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ON CONSECUTIVE PERFECT POWERS AND FRACTIONAL PARTS

RAFAEL JAKIMCZUK[1] *

Abstract. We obtain various formulae where consecutive perfect powers and fractional parts appear.

1. Introduction and Main Results

A positive integer of the form m^n where m and $n \geq 2$ are positive integers is called a perfect power. In contrary case the positive integer is called a not perfect power. The integer m is called basis of the perfect power and the integer n is called exponent of the perfect power. The first few terms of the integer sequence of perfect powers are

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, \ldots$$

The n-th perfect power will be denoted P_n.

Let us consider the prime factorization of a positive integer $m > 1$.

$$m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

where $p_1, p_2, \ldots, p_r$ are the different primes and $a_1, a_2, \ldots, a_r$ are the exponents. The integer m is a perfect power if and only if $\gcd(a_1, a_2, \ldots, a_r) > 1$. Therefore, if m is a perfect power then

$$m = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} = A^s,$$

where A is not a perfect power and $s = \gcd(a_1, a_2, \ldots, a_r) > 1$. The integer $A > 1$ will be called principal basis of the perfect power m and the integer $s > 1$ will be called principal exponent of the perfect power m.

Consequently we shall write $P_n = A_n^{s_n}$ where A_n is the principal basis and s_n is the principal exponent. For example $P_{11} = 64 = 2^6 = A_{11}^{s_{11}}$.

Let $A(n)$ be the number of perfect powers that are contained in the open interval $((n-1)^2, n^2)$, where $n \geq 2$ is a positive integer. It is well-known that $A(n) = 0$ for almost all intervals $((n-1)^2, n^2)$, since we have the theorem (see [1])

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* Corresponding author.

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Theorem 1.1. *Let us consider the n open intervals $(0, 1^2), (1^2, 2^2), \ldots, ((n-1)^2, n^2)$. Let $S(n)$ be the number of intervals that contain some perfect power. The following limit holds*

$$\lim_{n\to\infty} \frac{S(n)}{n} = 0.$$

An open interval $((n-1)^2, n^2)$ that contains some perfect power will be called S_1 open interval. An closed interval $[(n-1)^2, n^2]$ that contains some perfect power in its inner will be called S_1 closed interval. A S_1 closed interval $[(n-1)^2, n^2]$ contains the perfect powers $(n-1)^2$, n^2 and at least another perfect power. For example (see above), the intervals $[2^2, 3^2]$ and $[5^2, 6^2]$ are S_1 closed intervals.

Theorem 1.2. *The following inequality holds*

$$d_n = P_{n+1} - P_n < 4\sqrt{P_n} < 4\sqrt{P_{n+1}} \qquad (n \geq n_0) \tag{1.1}$$

Proof. It is well-known (see [2]) the inequality

$$d_n = P_{n+1} - P_n < 2n = 2\sqrt{n^2}. \tag{1.2}$$

On the other hand, it is well-known the asymptotic formula $P_n \sim n^2$ (see [3]). Hence

$$\sqrt{n^2} < 2\sqrt{P_n}. \tag{1.3}$$

Equations (1.2) and (1.3) give (1.1). □

Theorem 1.3. *The following formulae hold*

$$\lim_{n\to\infty} (A_n s_n) = \infty, \tag{1.4}$$

$$\log P_n \leq (A_n s_n) \leq \frac{1}{\log 2}\sqrt{P_n} \log P_n, \tag{1.5}$$

$$\sum_{i=2}^{n} (A_i s_i) \sim P_n \sim n^2, \tag{1.6}$$

$$\lim_{n\to\infty} (A_n + s_n) = \infty, \tag{1.7}$$

$$\lim_{n\to\infty} (\log A_n + \log s_n) = \infty. \tag{1.8}$$

Proof. We have the inequality $\log x < x$. Therefore

$$A_n s_n = \frac{A_n}{\log A_n} \log P_n \geq \log P_n. \tag{1.9}$$

On the other hand

$$A_n s_n = P_n^{1/s_n} \frac{\log P_n}{\log A_n} \leq \sqrt{P_n} \frac{\log P_n}{\log 2}. \tag{1.10}$$

Equations (1.9) and (1.10) give equation (1.5). Equation (1.4) is an immediate consequence of (1.9).

It is well-known the equation (see [4])

$$\sum_{P\leq x} e(P)b(P) \sim x, \tag{1.11}$$

where P denotes a perfect power, $e(P)$ is the principal exponent of P and $b(P)$ is the principal basis of P. If we replace $x = P_n$ into (1.11) then we obtain (1.6).

Equation (1.8) is an immediate consequence of equation (1.4) and equation (1.7) is an immediate consequence of equation (1.8). □

Theorem 1.4. *Let $a \geq 2$ an arbitrary but fixed positive integer. Let us consider the closed interval $[k^2,(k+1)^2]$ $(k \geq 3)$. The number of perfect powers, with basis a, that are contained in this interval is either 0 or 1.*

Proof. We have (mean value Theorem)

$$\begin{aligned} 2\frac{\log(k+1)}{\log a} - 2\frac{\log k}{\log a} &= \frac{2}{\log a}\left(\log(k+1) - \log k\right) \leq \frac{2}{\log 2}\frac{1}{k+\epsilon(k)} \\ &< \frac{2}{\log 2}\frac{1}{k} \leq \frac{2}{\log 2}\frac{1}{3} < 1, \end{aligned} \tag{1.12}$$

where $0 < \epsilon(k) < 1$.

Suppose that the closed interval $[k^2.(k+1)^2]$ contains two different perfect powers a^{h_1} and a^{h_2}. We have

$$k^2 \leq a^{h_1} < a^{h_2} \leq (k+1)^2.$$

That is

$$2\log k \leq h_1 \log a < h_2 \log a \leq 2\log(k+1).$$

That is

$$2\frac{\log k}{\log a} \leq h_1 < h_2 \leq 2\frac{\log(k+1)}{\log a}.$$

Therefore

$$2\frac{\log(k+1)}{\log a} - 2\frac{\log k}{\log a} \geq 1, \tag{1.13}$$

since h_1 and h_2 are different positive integers. Equation (1.13) is an evident contradiction (see equation (1.12)). □

Theorem 1.5. *Let $h \geq 3$ an arbitrary but fixed positive integer. Let us consider the closed interval $[k^2,(k+1)^2]$ $(k \geq 2)$. The number of h-th perfect powers that are contained in this interval is either 0 or 1.*

Proof. We have (mean value Theorem)

$$(k+1)^{2/h} - k^{2/h} = \frac{2}{h}(k+\epsilon(k))^{(2/h)-1} \leq \frac{2}{3}(2)^{(2/3)-1} < 1 \quad (k \geq 2) \quad (h \geq 3) \tag{1.14}$$

where $0 < \epsilon(k) < 1$.

