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RINGS WHOSE ELEMENTS ARE REPRESENTED BY AT MOST THREE COMMUTING IDEMPOTENTS

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Abstract. We completely characterize up to an isomorphism those rings whose elements are expressed as the sum of two, respectively three, commuting idempotents or are minus an idempotent. This strengthens well-known joint results in the subject due to Hirano-Tominaga (Bull. Austral. Math. Soc., 1988), Ahn-Anderson (Rocky Mount. J. Math., 2006), Danchev-McGovern (J. Algebra, 2015), Ying et al. (Can. Math. Bull., 2016), as well as own results established by Danchev (Bull. Iran. Math. Soc., 2019) and (Boll. Un. Mat. Ital., 2019).

1. Introduction and Motivation

Everywhere in the text of the present paper, suppose that all rings R into consideration be assumed associative, containing the identity element 1, which differs from the zero element 0. Our standard terminology and notation are mainly in agreement with [8]. For instance, $U(R)$ stands for the group of units in R, $J(R)$ stands for the Jacobson radical of R, $Nil(R)$ stands for the set of all nilpotents in R and $Id(R)$ stands for the set of all idempotents in R. All other unexplained explicitly notions and notations will be stated in the sequel.

The classical concept of a *Boolean* ring means that each element is an idempotent. These rings are known to be a subdirect sum of family of copies of the field of two elements $\mathbb{Z}_2$. In [7] this was enlarged to rings for which the elements are sums of two idempotents. To pay attention on the sign "-", in [1] were described rings whose elements are idempotents or minus idempotents (actually, this was slightly extended in [6]). This makes sense to explore rings for which their elements are representable by commuting idempotents only (see, e.g., [4], [5] and [9]).

So, motivated by the aforementioned principal results, we shall explore here the rings having elements represented either as a sum of at most three idempotents or elements which are minus an idempotents. These considerations allow us to obtain some new results in that topic as follows.

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2. Main Results

It is rather easy to verify that the next three conditions for any ring R are equivalent:

(1) $\forall x \in R$: $x = e + f$ for some two commuting idempotents $e, f \in R$.

(2) $\forall x \in R$: $x = e - f$ for some two commuting idempotents $e, f \in R$.

(3) $\forall x \in R$: $x = -e - f$ for some two commuting idempotents $e, f \in R$.

That is why, it is quite natural to consider the following new class of rings as follows:

2.1. **Rings whose elements are sums of two commuting idempotents or minus an idempotent.** We deal here with rings whose elements depend on commuting idempotents only. They can be expressed as sums of two idempotents or are minus an idempotent.

So, we come to our first main tool.

Definition 2.1. We shall say that the ring R belongs to the class $\mathcal{P}_1$ if, for every element r from R, there exist two idempotents e, f of R such that $r = e + f$ or $r = -e$.

This notion extends the concept of *weakly Boolean rings*, where each element is idempotent or minus an idempotent. These rings R are characterized thus: $R \cong B$ is Boolean or $R \cong \mathbb{Z}_3$ or $R \cong B \times \mathbb{Z}_3$ (see, for example, [1] and [6]).

Likewise, the above notion generalizes the considerations in [7] of rings whose elements are sums of two commuting idempotents.

We now proceed by giving up a spectacular proof of the following first main result.

Theorem 2.2. *A ring R lies in the class $\mathcal{P}_1$ if, and only if, $R \cong R_1 \times R_2$, where either*

(i) $R_1 = \{0\}$, or $R_1 \cong B$ is a Boolean ring (and so it is a subdirect product of family of copies of the field $\mathbb{Z}_2$), or $R_1 \cong \mathbb{Z}_4$, or $R \cong B \times \mathbb{Z}_4$;

and

(ii) $R_2 = \{0\}$ (which is mandatory when $J(R_1) \neq \{0\}$), or R_2 is a ring which is a subdirect product of family of copies of the field $\mathbb{Z}_3$.

