lambda$).

Definition 2.3. ([5]) A fuzzy set λ in a ring R is called an $(\in, \in\vee q)$-*fuzzy subring* of R if for all $x, y \in R$ and $t, r \in (0, \delta]$,

i) $x_t \in \lambda,\ y_r \in \lambda \ \Rightarrow\ (x + y)_{\min\{t,r\}} \in\vee\, q\, \lambda$,

ii) $x_t \in \lambda \ \Rightarrow\ (-x)_t \in\vee\, q\, \lambda$,

iii) $x_t \in \lambda,\ y_r \in \lambda \ \Rightarrow\ (xy)_{\min\{t,r\}} \in\vee\, q\, \lambda$.

Definition 2.4. ([5]) A fuzzy set λ in a ring R is called an $(\in, \in\vee q)$-*fuzzy ideal* of R if

i) λ is an $(\in, \in\vee q)$-fuzzy subring of R,

ii) $x_t \in \lambda,\ y \in R \ \Rightarrow\ (xy)_t \in\vee\, q\, \lambda,\ (yx)_t \in\vee\, q\, \lambda$.

Jun et al. [7] generalized a quasi-coincident fuzzy point. Let $\delta \in (0, 1]$. For a fuzzy point x_t and a fuzzy set λ in a set X, we say that

- x_t is a δ-*quasi-coincident* with λ, written $x_t\, q_0^\delta\, \lambda$, if $\lambda(x) + t > \delta$.
- $x_t \in\vee\, q_0^\delta\, \lambda$ if $x_t \in \lambda$ or $x_t\, q_0^\delta\, \lambda$.
- $x_t\, \overline{\alpha}\, \lambda$ if $x_t\, \alpha\, \lambda$ does not hold where $\alpha \in \{\in, q, \in\vee q, \in\wedge q, \in\vee q_0^\delta, \in\wedge q_0^\delta\}$.

Obviously, $x_t\, q\, \lambda$ implies $x_t\, q_0^\delta\, \lambda$. If $\delta = 1$, then the δ-quasi-coincident with λ is the quasi-coincident with λ, that is, $x_t\, q_0^1\, \lambda = x_t\, q\, \lambda$.

3. Main results

In what follows let δ and R denote an element of $(0, 1]$ and a ring, respectively, unless otherwise specified.

Definition 3.1. A fuzzy set λ in R is called an $(\in, \in\vee q_0^\delta)$-*fuzzy subring* of R if for all $x, y \in R$ and $t, r \in (0, \delta]$,

$$x_t \in \lambda,\ y_r \in \lambda \ \Rightarrow\ (x+y)_{\min\{t,r\}} \in\vee\, q_0^\delta\, \lambda, \tag{3.1}$$

$$x_t \in \lambda \ \Rightarrow\ (-x)_t \in\vee\, q_0^\delta\, \lambda, \tag{3.2}$$

$$x_t \in \lambda,\ y_r \in \lambda \ \Rightarrow\ (xy)_{\min\{t,r\}} \in\vee\, q_0^\delta\, \lambda. \tag{3.3}$$

We know that a fuzzy set λ in R satisfies two condition (3.1) and (3.2) if and only if it satisfies:

$$x_t \in \lambda,\ y_r \in \lambda \ \Rightarrow\ (x-y)_{\min\{t,r\}} \in\vee\, q_0^\delta\, \lambda$$

for all $x, y \in R$and $t, r \in (0, \delta]$.

Example 3.2. Let $R = \{(a, b) \mid a, b \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the ring of integers, with the additive operation and the multiplicative operation defined as follows:

$(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (0, 0)$

for any $(a, b), (c, d) \in R$. Then $(R, +, \cdot)$ forms a ring with zero $(0, 0)$ (see [13]). Define a fuzzy set λ in R as follows:

$$\lambda : R \to [0, 1],\ x \mapsto \begin{cases} 0.77 & \text{if } x = (-2, -4), \\ 0.45 & \text{if } x \in A, \\ 0.33 & \text{if } x \in B, \\ 0.22 & \text{otherwise}, \end{cases}$$

where $A = \{(a, 4b) \mid a, b \in \mathbb{Z}\} \setminus \{(-2, -4)\}$ and

$$B = \{(a, 2b) \mid a, b \in \mathbb{Z}\} \setminus \{(a, 4b) \mid a, b \in \mathbb{Z}\}.$$

It is routine to verify that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring of R with $\delta \in (0, 0.9]$. If $\delta = 0.94 \in (0.9, 1]$, then λ is not an $(\in, \in\vee q_0^\delta)$-fuzzy subring of R since $(-2, -4)_{0.47} \in \lambda$ and $(3, 6)_{0.49} \in \lambda$, but

$$((-2, -4) + (3, 6))_{\min\{0.47,\, 0.49\}} = (1, 2)_{0.47}\, \overline{\in\vee\, q_0^\delta}\, \lambda.$$

Note that every $(\in, \in\vee q_0^\delta)$-fuzzy subring with $\delta = 1$ is an $(\in, \in\vee q)$-fuzzy subring.

Note that every $(\in, \in\vee q_0^\delta)$-fuzzy subring with $\delta = 1$ is an $(\in, \in\vee q)$-fuzzy subring.