Suppose that the closed interval $[k^2.(k+1)^2]$ contains two different h-th perfect powers s^h and r^h. Hence we have

$$k^2 \leq s^h < r^h \leq (k+1)^2.$$

Therefore

$$k^{2/h} \leq s < r \leq (k+1)^{2/h}$$

and consequently

$$(k+1)^{2/h} - k^{2/h} \geq r - s \geq 1,$$

since r and s are different positive integers. This is an evident contradiction (see equation (1.14)). □

Theorem 1.6. *If the S_1 open interval $(k^2, (k+1)^2)$ contains $t \geq 1$ different perfect powers $m_1^{h_1}$, $m_2^{h_2}$, ..., $m_t^{h_t}$, where $m_1, m_2, \ldots, m_t$ are the basis and $h_1, h_2, \ldots, h_t$ are the exponents, then the exponents are odd numbers and if $t \geq 2$ then $\gcd(h_i, h_j) = 1$, $(i \neq j)$, $(i = 1, 2, \ldots, t)$, $(j = 1, 2, \ldots, t)$.*

Proof. It is an immediate consequence of Theorem 1.5. □

Theorem 1.7. *If the S_1 closed interval $[m_0^{h_0} = k^2, (k+1)^2 = m_{t+1}^{h_{t+1}}]$ contains $t+2 \geq 3$ different perfect powers $m_0^{h_0} = k^2 < m_1^{h_1} < m_2^{h_2} < \ldots < m_t^{h_t} < (k+1)^2 = m_{t+1}^{h_{t+1}}$, where $m_0, m_1, m_2, \ldots, m_t, m_{t+1}$ are the basis and $h_0, h_1, h_2, \ldots, h_t, h_{t+1}$ are the exponents, then*

$$m_i^{\frac{h_i}{h_{i-1}}} \qquad (i = 1, 2, \ldots, t+1)$$

is not an integer.

Proof. If $m_1^{\frac{h_1}{h_0}} = e_1$, where $e_1 > 1$ is an integer, then $m_1^{h_1} = e^{h_0}$, where e^{h_0} is a square. This is impossible, since k^2 and $(k+1)^2$ are consecutive squares. If $m_i^{\frac{h_i}{h_{i-1}}} = e_i$ $(i = 2, \ldots, t+1)$, where $e_i > 1$ is an integer, then $m_i^{h_i} = e^{h_{i-1}}$, and consequently the closed interval contains two perfect powers with the same exponent odd h_{i-1}, namely $m_{i-1}^{h_{i-1}}$ and $e^{h_{i-1}}$. This is impossible by Theorem 1.6. □

Theorem 1.8. *Suppose that the consecutive perfect powers $P_n = A_n^{s_n}$ and $P_{n+1} = A_{n+1}^{s_{n+1}}$ are contained in a S_1 closed interval. The following formulae hold*

$$A_n = \left\lfloor A_{n+1}^{\frac{s_{n+1}}{s_n}} \right\rfloor, \tag{1.15}$$

$$s_n = \left\lfloor \frac{s_{n+1} \log A_{n+1}}{\log A_n} \right\rfloor. \tag{1.16}$$

Proof. We have

$$A_n^{s_n} < A_{n+1}^{s_{n+1}}. \tag{1.17}$$

That is

$$A_n < A_{n+1}^{\frac{s_{n+1}}{s_n}}.$$

Consequently (see Theorem 1.7)

$$A_n \leq \left\lfloor A_{n+1}^{\frac{s_{n+1}}{s_n}} \right\rfloor < A_{n+1}^{\frac{s_{n+1}}{s_n}}.$$

Suppose that

$$A_n < \left\lfloor A_{n+1}^{\frac{s_{n+1}}{s_n}} \right\rfloor < A_{n+1}^{\frac{s_{n+1}}{s_n}}. \tag{1.18}$$

Equation (1.18) gives

$$A_n^{s_n} < \left\lfloor A_{n+1}^{\frac{s_{n+1}}{s_n}} \right\rfloor^{s_n} < A_{n+1}^{s_{n+1}}. \tag{1.19}$$

Consequently $A_n^{s_n}$ and $A_{n+1}^{s_{n+1}}$ are not consecutive, an evident contradiction. Therefore equation (1.15) is proved.

Equation (1.17) gives

$$s_n < \frac{s_{n+1} \log A_{n+1}}{\log A_n}.$$

Note that $\frac{s_{n+1} \log A_{n+1}}{\log A_n}$ is not an integer (see Theorem 1.4). Therefore we have

$$s_n \leq \left\lfloor \frac{s_{n+1} \log A_{n+1}}{\log A_n} \right\rfloor < \frac{s_{n+1} \log A_{n+1}}{\log A_n}.$$

Suppose that

$$s_n < \left\lfloor \frac{s_{n+1} \log A_{n+1}}{\log A_n} \right\rfloor < \frac{s_{n+1} \log A_{n+1}}{\log A_n}. \tag{1.20}$$

Equation (1.20) gives

$$A_n^{s_n} < A_n^{\left\lfloor \frac{s_{n+1} \log A_{n+1}}{\log A_n} \right\rfloor} < A_{n+1}^{s_{n+1}}.$$

Consequently $A_n^{s_n}$ and $A_{n+1}^{s_{n+1}}$ are not consecutive, an evident contradiction. Therefore equation (1.16) is proved. □

Now, we put

$$\epsilon_n = A_{n+1}^{\frac{s_{n+1}}{s_n}} - \left\lfloor A_{n+1}^{\frac{s_{n+1}}{s_n}} \right\rfloor = A_{n+1}^{\frac{s_{n+1}}{s_n}} - A_n. \tag{1.21}$$

That is, $0 < \epsilon_n < 1$ is the fractional part of $A_{n+1}^{\frac{s_{n+1}}{s_n}}$.

Lemma 1.9. *If b_n is a bounded sequence of real numbers, that is, there exists $K > 0$ such that $|b_n| < K$ and a_n is a sequence of real numbers such that $a_n \to 0$, then we have the following formula*

$$(1 + a_n)^{b_n} = 1 + b_n a_n + o(b_n a_n) = 1 + b_n a_n (1 + o(1)). \tag{1.22}$$

If b_n is a bounded sequence of real numbers, $\frac{1}{b_n - 1}$ is also a bounded sequence of real numbers and a_n is a sequence of real numbers such that $a_n \to 0$, then we have the following formula

$$\begin{aligned} (1 + a_n)^{b_n} &= 1 + b_n a_n + \frac{b_n(b_n - 1)}{2} a_n^2 + o\left(\frac{b_n(b_n - 1)}{2} a_n^2\right) \\ &= 1 + b_n a_n + \frac{b_n(b_n - 1)}{2} a_n^2 (1 + o(1)). \end{aligned} \tag{1.23}$$

Proof. We have the Taylor's formulae

$$\log(1+x) = x + o(x) \qquad (x \to 0),$$
$$e^x = 1 + x + o(x) \qquad (x \to 0).$$

Therefore we have

$$\begin{aligned}(1+a_n)^{b_n} &= \exp(b_n \log(1+a_n)) = \exp(b_n(a_n + o(a_n))) = \exp(b_n a_n(1+o(1))) \\ &= 1 + b_n a_n(1+o(1)) + o(b_n a_n(1+o(1))) = 1 + b_n a_n + o(b_n a_n).\end{aligned}$$

Consequently the first formula is proved.

