Gulf Journal of Mathematics
Vol 6, Issue 3 (2018) 78- 90

ON THE SUB-SUPERSOLUTION PRINCIPLE FOR $p(x)$-LAPLACIAN EQUATIONS

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ABSTRACT. This paper deals with the sub-supersolution method for the *p(x)*-Laplacian Dirichlet problem. A sub-supersolution principle for the Dirichlet problems involving the *p(x)*-Laplacian is established by using induction method.

1. Introduction

In recent years, the study of differential equations and variational problems with nonstandard $p(x)$-growth condition has been an interesting topic. The $p(x)$-Laplacian arises from the study of nonlinear elasticity, electrorheological fluids and image restoration etc. For example, electrorheological fluids have an extensive applications in robotics, aircraft and aerospace. We refer readers to [3, 18, 20] for more detailed background of applications. There are many reference papers related to the study of differential equations and variational problems with variable exponent. Far from being complete, we refer readers to [6, 9, 17] and references cited therein. For example, the regularity of weak solutions for differential equations with variable exponent was studied in [2, 12], and existence of solutions for variable exponent problems was studied in a series of papers [1, 4, 5, 14].

Clearly, if $p(x) \equiv p$, a constant, the operator is the well-known p-Laplacian, and (P) is the usual p-Laplacian equation, but for non-constant $p(x)$, $p(x)$-Laplacian problems are more complicated due to the non-homogeneity of $p(x)$-Laplacian.

Mohammed [16] considered the existence and uniqueness of weak solutions of the singular boundary value problem with constant exponent as follows

$$\begin{cases} -\Delta_p u = f(x,u), \text{ in } \Omega, \\ u(x) > 0, \text{ in } \Omega, \\ u(x) = 0, \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,\omega}$ boundary for some $0 < \omega < 1$, $\Delta_p u = \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian and singular nonlinearity term $f(x,t)$ could show up when $t \to 0^+$. Mohammed make the following two assumptions:

Date: Received: Apr 3, 2018; Accepted: Jun 5, 2018.
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2000 *Mathematics Subject Classification.* 35J25; 35J65; 35J70.
Key words and phrases. Variable exponent Lebesgue-Sobolev spaces, Subsolution-supersolution, *p(x)*-Laplacian, Dirichlet problem, Induction method.

(1) For each $\theta \in (0,1)$, there is a constant $C_\theta \geq 1$ such that $g(\theta t) \leq C_\theta g(t)$ for all $t > 0$;

(2) $f(x,s) \geq a(x)$ for any $(x,s) \in (0,1)$.

In [11], the authors studied the existence of solutions of the nonlinear elliptic problem with constant exponent,

$$\begin{cases} -\Delta_p u = \lambda f(x,u), \text{ in } \Omega, \\ u(x) > 0, \text{ in } \Omega, \\ u(x) = 0, \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1 < p < N$, $f : \Omega \times (0,\infty) \to [0,\infty)$ is a suitable function and $\lambda > 0$ is a real parameter. The nonlinearity term f is allowed to be either $f(x,s) \to +\infty$ or $f(x,s) \to +\infty$ as $s \to 0^+$ for each $x \in \Omega$ and the assumptions (1) and (2) are not assumed.

In [8], Fan studied the sub-supersolution method for the $p(x)$-Laplacian equations. Author established a sub-supersolution principle for the Dirichlet problems involving the $p(x)$-Laplacian. Fan proved that the local minimizers in the C^1 topology are also local minimizers in the $W^{1,p(x)}$ topology for given energy functionals. A strong comparison theorem for the $p(x)$-Laplacian equations is presented in this paper. Some applications of the abstract theorems obtained in this paper to the eigenvalue problems for the $p(x)$-Laplacian equations are given.

In [14], Liu et al studied the existence of positive solutions for the $p(x)$-Laplacian Dirichlet problem

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \lambda f(x,u), \text{ in } \Omega, \\ u(x) > 0, \text{ in } \Omega, \\ u(x) = 0, \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ bounded domain with smooth boundary $\partial\Omega$, and $\lambda > 0$ is a real parameter. The singular nonlinearity term f is allowed to be either $f(x;s) \to +\infty$, or $f(x;s) \to +\infty$ as $s \to 0^+$ for each $x \in \Omega$. Their main results generalize the results in [11] from constant exponents to variable exponents. They gave the asymptotic behavior of solutions of a simpler equation which is useful for finding supersolutions of differential equations with variable exponents, which is of independent interest.