Let $\delta_1 > \delta_2$ in $(0, 1]$. Then every $(\in, \in\vee q_0^\delta)$-fuzzy subring of R with $\delta = \delta_1$ is also an $(\in, \in\vee q_0^\delta)$-fuzzy subring of R with $\delta = \delta_2$. But, the converse is not true as seen in Example 3.2.

Obviously, every $(\in, \in\vee q)$-fuzzy subring is an $(\in, \in\vee q_0^\delta)$-fuzzy subring, but the converse is not true. In fact, the $(\in, \in\vee q_0^\delta)$-fuzzy subring of R with $\delta \in (0, 0.9]$ in Example 3.2 is not an $(\in, \in\vee q)$-fuzzy subring of R.

We provide a characterization of an $(\in, \in\vee q_0^\delta)$-fuzzy subring.

Theorem 3.3. *For a fuzzy set λ in R, the following are equivalent:*

i) λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring of R.

ii) $\lambda(x - y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ and $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in R$.

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring of R, and there exist $a, b \in R$ such that

$$\lambda(a+b) < \min\{\lambda(a), \lambda(b), \tfrac{\delta}{2}\}.$$

Then $\lambda(a+b) < t \leq \min\{\lambda(a), \lambda(b), \frac{\delta}{2}\}$ for some $t \in (0, \delta]$. It follows that $t \in (0, \frac{\delta}{2}]$, $a_t \in \lambda$ and $b_t \in \lambda$, but $(a+b)_t \overline{\in} \lambda$ and $\lambda(a+b) + t < 2t \leq \delta$, i.e., $(a+b)_t \overline{\in\vee q_0^\delta} \lambda$. This contradicts (3.1). Hence $\lambda(x+y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in R$. Similarly, $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in R$. Now, let $x \in R$ and $\lambda(x) = t < \frac{\delta}{2}$. Suppose that $r := \lambda(-x) < \lambda(x)$ and take $s \in (0, \delta]$ such that $r < s < t$ and $r + s < \delta$. Then $x_s \in \lambda$, but $(-x)_s \overline{\in} \lambda$ and $\lambda(-x) + s = r + s < \delta$, i.e., $(-x)_s \overline{\in\vee q_0^\delta} \lambda$, a contradiction. Hence $\lambda(-x) \geq \lambda(x) = \min\{\lambda(x), \frac{\delta}{2}\}$ for all $x \in R$. If $\lambda(x) \geq \frac{\delta}{2}$, then $x_{\frac{\delta}{2}} \in \lambda$. Assuming $\lambda(-x) < \min\{\lambda(x), \frac{\delta}{2}\}$ implies that $\lambda(-x) < \frac{\delta}{2}$ and $\lambda(-x) + \frac{\delta}{2} < \delta$, that is, $(-x)_{\frac{\delta}{2}} \overline{\in\vee q_0^\delta} \lambda$. This contradicts (3.2). Hence $\lambda(-x) \geq \lambda(x) = \min\{\lambda(x), \frac{\delta}{2}\}$ for all $x \in R$. Therefore we have $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ and $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in R$.

Conversely, suppose that the second condition is valid. Let $x_t \in \lambda$ and $y_r \in \lambda$ for $x, y \in R$ and $t, r \in (0, \delta]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq r$. It follows that

$$\begin{aligned}\lambda(x-y) &\geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \geq \min\{t, r, \tfrac{\delta}{2}\} \\ &= \begin{cases} \min\{t, r\} & \text{if } t \leq \frac{\delta}{2} \text{ or } r \leq \frac{\delta}{2}, \\ \frac{\delta}{2} & \text{if } t > \frac{\delta}{2} \text{ and } r > \frac{\delta}{2}, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\lambda(xy) &\geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \geq \min\{t, r, \tfrac{\delta}{2}\} \\ &= \begin{cases} \min\{t, r\} & \text{if } t \leq \frac{\delta}{2} \text{ or } r \leq \frac{\delta}{2}. \\ \frac{\delta}{2} & \text{if } t > \frac{\delta}{2} \text{ and } r > \frac{\delta}{2}. \end{cases}\end{aligned}$$

Hence $(x-y)_{\min\{t,r\}} \in\vee q_0^\delta \lambda$ and $(xy)_{\min\{t,r\}} \in\vee q_0^\delta \lambda$. Therefore λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring of R. □

Corollary 3.4. ([5]) *A fuzzy set* λ *in* R *is an* $(\in, \in\vee q)$*-fuzzy subring of* R *if and only if* $\lambda(x-y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ *and* $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ *for all* $x, y \in R$.

Definition 3.5. A fuzzy set λ in R is called an $(\in, \in\vee q_0^\delta)$*-fuzzy left* (resp. *right*) *ideal* of R if it is an $(\in, \in\vee q_0^\delta)$-fuzzy subring of R such that for all $x, a \in R$ and $t \in (0, \delta]$

$$x_t \in \lambda \;\Rightarrow (\mathrm{ax})_\mathrm{t} \in\vee \mathrm{q}_0^\delta \lambda (\mathrm{resp.} (\mathrm{xa})_\mathrm{t} \in\vee \mathrm{q}_0^\delta \lambda).$$

By an $(\in, \in\vee q_0^\delta)$*-fuzzy ideal*, we mean both an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal and an $(\in, \in\vee q_0^\delta)$-fuzzy right ideal.