We have the Taylor's formulae

$$\log(1+x) = x - \frac{x^2}{2} + o(x^2) \qquad (x \to 0),$$
$$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \qquad (x \to 0).$$

Therefore we have

$$\begin{aligned}(1+a_n)^{b_n} &= \exp(b_n \log(1+a_n)) = \exp(b_n(a_n - \frac{a_n^2}{2} + o(a_n^2))) \\ &= 1 + b_n(a_n - \frac{a_n^2}{2} + o(a_n^2)) + \frac{b_n^2(a_n - \frac{a_n^2}{2} + o(a_n^2))^2}{2} \\ &+ o(b_n^2(a_n - \frac{a_n^2}{2} + o(a_n^2))^2) = 1 + b_n(a_n - \frac{a_n^2}{2} + o(a_n^2)) \\ &+ \frac{b_n^2}{2}a_n^2(1+o(1)) + o(b_n^2 a_n^2(1+o(1))) = 1 + b_n a_n - \frac{b_n}{2}a_n^2 + \frac{b_n^2}{2}a_n^2 \\ &+ o(b_n a_n^2) = 1 + b_n a_n + \frac{b_n(b_n-1)}{2}a_n^2 + o(b_n a_n^2) = 1 + b_n a_n \\ &+ \frac{b_n(b_n-1)}{2}a_n^2 + o(\frac{2}{b_n-1}\frac{b_n(b_n-1)}{2}a_n^2) = 1 + b_n a_n \\ &+ \frac{b_n(b_n-1)}{2}a_n^2(1+o(1)).\end{aligned}$$

Consequently the second formula is proved. □

Theorem 1.10. *Suppose that the consecutive perfect powers $P_n = A_n^{s_n}$ and $P_{n+1} = A_{n+1}^{s_{n+1}}$ are contained in a S_1 closed interval. The following formulae hold*

$$\epsilon_n = \frac{A_n}{s_n}\frac{d_n}{A_n^{s_n}} + \frac{1}{2}\left(\frac{1}{s_n} - 1\right)\frac{A_n}{s_n}\left(\frac{d_n}{A_n^{s_n}}\right)^2(1+o(1)). \tag{1.24}$$

$$\epsilon_n \sim \frac{A_n}{s_n}\frac{d_n}{A_n^{s_n}}. \tag{1.25}$$

$$d_n \sim s_n A_n^{s_n-1}\epsilon_n. \tag{1.26}$$

$$0 < \epsilon_n < \frac{8}{s_n A_n^{\frac{s_n}{2}-1}} \qquad (s_n \geq 3). \tag{1.27}$$

$$\lim_{n\to\infty} \epsilon_n = 0 \qquad (s_n \geq 3). \tag{1.28}$$

Proof. We have (see (1.21), (1.23) and (1.1))

$$\begin{aligned} A_n &= A_{n+1}^{\frac{s_{n+1}}{s_n}} - \epsilon_n = (A_n^{s_n} + d_n)^{\frac{1}{s_n}} - \epsilon_n = A_n \left(1 + \frac{d_n}{A_n^{s_n}}\right)^{\frac{1}{s_n}} - \epsilon_n \\ &= A_n \left(1 + \frac{1}{s_n}\frac{d_n}{A_n^{s_n}} + \frac{1}{2}\frac{1}{s_n}\left(\frac{1}{s_n} - 1\right)\left(\frac{d_n}{A_n^{s_n}}\right)^2 (1 + o(1))\right) - \epsilon_n. \end{aligned}$$

That is, equation (1.24). Equation (1.25) is a weak consequence of equation (1.24) and equation (1.26) is an immediate consequence of equation (1.25).

Equation (1.26) and equation (1.1) give

$$\frac{1}{2} s_n A_n^{s_n - 1} \epsilon_n < d_n < 4 A_n^{\frac{s_n}{2}}. \tag{1.29}$$

Equation (1.29) gives

$$0 < \epsilon_n < \frac{8 A_n^{\frac{s_n}{2}}}{s_n A_n^{s_n - 1}} = \frac{8}{s_n A_n^{\frac{s_n}{2} - 1}} = \frac{8}{s_n A_n^{\frac{3+t_n}{2} - 1}} = \frac{8}{(s_n A_n)^{1/2} (s_n)^{1/2} (A_n)^{t_n/2}} \tag{1.30}$$

Consequently equation (1.27) is proved. On the other hand we have (see (1.4))

$$\lim_{n\to\infty} \left((s_n A_n)^{1/2} (s_n)^{1/2} (A_n)^{t_n/2}\right) = \infty, \tag{1.31}$$

since $(s_n)^{1/2} \geq 1$ and $(A_n)^{t_n/2} \geq 1$. Equations (1.30) and (1.31) give equation (1.28). □

Theorem 1.11. *Suppose that the consecutive perfect powers $P_n = A_n^{s_n}$ and $P_{n+1} = A_{n+1}^{s_{n+1}}$ are contained in a S_1 closed interval. Let us consider the function $f(x) = x^{1/s_n}$. We have*

$$\epsilon_n = f'(\alpha_n) d_n \qquad (P_n = A_n^{s_n} < \alpha_n < P_{n+1} = A_{n+1}^{s_{n+1}}). \tag{1.32}$$

Besides

$$f'(\alpha_n) = g(n) f'(P_n) = g(n) f'(A_n^{s_n}), \tag{1.33}$$

where $\lim_{n\to\infty} g(n) = 1$ *and*

$$\lim_{n\to\infty} f'(\alpha_n) = \lim_{n\to\infty} f'(A_n^{s_n}) = 0. \tag{1.34}$$

Also, in the interval $[P_n, P_{n+1}]$ the greatest derivative is

$$f'(P_n) = f'(A_n^{s_n}) = \frac{1}{s_n A_n^{s_n - 1}} \tag{1.35}$$

and the least derivative is

$$f'(P_{n+1}) = f'(A_{n+1}^{s_{n+1}}) = h(n) \frac{1}{s_n A_n^{s_n - 1}}, \tag{1.36}$$

where $h(n) \to 1$.