Proof. "$\Rightarrow$". First, let $3 = -e$ for some $e \in Id(R)$. By squaring, we get that $9 = -3$, i.e., $12 = 4.3 = 0$.

Let us now $3 = e + f$ for some two commuting $e, f \in R$. Again by squaring, we derive that $9 = 3 + 2f$, that is, $6 = 2ef$. Moreover, multiplying the initial equality by f, we deduce that $2f = ef$ and, by squaring, that $4f = 2f = 0$. Hence $6 = 2.3 = 0$, as expected.

Furthermore, with the Chinese Remainder Theorem at hand, one writes that $R \cong R_1 \times R_2$, where both rings R_1, R_2 remain from the class $\mathcal{P}_1$ with the properties $4 = 0$ in R_1 and $3 = 0$ in R_2.

We next will describe all direct factors in the following manner:

Describing R_1. Assume $R_1 \neq \{0\}$. Adapting the ideas for proofs from [4] and [5], one deduces that the quotient $R_1/J(R_1)$ is a Boolean ring with either zero $J(R_1)$ or nil $J(R_1) = \{0, 2\}$. Furthermore, exploiting the method of proof in [2, Proposition 17], it follows that R_1 is either a Boolean ring or the indecomposable ring $\mathbb{Z}_4$ or the direct product of two such rings.

Describing R_2. Assume $R_2 \neq \{0\}$. Since $3 = 0$, it is not too hard to verify that the equality $r^3 = r$ is valid for every element $r \in R_2$. Thus, in accordance with [7], R_2 has to be a subdirect product of copies of the $\mathbb{Z}_3$'s.

"$\Leftarrow$". Given an arbitrary element $x \in R$. By definition, in B every element is an idempotent. Besides, it is shown in [7] that any element from R_2 is a sum of two idempotents. Thus, simple arguments lead to the fact that this is also automatically true in $B \times R_2$ too.

What remains to show is that in $\mathbb{Z}_4$ and $B \times \mathbb{Z}_4$ every element is a sum of two idempotents or is minus an idempotent. But in the first situation this follows by a direct check, while in the second one this is almost obvious, bearing in mind that in B it is also fulfilled that $2 = 0$ (whence the signs "+" and "-" amounts), and that in the ring $\mathbb{Z}_4$ the elements $0, 1, 2$ are always sums of two idempotents and 3 is minus an idempotent, that is, $3 = -1$. □

Remark 2.3. It is worthy noticing that the rings from Theorem 2.2 (i) somewhat arisen also in [6, Proposition 1.19].

Likewise, it is not too surprised that the class $\mathcal{P}_1$ is simultaneously strictly contained in the class $\mathcal{C}$ of rings investigated in [5] and in the class of rings studied in Theorem 4.4 of [9], which classes are also different as, moreover, even that from [5] is more general containing the field $\mathbb{Z}_5$. By the way, concerning the latter ring class, the aforementioned characterization result in [2, Proposition 17] is better than that in [9, Theorem 4.4]. However, in the cited Proposition 17 from [2] there is a shortcoming, namely in condition (b) the possibility $R_2 = \{0\}$ is omitted. Indeed, the examples stated below specify that omission.

Example 2.4. The direct products $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ are from the class $\mathcal{P}_1$.

However, considering the direct product $\mathbb{Z}_4 \times \mathbb{Z}_3$, what can be said is that it is *not* in the class $\mathcal{P}_1$, since the element $(3, 1)$ is not presentable as the sum of two idempotents and it is also not minus an idempotent, bearing in mind that the only idempotents in both $\mathbb{Z}_3$ and $\mathbb{Z}_4$ are the trivial ones, namely $0, 1$. This clarifies why $R_2 = \{0\}$ whenever $J(R_1) \neq \{0\}$, because $J(\mathbb{Z}_4) = \{0, 2\}$. Nevertheless, it can be presented as $(3, 1) = (0, 1) - (1, 0) = -(0, 1) - (1, 1)$, as required.