Example 3.6. Consider the ring $R = \mathbb{Z}/(4)$ where $\mathbb{Z}$ is the ring of integers. Define a fuzzy set λ in R as follows:

$$\lambda : R \to [0,1],\ x \mapsto \begin{cases} 0.43 & \text{if } x = \bar{0}, \\ 0.25 & \text{if } x = \bar{1}, \\ 0.67 & \text{if } x = \bar{2}, \\ 0.25 & \text{if } x = \bar{3}. \end{cases}$$

Then λ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R with $\delta \in (0, 0.86]$. If $\delta = 0.9 \in (0.86, 1]$, then λ is not an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R since $\bar{2}_{0.44} \in \lambda$ but $(\bar{2}\cdot\bar{2})_{0.44} = \bar{0}_{0.44}\, \overline{\in\vee q_0^\delta}\, \lambda$. Also, it is not an $(\in, \in\vee q)$-fuzzy ideal of R since $\bar{2}_{0.52} \in \lambda$ but $(\bar{2}\cdot\bar{2})_{0.52} = \bar{0}_{0.52}\, \overline{\in\vee q}\, \lambda$.

Note that every $(\in, \in\vee q_0^\delta)$-fuzzy ideal with $\delta = 1$ is an $(\in, \in\vee q)$-fuzzy ideal. Let $\delta_1 > \delta_2$ in $(0, 1]$. Then every $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R with $\delta = \delta_1$ is also an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R with $\delta = \delta_2$. But, the converse is not true as seen in Example 3.6.

Obviously, every $(\in, \in\vee q)$-fuzzy ideal is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal, but the converse is not true as seen in Example 3.6.

Theorem 3.7. *For an $(\in, \in\vee q_0^\delta)$-fuzzy subring λ in R, the following are equivalent:*

i) λ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R.
ii) $\lambda(xy) \geq \min\{\lambda(x), \frac{\delta}{2}\}$ and $\lambda(yx) \geq \min\{\lambda(x), \frac{\delta}{2}\}$ for all $x, y \in R$.

Proof. The proof is straightforward by the similar way to the proof of Theorem 3.3. □

Definition 3.8. ([7]) For a subset B of R, a fuzzy set χ_B^δ in R defined by

$$\chi_B^\delta : R \to [0,1],\ x \mapsto \begin{cases} \delta & \text{if } x \in B, \\ 0 & \text{otherwise}, \end{cases}$$

is called a *δ-characteristic fuzzy set* of B in R.

Theorem 3.9. *For any subset S of R and the δ-characteristic fuzzy set χ_S^δ of S in R, the following are equivalent:*

i) S is a subring (ideal) of R.
ii) χ_S^δ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R.

Proof. $(i) \Rightarrow (ii)$ is straightforward.

Assume that χ_S^δ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R. Let $x, y \in S$. Then

$$\lambda(x - y) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} = \min\{\delta, \tfrac{\delta}{2}\} = \tfrac{\delta}{2},$$

and so $x - y \in S$. Similarly, if $x \in S$ and $a \in R$, then $ax \in S$ and $xa \in S$. Therefore S is a subring (ideal) of R. □

Theorem 3.10. *A fuzzy set λ in R is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R if and only if the set*

$$U(\lambda; t) := \{x \in R \mid \lambda(x) \geq t\}$$

is a subring (ideal) of R for all $t \in (0, \frac{\delta}{2}]$.

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R. Let $t \in (0, \frac{\delta}{2}]$ and $x, y \in U(\lambda; t)$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$, and so

$$\lambda(x - y) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} \geq \min\{t, \tfrac{\delta}{2}\} = t$$

by Theorem 3.3.(ii). Hence $x - y \in U(\lambda; t)$. Similarly, we have $xy \in U(\lambda; t)$, $ax \in U(\lambda; t)$ and $xa \in U(\lambda; t)$ for all $a, x, y \in R$. Therefore $U(\lambda; t)$ is a subring (ideal) of R for all $t \in (0, \frac{\delta}{2}]$.

Conversely, let $U(\lambda; t)$ be a subring (ideal) of R for all $t \in (0, \frac{\delta}{2}]$. Assume that there exist $x, y \in R$ such that $\lambda(x - y) < \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$. Choose $t \in (0, \delta]$ such that

$$\lambda(x - y) < t \leq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\}.$$

Then $x \in U(\lambda; t)$ and $y \in U(\lambda; t)$, and so $x - y \in U(\lambda; t)$. This is a contradiction, and thus $\lambda(x - y) \geq \min\{\lambda(x), \lambda(y), \frac{\delta}{2}\}$ for all $x, y \in R$. Similarly, we have

$$\lambda(xy) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\}$$

and

$$\lambda(yx) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\}$$

for all $x, y \in R$. Therefore λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R. □

Corollary 3.11. ([5]) *A fuzzy set λ in R is an $(\in, \in\vee q)$-fuzzy subring of R if and only if $\lambda(x - y) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and $\lambda(xy) \geq \min\{\lambda(x), \lambda(y), 0.5\}$ for all $x, y \in R$.*