Let $\beta_n = \alpha_n - P_n$. We have

$$\beta_n \sim \frac{d_n}{2}. \tag{1.37}$$

Besides

$$\int_{P_n}^{P_{n+1}} \left(x^{1/m} - A_n\right)\ dx \sim \frac{1}{2}\epsilon_n d_n. \tag{1.38}$$

Proof. We have (equation (1.21) and mean value Theorem)

$$\epsilon_n = A_{n+1}^{\frac{s_{n+1}}{s_n}} - A_n = f\left(A_{n+1}^{s_{n+1}}\right) - f\left(A_n^{s_n}\right) = f'(\alpha_n)d_n, \tag{1.39}$$

where $P_n = A_n^{s_n} < \alpha_n < P_{n+1} = A_{n+1}^{s_{n+1}}$. Consequently equation (1.32) is proved.

On the other hand, equation (1.25) gives

$$\epsilon_n = g(n)\frac{1}{s_n A_n^{s_n-1}} d_n = g(n) f'(A_n^{s_n}) d_n, \tag{1.40}$$

where $g(n) \to 1$.

Equation (1.33) is an immediate consequence of equations (1.39) and (1.40). Equation (1.34) is an immediate consequence of equations (1.40) and (1.4).

Note that $f'(x) = \frac{1}{s_n} x^{\frac{1}{s_n}-1}$ is a decreasing function, consequently the greatest derivative in the interval $[P_n, P_{n+1}]$ is

$$f'(P_n) = f'(A_n^{s_n}) = \frac{1}{s_n A_n^{s_n-1}}$$

and the least derivative is

$$\begin{aligned} f'(P_{n+1}) &= f'(A_{n+1}^{s_{n+1}}) = \frac{1}{s_n}\left(A_{n+1}^{s_{n+1}}\right)^{\frac{1}{s_n}-1} = \frac{1}{s_n}\left(A_n^{s_n} + d_n\right)^{\frac{1}{s_n}-1} \\ &= \frac{1}{s_n A_n^{s_n-1}}\left(1 + \frac{d_n}{A_n^{s_n}}\right)^{-\left(1-\frac{1}{s_n}\right)} \\ &= \frac{1}{s_n A_n^{s_n-1}} \exp\left(-\left(1 - \frac{1}{s_n}\right)\log\left(1 + \frac{d_n}{A_n^{s_n}}\right)\right) = \frac{h(n)}{s_n A_n^{s_n-1}}, \end{aligned}$$

where $h(n) \to 1$, since (see (1.1)) $\frac{d_n}{P_n} = \frac{d_n}{A_n^{s_n}} \to 0$ and $1 - \frac{1}{s_n}$ is bounded. Therefore equations (1.35) and (1.36) are proved.

Now, we have (see (1.32) and (1.22))

$$\begin{aligned} \frac{\epsilon_n}{d_n} &= f'(\alpha_n) = f'\left(P_n + \beta_n\right) = \frac{1}{s_n}\left(A_n^{s_n} + \beta_n\right)^{\frac{1}{s_n}-1} = \frac{1}{s_n} A_n^{1-s_n}\left(1 + \frac{\beta_n}{A_n^{s_n}}\right)^{\frac{1}{s_n}-1} \\ &= \frac{1}{s_n} A_n^{1-s_n}\left(1 + \left(\frac{1}{s_n} - 1\right)\frac{\beta_n}{A_n^{s_n}}(1 + o(1))\right) \\ &= \frac{A_n}{s_n}\frac{1}{A_n^{s_n}} + \beta_n\left(\frac{1}{s_n} - 1\right)\frac{A_n}{s_n}\frac{1}{A_n^{2s_n}}(1 + o(1)). \end{aligned}$$

That is

$$\epsilon_n = \frac{A_n}{s_n}\frac{d_n}{A_n^{s_n}} + \beta_n\left(\frac{1}{s_n} - 1\right)\frac{A_n}{s_n}\frac{d_n}{A_n^{2s_n}}(1 + o(1)). \tag{1.41}$$

On the other hand, we have (see (1.24))

$$\epsilon_n = \frac{A_n}{s_n}\frac{d_n}{A_n^{s_n}} + \frac{1}{2}\left(\frac{1}{s_n} - 1\right)\frac{A_n}{s_n}\left(\frac{d_n}{A_n^{s_n}}\right)^2(1 + o(1)). \tag{1.42}$$

Equation (1.37) is an immediate consequence of (1.41) and (1.42).

Since $f'(x)$ is decreasing the area (1.38) is between the areas of two triangles, namely

$$\frac{(f'(P_{n+1})d_n)d_n}{2} < \int_{P_n}^{P_{n+1}} \left(x^{1/m} - A_n\right)\, dx < \frac{(f'(P_n)d_n)d_n}{2}. \tag{1.43}$$

Substituting (1.35), (1.36) and (1.26) into (1.43) we obtain (1.38). □

The former theorems have been established in terms of $A_n^{s_n}$. In the following theorems we establish the theorems in terms of $A_{n+1}^{s_{n+1}}$.

Theorem 1.12. *Suppose that the consecutive perfect powers $P_n = A_n^{s_n}$ and $P_{n+1} = A_{n+1}^{s_{n+1}}$ are contained in a S_1 closed interval. We have following formulae*

$$A_{n+1} = \left\lfloor A_n^{\frac{s_n}{s_{n+1}}} \right\rfloor + 1, \tag{1.44}$$

$$s_{n+1} = \left\lfloor \frac{s_n \log A_n}{\log A_{n+1}} \right\rfloor + 1.$$

Proof. The proof is the same as the proof of Theorem 1.8. □

Now, we put

$$\gamma_n = A_n^{\frac{s_n}{s_{n+1}}} - \left\lfloor A_n^{\frac{s_n}{s_{n+1}}} \right\rfloor. \tag{1.45}$$

That is, $0 < \gamma_n < 1$ is the fractional part of $A_n^{\frac{s_n}{s_{n+1}}}$.