Next, continuing in a direction of common generalization of the previous concept $\mathcal{P}_1$, it is also pretty easy to check that the next four conditions for any ring R are equivalent:

(1) $\forall x \in R$: $x = e + f + h$ for some three commuting idempotents $e, f, h \in R$.

(2) $\forall x \in R$: $x = e + f - h$ for some three commuting idempotents $e, f, h \in R$.

(3) $\forall x \in R$: $x = e - f - h$ for some three commuting idempotents $e, f, h \in R$.

(4) $\forall x \in R$: $x = -e - f - h$ for some three commuting idempotents $e, f, h \in R$.

That is why, it is quite natural to consider the following new class of rings as follows:

2.2. **Rings whose elements are sums of three commuting idempotents or minus an idempotent.** We can now enlarge somewhat the above Definition 2.1 to the following new point of view:

Definition 2.5. We shall say that the ring R belongs to the class $\mathcal{P}_2$ if, for every element r from R, there exist three idempotents e, f, h of R such that $r = e+f+h$ or $r = -e$.

We are now in a position to proceed by proving the following second main result.

Theorem 2.6. *A ring R lies in the class $\mathcal{P}_2$ if, and only if, $R \cong R_1 \times R_2 \times R_3$, where either*

(i) $R_1 = \{0\}$, or R_1 is a commutative ring of even characteristic not exceeding 4 such that the factor-ring $R_1/J(R_1)$ is a Boolean ring (and thus it is a subdirect product of family of copies of the field $\mathbb{Z}_2$), and either $J(R_1) = \{0\}$ or $J(R_1) = 2Id(R_1)$;

and

(ii) $R_2 = \{0\}$, or R_2 is a ring which is a subdirect product of family of copies of the field $\mathbb{Z}_3$;

and

(iii) $R_3 = \{0\}$ (which is mandatory when $J(R_1) \neq \{0\}$), or $R_3 \cong \mathbb{Z}_5$ as either $R_1 = \{0\}$ or R_1 is Boolean, and $R_2 = \{0\}$.

Proof. "$\Rightarrow$". Write $-3 = -e$, that is, $3 = e$ and $9 = 3$, whence $6 = 2.3 = 0$.

If now $-3 = e + f + h$, then by a multiplication of both sides by e, we find that $-4e = ef + eh$ and, multiplying by f, we have $-5ef = efh$ and finally, multiplying by h, it follows that $6efh = 0$.

On the other hand, squaring $-3 = e+f+h$, we obtain that $12+8e = 2fh$ which multiplied by $3e$ gives that $60e = 0$. Similarly, $60f = 60h = 0$. Consequently, $-3 = e + f + h$ implies that $-180 = 60e + 60f + 60h = 0$, i.e., $4.9.5 = 0$, as expected.

Henceforth, the Chinese Remainder Theorem applies to write that $R \cong R_1 \times R_2 \times R_3$, where all of the three rings R_1, R_2, R_3 are also of the class $\mathcal{P}_2$ as well as $4 = 0$ in R_1, $9 = 0$ in R_2 and $5 = 0$ in R_3.

However, we claim even that $3 = 0$ in R_2. Indeed, assume first that $4 = -e$. Thus, by squaring, it follows that $16 = e = -4$ and hence $20 = 0$. Since $20 \in U(R_2)$ as $9 = 0$, we get the desired contradiction $1 = 0$.

After that, suppose $4 = e+f+h$. By a subsequent multiplication of both sides with e, f and h, respectively, one infers that $3e = ef+eh$, that $2ef = efh$ and that $efh = 0$. We, therefore, arrive at $2ef = 2fh = 2eh = 0$. Taking into account that $9 = 0$, the multiplication by 5 of these equalities gives that $ef = fh = eh = 0$.

Consequently, $3e = 3f = 3h = 0$ and, finally, $12 = 3e + 3f + 3h = 0$. Since $4 \in U(R_2)$ as $3 \in Nil(R_2)$, we deduce after all the wanted equality that $3 = 0$.

We next will describe all direct factors in the following manner:

Describing R_1. Here $4 = 0$. As in [4, Theorem 2.12], we can derive the wanted isomorphic structure.