Definition 3.12. A fuzzy set λ in R is called an $(\in, \in\vee q_0^\delta)$-*fuzzy left* (resp. *right*) *ideal* of R if it is an $(\in, \in\vee q_0^\delta)$-fuzzy subring of R such that for all $x, a \in R$ and $t \in (0, \delta]$

$$x_t \in \lambda \;\Rightarrow (\mathrm{ax})_\mathrm{t} \in\vee\, \mathrm{q}_0^\delta\, \lambda (\mathrm{resp.} (\mathrm{xa})_\mathrm{t} \in\vee\, \mathrm{q}_0^\delta\, \lambda).$$

By an $(\in, \in\vee q_0^\delta)$-*fuzzy ideal*, we mean both an $(\in, \in\vee q_0^\delta)$-fuzzy left ideal and an $(\in, \in\vee q_0^\delta)$-fuzzy right ideal.

Theorem 3.13. *A fuzzy set λ in R is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R if and only if the set*

$$\Omega(\lambda; t) := \{x \in R \mid x_t \in\vee\, q_0^\delta\, \lambda\}$$

is a subring (ideal) of R for all $t \in (0, \delta]$.

Proof. Similar to the proof of Theorem 3.10. □

Proposition 3.14. *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R, then for all $x \in R$ and $m, n \in \mathbb{N}$ where $\mathbb{N}$ is the set of all natural numbers,*

i) $\lambda(mx) \geq \min\{\lambda(x), \frac{\delta}{2}\}$,

ii) $m \geq n \;\Rightarrow\; \lambda(x^m) \geq \min\{\lambda(x^n), \frac{\delta}{2}\}$.

Proof. i) We have $\lambda(2x) \geq \min\left\{\lambda(x), \frac{\delta}{2}\right\}$ for all $x \in R$. Thus (i) is true for $m = 2$. Assume that (i) is true for $m = r$. Then

$$\lambda\left((r+1)x\right) \geq \min\left\{\lambda(rx), \lambda(x), \tfrac{\delta}{2}\right\} \geq \min\left\{\lambda(x), \tfrac{\delta}{2}\right\}.$$

Thus (i) is valid by the Mathematical Induction.

ii) Let $m, n \in \mathbb{N}$ be such that $m \geq n$. Then

$$\lambda\left(x^m\right) = \lambda\left(x^{m-n}x^n\right) \geq \min\left\{\lambda\left(x^n\right), \tfrac{\delta}{2}\right\}$$

for all $x \in R$. □

Definition 3.15. Let λ be an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of a commutative ring R. The fuzzy set $\mathrm{Rad}\lambda$ in R defined by

$$(\mathrm{Rad}\lambda)(x) := \begin{cases} \min\left\{\sup\{\lambda(x^n) \mid n \in \mathbb{N}\}, \frac{\delta}{2}\right\} & \text{if } \lambda(x) < \frac{\delta}{2}, \\ \lambda(x) & \text{if } \lambda(x) \geq \frac{\delta}{2} \end{cases}$$

is called the $(\in, \in\vee q_0^\delta)$-*fuzzy radical* of λ.

Theorem 3.16. *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of a commutative ring R, then its $(\in, \in\vee q_0^\delta)$-fuzzy radical is also an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R.*

Proof. For any $x, y \in R$, either $\lambda(x-y) < \frac{\delta}{2}$ or $\lambda(x-y) \geq \frac{\delta}{2}$. Assume that $\lambda(x-y) < \frac{\delta}{2}$. Since R is commutative, we have $(x-y)^{m+n} = ax^m + by^n$ where $a, b \in R$ and $m, n \in \mathbb{N}$. Thus

$$\begin{aligned}(\mathrm{Rad}\lambda)(x-y) &= \min\left\{\sup\left\{\lambda((x-y)^r) \mid r \in \mathbb{N}\right\}, \tfrac{\delta}{2}\right\} \\ &\geq \sup\left\{\min\left\{\lambda((x-y)^r), \tfrac{\delta}{2}\right\} \mid r \in \mathbb{N}\right\} \\ &\geq \min\left\{\lambda((x-y)^{m+n}), \tfrac{\delta}{2}\right\} \\ &= \min\left\{\lambda(ax^m + by^n), \tfrac{\delta}{2}\right\} \\ &\geq \min\left\{\lambda(ax^m), \lambda(by^n), \tfrac{\delta}{2}\right\} \\ &\geq \min\left\{\lambda(x^m), \lambda(y^n), \tfrac{\delta}{2}\right\}.\end{aligned} \tag{3.4}$$

Since $\lambda(x-y) < \frac{\delta}{2}$, we can consider the following three cases:

(i) $\lambda(x) < \frac{\delta}{2}$ and $\lambda(y) < \frac{\delta}{2}$,
(ii) $\lambda(x) \geq \frac{\delta}{2}$ and $\lambda(y) < \frac{\delta}{2}$,
(iii) $\lambda(x) < \frac{\delta}{2}$ and $\lambda(y) \geq \frac{\delta}{2}$.