Equations (1.44) and (1.45) give

$$A_{n+1} = A_n^{\frac{s_n}{s_{n+1}}} + (1 - \gamma_n) = A_n^{\frac{s_n}{s_{n+1}}} + \alpha_n, \tag{1.46}$$

where $0 < \alpha_n = 1 - \gamma_n < 1$

Theorem 1.13. *Suppose that the consecutive perfect powers $P_n = A_n^{s_n}$ and $P_{n+1} = A_{n+1}^{s_{n+1}}$ are contained in a S_1 closed interval. The following formulae hold*

$$\alpha_n = \frac{A_{n+1}}{s_{n+1}} \frac{d_n}{A_{n+1}^{s_{n+1}}} - \frac{1}{2}\left(\frac{1}{s_{n+1}} - 1\right) \frac{A_{n+1}}{s_{n+1}} \left(\frac{d_n}{A_{n+1}^{s_{n+1}}}\right)^2 (1 + o(1)),$$

$$\alpha_n \sim \frac{A_{n+1}}{s_{n+1}} \frac{d_n}{A_{n+1}^{s_{n+1}}},$$

$$d_n \sim s_{n+1} A_{n+1}^{s_{n+1}-1} \alpha_n, \tag{1.47}$$

$$0 < \alpha_n < \frac{8}{s_{n+1} A_{n+1}^{\frac{s_{n+1}}{2}-1}} \qquad (s_{n+1} \geq 3),$$

$$\lim_{n\to\infty} \alpha_n = 0 \qquad (s_{n+1} \geq 3). \tag{1.48}$$

Proof. The proof is the same as the proof of Theorem 1.10. □

Theorem 1.14. *Suppose that the consecutive perfect powers $P_n = A_n^{s_n}$ and $P_{n+1} = A_{n+1}^{s_{n+1}}$ are contained in a S_1 closed interval. Let us consider the function $g(x) = x^{1/s_{n+1}}$. We have*

$$\alpha_n = g'(\mu_n)d_n \qquad (P_n = A_n^{s_n} < \mu_n < P_{n+1} = A_{n+1}^{s_{n+1}}). \tag{1.49}$$

Besides

$$g'(\mu_n) = j(n)g'(P_{n+1}) = j(n)g'(A_{n+1}^{s_{n+1}}), \tag{1.50}$$

where $\lim_{n\to\infty} j(n) = 1$ and

$$\lim_{n\to\infty} g'(\mu_n) = \lim_{n\to\infty} g'(A_{n+1}^{s_{n+1}}) = 0. \tag{1.51}$$

Also, in the interval $[P_n, P_{n+1}]$ the greatest derivative is

$$g'(P_n) = g'(A_n^{s_n}) = h(n)\frac{1}{s_{n+1}A_{n+1}^{s_{n+1}-1}} \tag{1.52}$$

and the least derivative is

$$g'(P_{n+1}) = g'(A_{n+1}^{s_{n+1}}) = \frac{1}{s_{n+1}A_{n+1}^{s_{n+1}-1}}, \tag{1.53}$$

where $h(n) \to 1$.

Let $\delta_n = \mu_n - P_n$. We have

$$\delta_n \sim \frac{d_n}{2}.$$

Besides

$$\int_{P_n}^{P_{n+1}} \left(x^{\frac{1}{s_{n+1}}} - A_n^{\frac{s_n}{s_{n+1}}} \right) \, dx \sim \frac{1}{2}\alpha_n d_n.$$

Proof. The proof is the same as the proof of Theorem 1.11. □

Theorem 1.15. *The following formulae hold*

$$\frac{\epsilon_n}{\alpha_n} = \frac{P_{n+1}^{\frac{1}{s_n}} - P_n^{\frac{1}{s_n}}}{P_{n+1}^{\frac{1}{s_{n+1}}} - P_n^{\frac{1}{s_{n+1}}}}, \tag{1.54}$$

$$\frac{s_{n+1}}{s_n}\frac{A_{n+1}^{\frac{s_{n+1}}{s_n}}}{A_n^{\frac{s_n}{s_{n+1}}}}\frac{A_n^{s_n}}{A_{n+1}^{s_{n+1}}} < \frac{\epsilon_n}{\alpha_n} < \frac{s_{n+1}}{s_n}\frac{A_n}{A_{n+1}}\frac{A_{n+1}^{s_{n+1}}}{A_n^{s_n}}, \tag{1.55}$$

$$\frac{\epsilon_n}{\alpha_n} \sim \frac{s_{n+1}A_n}{s_nA_{n+1}} \sim \frac{A_n \log A_n}{A_{n+1}\log A_{n+1}} \sim \frac{s_{n+1}}{s_n}A_{n+1}^{\frac{s_{n+1}}{s_n}-1} \sim \frac{s_{n+1}}{s_n}\frac{1}{A_n^{\frac{s_n}{s_{n+1}}-1}}. \tag{1.56}$$

Proof. We have (see (1.21) and (1.46))

$$\frac{\epsilon_n}{\alpha_n} = \frac{A_{n+1}^{\frac{s_{n+1}}{s_n}} - A_n}{A_{n+1} - A_n^{\frac{s_n}{s_{n+1}}}} = \frac{P_{n+1}^{\frac{1}{s_n}} - P_n^{\frac{1}{s_n}}}{P_{n+1}^{\frac{1}{s_{n+1}}} - P_n^{\frac{1}{s_{n+1}}}}.$$

Hence, equation (1.54) is proved.

Now, we have (see Theorem 1.11 and Theorem 1.14)

$$\frac{f'(P_{n+1})}{g'(P_n)} < \frac{\epsilon_n}{\alpha_n} = \frac{f'(\alpha_n)d_n}{g'(\mu_n)d_n} = \frac{f'(\alpha_n)}{g'(\mu_n)} < \frac{f'(P_n)}{g'(P_{n+1})}. \tag{1.57}$$

Substituting (1.35), (1.36), (1.52) and (1.53) into (1.57) we obtain equation (1.55).

We have (see (1.26) and (1.47))

$$1 = \frac{d_n}{d_n} = \frac{f(n)s_n A_n^{s_n-1}\epsilon_n}{g(n)s_{n+1}A_{n+1}^{s_{n+1}-1}\alpha_n} = \frac{f(n)s_n A_n^{s_n}A_n^{-1}\epsilon_n}{g(n)s_{n+1}A_{n+1}^{s_{n+1}}A_{n+1}^{-1}\alpha_n},$$

where $f(n) \to 1$ and $g(n) \to 1$.

Now, $P_{n+1} = A_{n+1}^{s_{n+1}} \sim P_n = A_n^{s_n}$ since $P_n \sim n^2$ (see [3]). Besides, since $A_{n+1}^{s_{n+1}} \sim A_n^{s_n}$ we obtain $s_{n+1}\log A_{n+1} \sim s_n \log A_n$. Consequently the first and second equations are proved.