Describing R_2. Here $3 = 0$. It is routinely checked that $y^3 = y$ for all $y \in R_2$, so that [7] is applicable to infer to wanted description.

Describing R_3. Here $5 = 0$. If now, an arbitrary $z \in R_3$ is minus an idempotent, then $z^2 = -z$ and hence $z^3 = z$. If reciprocally z is a sum of three idempotents, then processing as in [4], we can get that $z^3 = -z$ and, henceforth, [3] applies to conclude that R_3 is exactly the five element field, as desired.

"$\Leftarrow$". Given an arbitrary element $x \in R$. Referring to [4] and [7], respectively, in R_1 each element x_1 is a sum of three idempotents, whereas in R_2 each element x_2 is a sum of two idempotents. Regarding $R_3 \cong \mathbb{Z}_5 = \{0, 1, 2, 3, 4 \mid 5 = 0\}$, the elements $0, 1, 2, 3$ are sums of three idempotents, while $4 = -1$ is obviously minus an idempotent. That is why, plain calculations demonstrate that $x = (x_1, x_2, 0)$ with $x_1 \neq 0$ is a sum of three idempotents as well as that $x = (x_1, 0, x_3)$ with $2x_1 = 0$ and $x_3 \in R_3$ is either a sum of three idempotents or is minus an idempotent, too, as expected. □

Remark 2.7. It is worthy noticing that the proved above last theorem somewhat improves on [4, Theorem 2.2].

Moreover, the above statement can be reformulated in the following way: Either $R = \{0\}$, or $R \cong K \times P$ for some two rings K and P having the properties that $K = \{0\}$ or $K \cong B$ is Boolean (and thus $K \subseteq \prod_\lambda \mathbb{Z}_2$ for some ordinal λ) or $K/J(K) \cong B$ with $J(K) = 2Id(K)$ as $\mathrm{char}(K) = 4$, and $P = \{0\}$ or $P \subseteq \prod_\mu \mathbb{Z}_3$ for some ordinal μ, or $R \cong \mathbb{Z}_5$, or $R \cong B \times \mathbb{Z}_5$.

Example 2.8. Certainly, the rings $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_5$ lie in the class $\mathcal{P}_2$.

However, considering the direct products $\mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_4 \times \mathbb{Z}_5$, one says that they are both *not* in the class $\mathcal{P}_2$, because the elements $(1, 4)$ and $(2, 4)$, respectively, are not presentable as the sum of three idempotents or minus an idempotent since the only idempotents in $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$ are the trivial ones, namely 0 and 1.

So, inspired by the examples alluded to above, we end our work with the following three problems of some interest and importance:

Problem 2.9. Classify the structure of those rings R for which, for each $x \in R$, there are commuting $e_1, e_2, e_3 \in Id(R)$ with $x = e_1 + e_2 + e_3$ or $x = e_1 + e_2 - e_3$.

It is not too hard to see that any element of R is a sum of four idempotents. In fact, one writes that $x - 1 = e_1 + e_2 + e_3$ or $x - 1 = e_1 + e_2 - e_3$. Thus $x = 1 + e_1 + e_2 + e_3$ or $x = e_1 + e_2 + (1 - e_3)$.

Problem 2.10. Classify the structure of those rings R for which, for each $x \in R$, there are commuting $e_1, e_2, e_3 \in Id(R)$ with $x = e_1 + e_2 + e_3$ or $x = e_1 - e_2 - e_3$.

Using the above tactic for the element $x-2$, it is not too difficult to observe that any element of R is a sum of five idempotents.

Problem 2.11. Classify the structure of those rings R for which, for each $x \in R$, there are commuting $e_1, e_2, e_3 \in Id(R)$ with $x = e_1 + e_2 + e_3$ or $x = -e_1 - e_2 - e_3$.

The meaning of the last problem is to describe those rings having an element x or $-x$ as a sum of three commuting idempotents. Here, incidentally, the field $\mathbb{Z}_7$ will occur.

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