For the first case, it follows from (3.4) that

$$\begin{aligned}(\mathrm{Rad}\lambda)(x-y) &\geq \min\Big\{\min\left\{\sup\left\{\lambda(x^m) \mid m \in \mathbb{N}\right\}, \tfrac{\delta}{2}\right\}, \\ &\qquad \min\left\{\sup\left\{\lambda(y^n) \mid n \in \mathbb{N}\right\}, \tfrac{\delta}{2}\right\}, \tfrac{\delta}{2}\Big\} \\ &= \min\left\{(\mathrm{Rad}\lambda)(x), (\mathrm{Rad}\lambda)(y), \tfrac{\delta}{2}\right\}.\end{aligned}$$

The second case implies that $(\mathrm{Rad}\lambda)(x) = \lambda(x)$. Using (3.4), we have

$$\begin{aligned}(\mathrm{Rad}\lambda)(x-y) &\geq \min\left\{\lambda(x), \min\left\{\sup\left\{\lambda(y^n) \mid n \in \mathbb{N}\right\}, \tfrac{\delta}{2}\right\}, \tfrac{\delta}{2}\right\} \\ &= \min\left\{(\mathrm{Rad}\lambda)(x), (\mathrm{Rad}\lambda)(y), \tfrac{\delta}{2}\right\}.\end{aligned}$$

The third case is similar to the second case. Suppose that $\lambda(x-y) \geq \frac{\delta}{2}$. Then $\lambda(x) < \frac{\delta}{2}$ and $\lambda(y) < \frac{\delta}{2}$. Hence

$$(\mathrm{Rad}\lambda)(x-y) = \lambda(x-y) \geq \tfrac{\delta}{2} \geq \min\left\{(\mathrm{Rad}\lambda)(x), (\mathrm{Rad}\lambda)(y), \tfrac{\delta}{2}\right\}.$$

Now, if $\lambda(xy) < \frac{\delta}{2}$ then

$$\begin{aligned}(\mathrm{Rad}\lambda)(xy) &= \min\left\{\sup\left\{\lambda((xy)^n) \mid n \in \mathbb{N}\right\}, \tfrac{\delta}{2}\right\}\\ &= \min\left\{\sup\left\{\lambda(x^n y^n) \mid n \in \mathbb{N}\right\}, \tfrac{\delta}{2}\right\}\\ &\geq \min\left\{\sup\left\{\min\left\{\lambda(x^n), \tfrac{\delta}{2}\right\} \mid n \in \mathbb{N}\right\}, \tfrac{\delta}{2}\right\}\\ &= \min\left\{\min\left\{\sup\left\{\lambda(x^n) \mid n \in \mathbb{N}\right\}, \tfrac{\delta}{2}\right\}, \tfrac{\delta}{2}\right\}\\ &= \min\left\{(\mathrm{Rad}\lambda)(x), \tfrac{\lambda}{2}\right\}.\end{aligned}$$

Finally assume that $\lambda(xy) \geq \frac{\delta}{2}$. Then

$$(\mathrm{Rad}\lambda)(xy) = \lambda(xy) \geq \tfrac{\delta}{2} \geq \min\left\{ \mathrm{Rad}\lambda(x), \tfrac{\delta}{2}\right\}.$$

Therefore $\mathrm{Rad}\lambda$ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R. □

Corollary 3.17. ([5]) *If λ is an $(\in, \in\vee q)$-fuzzy ideal of a commutative ring R, then its $(\in, \in\vee q)$-fuzzy radical is also an $(\in, \in\vee q)$-fuzzy ideal of R.*

Theorem 3.18. *Let λ be an $(\in, \in\vee q)$-fuzzy ideal of a commutative ring R such that*

$$(\forall S(\neq \emptyset) \subseteq R)(\exists\, a \in S)\,(\lambda(a) = \sup\{\lambda(b) \mid b \in S\})\,. \tag{3.5}$$

Then $\mathrm{Rad}U(\lambda;t) = U(\mathrm{Rad}\lambda;t)$ *for all* $t \in (0, \frac{\delta}{2}]$, *and* $\mathrm{Rad}\Omega(\lambda;t) = \Omega(\mathrm{Rad}\lambda;t)$ *for all* $t \in (0, \delta]$.

Proof. Let $t \in (0, \frac{\delta}{2}]$. If $x \in \mathrm{Rad}U(\lambda;t)$, then $x^n \in U(\lambda;t)$ for some $n \in \mathbb{N}$. Thus $\lambda(x^n) \geq t$, and so $(\mathrm{Rad}\lambda)(x) \geq t$, i.e., $x \in U(\mathrm{Rad}\lambda;t)$. This shows that $\mathrm{Rad}U(\lambda;t) \subseteq U(\mathrm{Rad}\lambda;t)$. Next, if $x \in U(\mathrm{Rad}\lambda;t)$ then $(\mathrm{Rad}\lambda)(x) \geq t$. Assume that $\lambda(x) < \frac{\delta}{2}$. Then

$$(\mathrm{Rad}\lambda)(x) = \min\{\sup\{\lambda(x^n) \mid n \in \mathbb{N}\}, \tfrac{\delta}{2}\} \geq t.$$