We have (see (1.46))

$$A_n^{s_n} + d_n = \left(A_n^{\frac{s_n}{s_{n+1}}} + \alpha_n\right)^{s_{n+1}}. \tag{1.58}$$

Consequently (see (1.58), (1.22) and (1.26))

$$\begin{aligned}
\alpha_n &= (A_n^{s_n} + d_n)^{\frac{1}{s_{n+1}}} - A_n^{\frac{s_n}{s_{n+1}}} = A_n^{\frac{s_n}{s_{n+1}}}\left(1 + \frac{d_n}{A_n^{s_n}}\right)^{\frac{1}{s_{n+1}}} - A_n^{\frac{s_n}{s_{n+1}}} \\
&= A_n^{\frac{s_n}{s_{n+1}}}\left(1 + \frac{1}{s_{n+1}}\frac{d_n}{A_n^{s_n}}(1+o(1))\right) - A_n^{\frac{s_n}{s_{n+1}}} = \frac{A_n^{\frac{s_n}{s_{n+1}}}}{s_{n+1}}\frac{d_n}{A_n^{s_n}}(1+o(1)) \\
&= \frac{A_n^{\frac{s_n}{s_{n+1}}}}{s_{n+1}}\frac{s_n A_n^{s_n-1}\epsilon_n}{A_n^{s_n}}(1+o(1)) = \frac{s_n}{s_{n+1}}\frac{A_n^{\frac{s_n}{s_{n+1}}}}{A_n}\epsilon_n(1+o(1))
\end{aligned}$$

That is

$$\frac{\epsilon_n}{\alpha_n} \sim \frac{s_{n+1}}{s_n}\frac{1}{A_n^{\frac{s_n}{s_{n+1}}-1}}.$$

As we desired.

On the other hand, we have (see (1.21))

$$\left(A_{n+1}^{\frac{s_{n+1}}{s_n}} - \epsilon_n\right)^{s_n} + d_n = A_{n+1}^{s_{n+1}}. \tag{1.59}$$

Consequently (see (1.59), (1.22) and (1.47)), as in the former case we obtain

$$\epsilon_n = A_{n+1}^{\frac{s_{n+1}}{s_n}} - \left(A_{n+1}^{s_{n+1}} - d_n\right)^{\frac{1}{s_n}} = \cdots = \frac{s_{n+1}}{s_n}A_{n+1}^{\frac{s_{n+1}}{s_n}-1}\alpha_n(1+o(1)).$$

That is

$$\frac{\epsilon_n}{\alpha_n} \sim \frac{s_{n+1}}{s_n}A_{n+1}^{\frac{s_{n+1}}{s_n}-1}.$$

As we desired. □

Remark 1.16. Using equations (1.21) and (1.46) the equation

$$P_n = A_n^{s_n} < P_{n+1} = A_{n+1}^{s_{n+1}}$$

can be written in the form

$$\left(A_{n+1}^{\frac{s_{n+1}}{s_n}} - \epsilon_n\right)^{s_n} < \left(A_n^{\frac{s_n}{s_{n+1}}} + \alpha_n\right)^{s_{n+1}}.$$

Theorem 1.17. *We have the following formulae*

$$A_n \sim A_{n+1}^{\frac{s_{n+1}}{s_n}}, \qquad A_{n+1} \sim A_n^{\frac{s_n}{s_{n+1}}}.$$

Proof. We have (see (1.56))

$$\frac{s_{n+1}A_n}{s_n A_{n+1}} \sim \frac{s_{n+1}}{s_n} \frac{1}{A_n^{\frac{s_n}{s_{n+1}}-1}}.$$

That is

$$A_{n+1} \sim A_n^{\frac{s_n}{s_{n+1}}}.$$

The another formula is proved in the same way (see (1.56)). □

Theorem 1.18. *We have the following formulae*

$$\left(A_{n+1}^{\frac{1}{s_n}} - A_n^{\frac{1}{s_{n+1}}}\right) \to 0, \tag{1.60}$$

$$A_{n+1}^{\frac{1}{s_n}} \sim A_n^{\frac{1}{s_{n+1}}}, \tag{1.61}$$

$$A_{n+1}^{\frac{1}{s_n}} > A_n^{\frac{1}{s_{n+1}}}. \tag{1.62}$$

Proof. If $s_n \geq 3$ then we have (see (1.21), (1.28) and (1.22))

$$A_n^{\frac{1}{s_{n+1}}} = \left(A_{n+1}^{\frac{s_{n+1}}{s_n}} - \epsilon_n\right)^{\frac{1}{s_{n+1}}} = A_{n+1}^{\frac{1}{s_n}} \left(1 - \frac{\epsilon_n}{A_{n+1}^{\frac{s_{n+1}}{s_n}}}\right)^{\frac{1}{s_{n+1}}}$$

$$= A_{n+1}^{\frac{1}{s_n}} \left(1 - \frac{1}{s_{n+1}} \frac{\epsilon_n}{A_{n+1}^{\frac{s_{n+1}}{s_n}}}(1+o(1))\right) = A_{n+1}^{\frac{1}{s_n}} - \frac{1}{s_{n+1}} \frac{\epsilon_n A_{n+1}^{\frac{1}{s_n}}}{A_{n+1}^{\frac{s_{n+1}}{s_n}}}(1+o(1))$$

That is

$$A_{n+1}^{\frac{1}{s_n}} - A_n^{\frac{1}{s_{n+1}}} = \frac{1}{s_{n+1}} \frac{\epsilon_n A_{n+1}^{\frac{1}{s_n}}}{A_{n+1}^{\frac{s_{n+1}}{s_n}}}(1+o(1)) \to 0. \tag{1.63}$$

On the other hand, if $s_{n+1} \geq 3$ then we have (see (1.46), (1.48) and (1.22))

$$A_{n+1}^{\frac{1}{s_n}} = \left(A_n^{\frac{s_n}{s_{n+1}}} + \alpha_n\right)^{\frac{1}{s_n}} = A_n^{\frac{1}{s_{n+1}}}\left(1 + \frac{\alpha_n}{A_n^{\frac{s_n}{s_{n+1}}}}\right)^{\frac{1}{s_n}}$$

$$= \quad A_n^{\frac{1}{s_{n+1}}}\left(1 + \frac{1}{s_n}\frac{\alpha_n}{A_n^{\frac{s_n}{s_{n+1}}}}(1+o(1))\right) = A_n^{\frac{1}{s_{n+1}}} + \frac{1}{s_n}\frac{\alpha_n A_n^{\frac{1}{s_{n+1}}}}{A_n^{\frac{s_n}{s_{n+1}}}}(1+o(1))$$

That is

$$A_{n+1}^{\frac{1}{s_n}} - A_n^{\frac{1}{s_{n+1}}} = \frac{1}{s_n}\frac{\alpha_n A_n^{\frac{1}{s_{n+1}}}}{A_n^{\frac{s_n}{s_{n+1}}}}(1+o(1)) \to 0. \tag{1.64}$$

Equations (1.63) and (1.64) give equation (1.60) and equation (1.61) is an immediate consequence of equation (1.60). Equation (1.62) is an immediate consequence of (1.63) and (1.64). □

Now, we define two new fractional parts. Equation (1.16) give us the fractional part

$$0 < \delta_n = \frac{s_{n+1}\log A_{n+1}}{\log A_n} - s_n < 1. \tag{1.65}$$

On the other hand, the equation below equation (1.44) give us the fractional part

$$\frac{s_n \log A_n}{\log A_{n+1}} + 1 - \chi_n = s_{n+1}.$$

Therefore, we have

$$0 < \gamma_n = 1 - \chi_n = s_{n+1} - \frac{s_n \log A_n}{\log A_{n+1}} < 1. \tag{1.66}$$

Theorem 1.19. *The following formulae hold*

$$\delta_n \sim \frac{d_n}{A_n^{s_n}\log A_n}, \tag{1.67}$$

$$\lim_{n\to\infty} \delta_n = 0, \tag{1.68}$$

$$d_n \sim A_n^{s_n}\log A_n \delta_n, \tag{1.69}$$

$$\gamma_n \sim \frac{d_n}{A_{n+1}^{s_{n+1}}\log A_{n+1}}, \tag{1.70}$$

$$\lim_{n\to\infty} \gamma_n = 0, \tag{1.71}$$

$$d_n \sim A_{n+1}^{s_{n+1}}\log A_{n+1}\gamma_n, \tag{1.72}$$

$$\frac{\delta_n}{\gamma_n} = \frac{\log A_{n+1}}{\log A_n} \sim \frac{s_n}{s_{n+1}}, \tag{1.73}$$

$$\frac{\alpha_n\gamma_n}{\epsilon_n\delta_n} \sim \frac{A_{n+1}}{A_n}, \tag{1.74}$$

$$\frac{\epsilon_n}{\delta_n} \sim \frac{A_n \log A_n}{s_n}, \tag{1.75}$$

$$\frac{\alpha_n}{\gamma_n} \sim \frac{A_{n+1} \log A_{n+1}}{s_{n+1}}, \tag{1.76}$$

$$s_n \sim \frac{s_{n+1} \log A_{n+1}}{\log A_n}, \tag{1.77}$$

$$s_{n+1} \sim \frac{s_n \log A_n}{\log A_{n+1}}, \tag{1.78}$$

$$\delta_n = \frac{d_n}{A_n^{s_n} \log A_n} - \frac{1}{2}\frac{1}{\log A_n}\left(\frac{d_n}{A_n^{s_n}}\right)^2 (1 + o(1)), \tag{1.79}$$

$$\gamma_n = \frac{d_n}{A_{n+1}^{s_{n+1}} \log A_{n+1}} + \frac{1}{2}\frac{1}{\log A_{n+1}}\left(\frac{d_n}{A_{n+1}^{s_{n+1}}}\right)^2 (1 + o(1)). \tag{1.80}$$

Proof. We have

$$\frac{A_{n+1}^{s_{n+1}}}{A_n^{s_n}} = \left(1 + \frac{d_n}{A_n^{s_n}}\right) \to 1. \tag{1.81}$$

Consequently

$$s_{n+1} \log A_{n+1} \sim s_n \log A_n. \tag{1.82}$$

Equation (1.82) give (1.77) and (1.78) (see (1.65) and (1.66)).

Note that if $x \to 0$ then $\log(1+x) \sim x$ and therefore equation (1.81) gives

$$\log\left(\frac{A_{n+1}^{s_{n+1}}}{A_n^{s_n}}\right) = s_{n+1}\log A_{n+1} - s_n \log A_n = \log\left(1 + \frac{d_n}{A_n^{s_n}}\right) \sim \frac{d_n}{A_n^{s_n}}. \tag{1.83}$$

Equations (1.65), (1.66) and (1.83) give equations (1.67) and (1.70)(use also (1.79)). Equations (1.68) and (1.69) are an immediate consequence of equation (1.67). Equations (1.71) and (1.72) are an immediate consequence of equation (1.70). Equation (1.73) is an immediate consequence of equations (1.65), (1.66) and (1.82). Equation (1.74) is an immediate consequence of equations (1.73) and (1.56). Equation (1.75) is an immediate consequence of equations (1.67) and (1.25). Equation (1.76) is an immediate consequence of equation (1.70) and the equation above of equation (1.47). Equation (1.65) can be written in the form (see also (1.83))

$$\delta_n = \frac{1}{\log A_n}\log\left(\frac{A_{n+1}^{s_{n+1}}}{A_n^{s_n}}\right) = \frac{1}{\log A_n}\log\left(1 + \frac{d_n}{A_n^{s_n}}\right). \tag{1.84}$$

Now, if we use the more precise formula

$$\log(1+x) = x - \frac{1}{2}x^2(1+o(1)), \qquad (x \to 0)$$

equation (1.84) gives (1.79). Equation (1.80) can be proved in the same way from equation (1.66). If we wish more precise formulae than (1.79) and (1.80) we can use a Taylor's polynomial with more terms. □

Remark 1.20. The inequality (see equations (1.21), (1.46), (1.65) and (1.66))

$$P_n = A_n^{s_n} < P_{n+1} = A_{n+1}^{s_{n+1}}$$

can be written in the form (see Remark 1.16)

$$\left(A_{n+1}^{\frac{s_{n+1}}{s_n}} - \epsilon_n\right)^{\left(\frac{s_{n+1}\log A_{n+1}}{\log A_n} - \delta_n\right)} < \left(A_n^{\frac{s_n}{s_{n+1}}} + \alpha_n\right)^{\left(\frac{s_n \log A_n}{\log A_{n+1}} + \gamma_n\right)}.$$

where the four fractional parts defined in this article appear.

Theorem 1.21. *The following formulae hold*

$$d_n - s_n A_n^{s_n - 1}\epsilon_n \sim \frac{1}{2}\left(1 - \frac{1}{s_n}\right)\frac{d_n^2}{A_n^{s_n}}, \tag{1.85}$$

$$0 \leq \frac{1}{2}\left(1 - \frac{1}{s_n}\right)\frac{d_n^2}{A_n^{s_n}} \leq 2 + \lambda_1, \tag{1.86}$$

$$s_{n+1}A_{n+1}^{s_{n+1}-1}\alpha_n - d_n \sim \frac{1}{2}\left(1 - \frac{1}{s_{n+1}}\right)\frac{d_n^2}{A_{n+1}^{s_{n+1}}}, \tag{1.87}$$

$$0 \leq \frac{1}{2}\left(1 - \frac{1}{s_{n+1}}\right)\frac{d_n^2}{A_{n+1}^{s_{n+1}}} \leq 2 + \lambda_2, \tag{1.88}$$

$$d_n - A_n^{s_n}\log A_n \delta_n \sim \frac{1}{2}\frac{d_n^2}{A_n^{s_n}}, \tag{1.89}$$

$$0 \leq \frac{1}{2}\frac{d_n^2}{A_n^{s_n}} \leq 2 + \lambda_3, \tag{1.90}$$

$$A_{n+1}^{s_{n+1}}\log A_{n+1}\gamma_n - d_n \sim \frac{1}{2}\frac{d_n^2}{A_{n+1}^{s_{n+1}}}, \tag{1.91}$$

$$0 \leq \frac{1}{2}\frac{d_n^2}{A_{n+1}^{s_{n+1}}} \leq 2 + \lambda_4. \tag{1.92}$$

Proof. If we multiply both sides of equation (1.24) by $s_n A_n^{s_n-1}$ (see also equation (1.26)) then we obtain equation (1.85). Equation (1.87) is proved in the same way from the first and third equation in Theorem 1.13. Equations (1.69) and (1.79) give equation (1.89) and equations (1.72) and (1.80) give equation (1.91).