By the condition (3.5), there exists $r \in \mathbb{N}$ such that $\lambda(x^r) = \sup\{\lambda(x^n) \mid n \in \mathbb{N}\}$. Hence $x^r \in U(\lambda;t)$, and so $x \in \mathrm{Rad}U(\lambda;t)$. Thus $U(\mathrm{Rad}\lambda;t) \subseteq \mathrm{Rad}U(\lambda;t)$. Now, it is clear that if $\lambda(x) \geq \frac{\delta}{2}$ then $U(\mathrm{Rad}\lambda;t) \subseteq \mathrm{Rad}U(\lambda;t)$. Therefore $\mathrm{Rad}U(\lambda;t) = U(\mathrm{Rad}\lambda;t)$ for all $t \in (0, \frac{\delta}{2}]$. If $x \in \mathrm{Rad}\Omega(\lambda;t)$, then $x^n \in \Omega(\lambda;t)$ for some $n \in \mathbb{N}$, and so $(x^n)_t \in\vee q_0^\delta \lambda$. Since $(\mathrm{Rad}\lambda)(x) \geq \lambda(x)$ and $\lambda(x^n) \geq \min\{\lambda(x), \frac{\delta}{2}\}$ for all $n \in \mathbb{N}$, it follows that $x_t \in\vee q_0^\delta (\mathrm{Rad}\lambda)$ and so that $x \in \Omega(\mathrm{Rad}\lambda;t)$. Now, let $x \in \Omega(\mathrm{Rad}\lambda;t)$. Then $x_t \in\vee q_0^\delta (\mathrm{Rad}\lambda)$, that is, $(\mathrm{Rad}\lambda)(x) \geq t$ or $(\mathrm{Rad}\lambda)(x) + t > \delta$.

Case 1. Assume that $(\mathrm{Rad}\lambda)(x) \geq t$. Let $t \leq \frac{\delta}{2}$. If $\lambda(x) < \frac{\delta}{2}$, then there exists $r \in \mathbb{N}$ such that

$$(\mathrm{Rad})(x) = \min\{\lambda(x^r), \tfrac{\delta}{2}\} \geq t$$

by the condition (3.5). It follows that $\lambda(x^r) \geq t$ and that $x^r \in U(\lambda; t) \subseteq \Omega(\lambda; t)$. Thus $x \in \mathrm{Rad}\Omega(\lambda; t)$. If $\lambda(x) \geq \frac{\delta}{2}$, then $(\mathrm{Rad}\lambda)(x) = \lambda(x) \geq t$ and so $x \in \mathrm{Rad}\Omega(\lambda; t)$. Next, let $t > \frac{\delta}{2}$. Then $(\mathrm{Rad}\lambda)(x) = \lambda(x) \geq t$ and so $x \in \mathrm{Rad}\Omega(\lambda; t)$.

Case 2. Assume that $(\mathrm{Rad}\lambda)(x) + t > \delta$. Then $\lambda(x) + t > \delta$, i.e., $x_t\, q_0^\delta \lambda$ or

$$\min\{\lambda(x^r), \tfrac{\delta}{2}\} + t > \delta, \text{ i.e., } x^r\, q_0^\delta \lambda$$

for some $r \in \mathbb{N}$. It follows that $x \in \Omega(\lambda; t)$ or $x^r \in \Omega(\lambda; t)$. Hence $x \in \mathrm{Rad}\Omega(\lambda; t)$. This completes the proof. □

Definition 3.19. Let λ be a fuzzy set in R. Given $a \in R$, a fuzzy set λ_a in R defined by

$$\lambda_a : R \to [0, 1],\ x \mapsto \min\{\lambda(x - a), \tfrac{\delta}{2}\}$$

is called the $(\in, \in\vee q_0^\delta)$-*fuzzy coset* of λ in R determined by a.

Let λ be an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R, and denote by $R_\delta^\lambda = \{\lambda_a \mid a \in R\}$, the set of all $(\in, \in\vee q_0^\delta)$-fuzzy cosets of λ in R. We provide two operations $\oplus$ and $\odot$ into R_δ^λ as follows:

$$\lambda_x \oplus \lambda_y = \lambda_{x+y} \text{ and } \lambda_x \odot \lambda_y = \lambda_{xy}$$

for all $\lambda_x, \lambda_y \in R_\delta^\lambda$. We first show that the operations are well defined. Let $a, b, x, y \in R$ such that $\lambda_a = \lambda_x$ and $\lambda_b = \lambda_y$. Then $\lambda_a(r) = \lambda_x(r)$ and $\lambda_b(r) = \lambda_y(r)$ for all $r \in R$, that is,

$$\min\{\lambda(r - a), \tfrac{\delta}{2}\} = \min\{\lambda(r - x), \tfrac{\delta}{2}\} \tag{3.6}$$

and

$$\min\{\lambda(r - b), \tfrac{\delta}{2}\} = \min\{\lambda(r - y), \tfrac{\delta}{2}\}. \tag{3.7}$$

If we take $r = a$ and $r = b$ in (3.6) and (3.7), respectively, then

$$\min\{\lambda(a - x), \tfrac{\delta}{2}\} = \min\{\lambda(0), \tfrac{\delta}{2}\} = \tfrac{\delta}{2} \tag{3.8}$$

and

$$\min\{\lambda(b - y), \tfrac{\delta}{2}\} = \min\{\lambda(0), \tfrac{\delta}{2}\} = \tfrac{\delta}{2}. \tag{3.9}$$

Taking $r = a + b - y$ in (3.6) and using (3.9) induce

$$\min\{\lambda(a + b - y - x), \tfrac{\delta}{2}\} = \min\{\lambda(b - y), \tfrac{\delta}{2}\} = \tfrac{\delta}{2}$$

and so $\lambda(a + b - y - x) \geq \frac{\delta}{2}$. It follows from Theorem 3.3 that

$$\begin{aligned}(\lambda_a \oplus \lambda_b)(r) = \lambda_{a+b}(r) &= \min\{\lambda(r - a - b), \tfrac{\delta}{2}\} \\ &= \min\{\lambda((r - x - y) - (a + b - x - y)), \tfrac{\delta}{2}\} \\ &\geq \min\{\lambda(r - x - y), \lambda(a + b - x - y), \tfrac{\delta}{2}\} \\ &= \min\{\lambda(r - x - y), \tfrac{\delta}{2}\} \\ &= \lambda_{x+y}(r) = (\lambda_x \oplus \lambda_y)(r).\end{aligned}$$

Similarly, we have $(\lambda_a \oplus \lambda_b)(r) \leq (\lambda_x \oplus \lambda_y)(r)$ for all $r \in R$. Hence $\lambda_a \oplus \lambda_b = \lambda_x \oplus \lambda_y$, and the addition is well defined. Using Theorem 3.3, (3.8) and (3.9) induces

$$\begin{aligned}(\lambda_a \odot \lambda_b)(r) = \lambda_{ab}(r) &= \min\{\lambda(r-ab), \tfrac{\delta}{2}\} \\ &= \min\{\lambda((r-xy)-(ab-xy)), \tfrac{\delta}{2}\} \\ &\geq \min\{\lambda(r-xy), \lambda(ab-xy), \tfrac{\delta}{2}\} \\ &= \min\{\lambda(r-xy), \lambda((a-x)b - x(y-b)), \tfrac{\delta}{2}\} \\ &\geq \min\{\lambda(r-xy), \lambda((a-x)b), \lambda(x(y-b)), \tfrac{\delta}{2}\} \\ &\geq \min\{\lambda(r-xy), \lambda(a-x), \lambda(b-y), \tfrac{\delta}{2}\} \\ &= \min\{\lambda(r-xy), \tfrac{\delta}{2}\} = \lambda_{xy}(r) = (\lambda_x \odot \lambda_y)(r)\end{aligned}$$

for all $r \in R$. Similarly, we get $(\lambda_a \odot \lambda_b)(r) \leq (\lambda_x \odot \lambda_y)(r)$ for all $r \in R$. Thus the multiplication is also well defined. We can easily check that R_δ^λ is a ring with λ_0 as the null element and λ_{-x} is the negative of λ_x for all $x \in R$. Therefore we have the following theorem.

Theorem 3.20. *For any $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R, the set of all $(\in, \in\vee q_0^\delta)$-fuzzy cosets of λ in R is a ring under operations $\oplus$ and $\odot$.*

For a fuzzy set λ in R, we define a fuzzy set $\widetilde{\lambda}$ in R_δ^λ as follows:

$$\widetilde{\lambda} : R_\delta^\lambda \to [0,1], \;\; \lambda_x \mapsto \lambda(x).$$

Theorem 3.21. *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R, then $\widetilde{\lambda}$ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R_δ^λ.*

Proof. Assume that λ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal in R and let $x, y \in R$. Then

$$\widetilde{\lambda}(\lambda_x \ominus \lambda_y) = \widetilde{\lambda}(\lambda_{x-y}) = \lambda(x-y) \geq \min\{\lambda(x), \lambda(y), \tfrac{\delta}{2}\} = \min\{\widetilde{\lambda}(\lambda_x), \widetilde{\lambda}(\lambda_y), \tfrac{\delta}{2}\}$$

and

$$\widetilde{\lambda}(\lambda_x \odot \lambda_y) = \widetilde{\lambda}(\lambda_{xy}) = \lambda(xy) \geq \min\{\lambda(x), \tfrac{\delta}{2}\} = \min\{\widetilde{\lambda}(\lambda_x), \tfrac{\delta}{2}\}.$$

Similarly, $\widetilde{\lambda}(\lambda_x \odot \lambda_y) \geq \min\{\widetilde{\lambda}(\lambda_y), \frac{\delta}{2}\}$. Therefore $\widetilde{\lambda}$ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R_δ^λ. □

Lemma 3.22. *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R, then*

$$U(\lambda; \tfrac{\delta}{2}) = \{x \in R \mid \lambda_x \supseteq \lambda_0\} = \{x \in R \mid \lambda_x = \lambda_0\}.$$

Proof. Let $A = \{x \in \lambda \mid \lambda_x \supseteq \lambda_0\}$ and $B = \{x \in R \mid \lambda_x = \lambda_0\}$. If $x \in A$, then $\lambda_x(r) \geq \lambda_0(r)$ for all $r \in R$. In particular, if we take $r = 0$ then $\lambda(x) \geq \frac{\delta}{2}$, i.e., $x \in U(\lambda; \frac{\delta}{2})$. Now, we have

$$\begin{aligned}\lambda_0(r) = \min\{\lambda(r), \tfrac{\delta}{2}\} &= \min\{\lambda(r-x+x), \tfrac{\delta}{2}\} \\ &\geq \min\{\lambda(r-x), \lambda(x), \tfrac{\delta}{2}\} = \min\{\lambda(r-x), \tfrac{\delta}{2}\} = \delta_x(r)\end{aligned}$$

for all $r \in R$, and so $\lambda_x = \lambda_0$, i.e., $x \in B$. Let $x \in U(\lambda; \frac{\delta}{2})$. Then $\lambda(x) \geq \frac{\delta}{2}$, and so

$$\lambda(r-x) \geq \min\{\lambda(r), \lambda(x), \tfrac{\delta}{2}\} = \min\{\lambda(r), \tfrac{\delta}{2}\}.$$