Note that equation (1.1) can be written in the more precise way (the proof is the same)

$$d_n = P_{n+1} - P_n \leq (2+\theta)\sqrt{P_n} < (2+\theta)\sqrt{P_{n+1}}, \tag{1.93}$$

where $\theta > 0$ can be arbitrarily small if n is sufficiently large.

Therefore

$$0 \leq \frac{1}{2}\left(1-\frac{1}{s_n}\right)\frac{d_n^2}{A_n^{s_n}} \leq \frac{1}{2}\frac{(2+\theta)^2 P_n}{P_n} \leq 2+\lambda_1,$$

where λ_1 is a positive number. Equation (1.86) is proved. Equations (1.88), (1.90) and (1.92) can be proved in the same way from equation (1.93). □

Lemma 1.22. *We have $A(n) = 1$ for infinite values of n. That is, there are infinite S_1 closed intervals with only one perfect power in their inner.*

Proof. This lemma is proved in [2]. □

Lemma 1.23. *There exists a sequence of S_1 closed intervals $[a^2 = P_n, P_{n+k} = (a+1)^2]$ such that $\lim \frac{d_n}{2n} = \lim \frac{P_{n+1}-P_n}{2n} = 0$. The sequence $\frac{d_n}{\sqrt{P_n}}$ is bounded.*

Proof. The first proposition is proved in [5]. The second proposition is an immediate consequence of Theorem 1.2. □

Theorem 1.24. *The following formulae hold*

$$\liminf \frac{d_n}{\sqrt{P_n}} = 0, \qquad \limsup \frac{d_n}{\sqrt{P_n}} = h > 0. \tag{1.94}$$

$$\liminf \frac{1}{2}\left(1-\frac{1}{s_n}\right)\frac{d_n^2}{A_n^{s_n}} = 0, \qquad \limsup \frac{1}{2}\left(1-\frac{1}{s_n}\right)\frac{d_n^2}{A_n^{s_n}} = h_1 > 0. \tag{1.95}$$

$$\liminf \frac{1}{2}\left(1-\frac{1}{s_{n+1}}\right)\frac{d_n^2}{A_{n+1}^{s_{n+1}}} = 0, \qquad \limsup \frac{1}{2}\left(1-\frac{1}{s_{n+1}}\right)\frac{d_n^2}{A_{n+1}^{s_{n+1}}} = h_2 > 0. \tag{1.96}$$

$$\liminf \frac{1}{2}\frac{d_n^2}{A_n^{s_n}} = 0, \qquad \limsup \frac{1}{2}\frac{d_n^2}{A_n^{s_n}} = h_3 > 0. \tag{1.97}$$

$$\liminf \frac{1}{2}\frac{d_n^2}{A_{n+1}^{s_{n+1}}} = 0, \qquad \limsup \frac{1}{2}\frac{d_n^2}{A_{n+1}^{s_{n+1}}} = h_4 > 0. \tag{1.98}$$

Proof. We shall prove (1.94). Note that $n \sim \sqrt{P_n}$. Hence, we have

$$\frac{d_n}{2n} = \frac{d_n}{f(n)2\sqrt{P_n}},$$

where $f(n) \to 1$. Consequently (Lemma 1.23)

$$\liminf \frac{d_n}{\sqrt{P_n}} = 0. \tag{1.99}$$

Suppose that

$$\limsup \frac{d_n}{\sqrt{P_n}} = 0,$$

then

$$\lim \frac{d_n}{\sqrt{P_n}} = 0, \tag{1.100}$$

and consequently if $\alpha > 0$ there exists n_0 such that if $n \geq n_0$ we have

$$\frac{d_n}{\sqrt{P_n}} < \alpha. \tag{1.101}$$

Let us consider the S_1 closed interval $[a^2 = P_n, P_{n+k} = (a+1)^2]$. This S_1 closed interval contains $k-1$ perfect powers in its inner, namely, $P_{n+1}, P_{n+2}, \dots, P_{n+k-1}$. Equation (1.101) gives

$$\frac{d_n}{\sqrt{P_n}} < \alpha, \qquad \frac{d_{n+1}}{\sqrt{P_{n+1}}} < \alpha, \quad \dots \quad \frac{d_{n+k-1}}{\sqrt{P_{n+k-1}}} < \alpha. \tag{1.102}$$

Therefore (see equation (1.102))

$$\begin{aligned} 2a+1 &= (a+1)^2 - a^2 = P_{n+k} - P_n = d_n + d_{n+1} + \cdots + d_{n+k-1} \\ &< \alpha\left(\sqrt{P_n} + \sqrt{P_{n+1}} + \cdots + \sqrt{P_{n+k-1}}\right) < \alpha k \sqrt{P_{n+k}} = \alpha k(a+1). \end{aligned}$$

That is

$$k > \frac{1}{\alpha}\frac{2a+1}{a+1} > \frac{1}{\alpha}, \tag{1.103}$$

since $\lim_{a\to\infty} \frac{2a+1}{a+1} = 2$. Now, α can be arbitrarily small, consequently equation (1.103) implies $(k-1) \to \infty$, an evident contradiction with Lemma 1.22. Hence (1.100) is false. That is, we have

$$\limsup \frac{d_n}{\sqrt{P_n}} = h > 0. \tag{1.104}$$

This proves (1.94)(see (1.99)). Now, we shall prove (1.95), the proof of (1.96), (1.97) and (1.98) are the same.

Note that

$$\liminf \frac{1}{2}\left(1 - \frac{1}{s_n}\right)\frac{d_n^2}{A_n^{s_n}} = 0 \tag{1.105}$$

is an immediate consequence of (1.99). Suppose that

$$\limsup \frac{1}{2}\left(1 - \frac{1}{s_n}\right)\frac{d_n^2}{A_n^{s_n}} = 0, \tag{1.106}$$

then

$$\lim \frac{1}{2}\left(1 - \frac{1}{s_n}\right)\frac{d_n^2}{A_n^{s_n}} = 0. \tag{1.107}$$

Limit (1.107) implies limit (1.100), but limit (1.100) is false. Therefore

$$\limsup \frac{1}{2}\left(1 - \frac{1}{s_n}\right)\frac{d_n^2}{A_n^{s_n}} = h_1 > 0. \tag{1.108}$$

This proves (1.95) (see (1.105)). □

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[1] División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.

Email address: jakimczu@mail.unlu.edu.ar