Thus $\lambda_x(r) = \min\{\lambda(r-x), \frac{\delta}{2}\} \geq \min\{\lambda(r), \frac{\delta}{2}\} = \lambda_0(r)$ for all $r \in R$, which implies that $x \in A$. This completes the proof. □

Theorem 3.23. *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy ideal of R, then the mapping*

$$f : R \to R_\delta^\lambda,\ x \mapsto \lambda_x$$

is a homomorphism with $\ker(f) = U(\lambda; \frac{\delta}{2})$.

Proof. For any $x, y \in R$, we have

$$f(x+y) = \lambda_{x+y} = \lambda_x \oplus \lambda_y = f(x) \oplus f(y)$$

and

$$f(xy) = \lambda_{xy} = \lambda_x \odot \lambda_y = f(x) \odot f(y).$$

Hence f is a homomorphism. Using Lemma 3.22, we have

$$\ker(f) = \{x \in R \mid f(x) = f(0)\} = \{x \in R \mid \lambda_x = \lambda_0\} = U(\lambda; \tfrac{\delta}{2}).$$

This completes the proof. □

Obviously, the homomorphism f in Theorem 3.23 is onto. Hence, by the first isomorphism theorem, we know that $R/\ker(f)$ is isomorphic to R_δ^λ.

Theorem 3.24. *If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R, then the fuzzy set*

$$\gamma : R \to [0,1],\ x \mapsto \widetilde{\lambda}(\lambda_x)$$

is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R.

Proof. If λ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R, then $\widetilde{\lambda}$ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R by Theorem 3.21. Let $x, y \in R$. Then

$$\gamma(x+y) = \widetilde{\lambda}(\lambda_{x+y}) = \widetilde{\lambda}(\lambda_x \oplus \lambda_y) \geq \min\{\widetilde{\lambda}(\lambda_x), \widetilde{\lambda}(\lambda_y), \tfrac{\delta}{2}\} = \min\{\gamma(x), \gamma(y), \tfrac{\delta}{2}\}.$$

Similarly, we have $\gamma(xy) \geq \min\{\gamma(x), \gamma(y), \frac{\delta}{2}\}$, $\gamma(xy) \geq \min\{\gamma(x), \frac{\delta}{2}\}$, and $\gamma(xy) \geq \min\{\gamma(y), \frac{\delta}{2}\}$. Therefore γ is an $(\in, \in\vee q_0^\delta)$-fuzzy subring (ideal) of R. □

Theorem 3.25. *For a ring homomorphism $f : R \to Q$, let λ and ν be $(\in, \in\vee q_0^\delta)$-fuzzy ideals of R and Q, respectively. Then the mapping*

$$\varphi : R_\delta^\lambda \to Q_\delta^\nu,\ \lambda_x \mapsto \nu_{f(x)}$$

is a homomorphism.

Proof. For any $x, y \in R$, we have

$$\varphi(\lambda_x \oplus \lambda_y) = \varphi(\lambda_{x+y}) = \nu_{f(x+y)} = \nu_{f(x)+f(y)} = \nu_{f(x)} \oplus \nu_{f(y)} = \varphi(\lambda_x) \oplus \varphi(\lambda_y)$$

and

$$\varphi(\lambda_x \odot \lambda_y) = \varphi(\lambda_{xy}) = \nu_{f(xy)} = \nu_{f(x)f(y)} = \nu_{f(x)} \odot \nu_{f(y)} = \varphi(\lambda_x) \odot \varphi(\lambda_y).$$

Hence φ is a homomorphism. □

4. Conclusion

Using more general form, so called δ-quasi-coincident with a fuzzy set, of "quasi-coincident with" relation (q) of a fuzzy point with a fuzzy set, we have introduced the concepts of $(\in, \in \vee q_0^\delta)$-fuzzy subrings/ideals, $(\in, \in \vee q_0^\delta)$-fuzzy radicals and $(\in, \in \vee q_0^\delta)$-fuzzy coset of a fuzzy set determined by an element of a ring. We have considered generalizations of the paper [5]. We have discussed relations between an $(\in, \in \vee q)$-fuzzy subring/ideal and an $(\in, \in \vee q_0^\delta)$-fuzzy subring/ideal, and have considered characterizations of an $(\in, \in \vee q_0^\delta)$-fuzzy subring and ideal. We have shown that for any $(\in, \in \vee q_0^\delta)$-fuzzy ideal of a ring R, the set of all $(\in, \in \vee q_0^\delta)$-fuzzy cosets of λ in R is a ring under operations $\oplus$ and $\odot$. We have induced a homomorphism between a given ring and a new ring, and have investigated related properties.

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