In [17], Motreanu consider a nonlinear Dirichlet boundary value problem involving the $p(x)$-Laplacian and a concave term:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = m(x,\lambda)\,|u|^{q(x)-2}u + f(x,u,\lambda), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is an open, bounded domain with smooth boundary $\partial\Omega$, $\lambda \in \left(0,\overline{\lambda}\right)$ (for some $\overline{\lambda} > 0$) is a parameter and f satisfies some growth conditions. Their main result show that the existence of at least three nontrivial solutions. They used truncation techniques and the method of sub-supersolutions.

In this paper we study the sub-supersolution method for the $p(x)$-Laplacian Dirichlet problem:

$$\begin{cases} -\Delta_{p(x)}u = f(x,u), \text{ in } \Omega, \\ u(x) > 0, \text{ in } \Omega, \\ u(x) = 0, \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with $C^2(\Omega)$ boundary,

$$\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$$

is the $p(x)$-Laplacian and $\nabla u = (\partial_{x_1}u, \partial_{x_2}u, \partial_{x_3}u)$. The goal of this paper is to study the sub-supersolution method by using induction principle for the Dirichlet problem involving the $p(x)$-Laplacian, which is a new research topic.

This paper is organized as follows. In Section 2, we recall some basic facts about the variable Lebesgue-Sobolev spaces and the degree theory for operators of (S_+) type. Section 3, is devoted to the main results of the paper.

2. Preliminaries

For completeness, we first recall some facts on the variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For more details see [7, 10, 13, 15, 19]. Suppose that Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Write $C_+\left(\overline{\Omega}\right) = \left\{p \in C\left(\overline{\Omega}\right) : p(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}$, $p^- := \inf_{x\in\overline{\Omega}} p(x)$, $p^+ := \sup_{x\in\overline{\Omega}} p(x)$ for any $p \in C_+\left(\overline{\Omega}\right)$. Denote by $\Re(\Omega)$ the set of all measurable real-valued functions defined on Ω. Note that two measurable functions in $\Re(\Omega)$ are considered as the same element of $\Re(\Omega)$ when they are equal almost everywhere.

Let $p \in C_+\left(\overline{\Omega}\right)$. We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u \in \Re(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < \infty\right\},$$

then $L^{p(x)}(\Omega)$ endowed with the norm

$$|u|_{p(x)} = \inf\left\{\delta > 0 : \int_\Omega \left|\frac{u(x)}{\delta}\right|^{p(x)} dx \le 1\right\},$$

becomes a Banach space. We refer the reader to [7, 10, 13] for the basic properties of the variable exponent Lebesgue and Sobolev spaces.

The modular of $L^{p(x)}(\Omega)$ which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ is defined by

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx,\ \forall u \in L^{p(x)}(\Omega).$$

Proposition 2.1 ([7, 10, 13]). *If* $u, u_n \in L^{p(x)}(\Omega)$, $n = 1, 2, \ldots$, *we have*
(*i*) $|u|_{p(x)} < 1\,(=1; >1) \Leftrightarrow \rho_{p(x)}(u) < 1\,(=1; >1)$;
(*ii*) $|u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+}$;
(*iii*) $|u|_{p(x)} < 1 \implies |u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-}$;
(*iv*) $\lim_{n\to\infty} |u_n|_{p(x)} = 0\,(\to\infty) \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n) = 0\,(\to\infty)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \ \forall u \in W^{1,p(x)}(\Omega).$$

We note that we can use the following equivalent norm on $W^{1,p(x)}(\Omega)$: The *modular* of $W^{1,p(x)}(\Omega)$ which is the mapping $\Lambda_{p(x)} : W^{1,p(x)}(\Omega) \to \mathbb{R}$ is defined by

$$\Lambda_{p(x)}(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)}\right) dx$$

for all $u \in W^{1,p(x)}(\Omega)$.

Proposition 2.2 ([7, 10, 13]). Let $u, u_n \in W^{1,p(x)}(\Omega)$:
(*i*) $\|u\|_{1,p(x)} < 1\,(=1; >1) \Leftrightarrow \Lambda_{p(x)}(u) < 1\,(=1; >1)$;
(*ii*) $\|u\|_{1,p(x)} > 1 \implies \|u\|_{1,p(x)}^{p^-} \leq \Lambda_{p(x)}(u) \leq \|u\|_{1,p(x)}^{p^+}$;
$\|u\|_{1,p(x)} < 1 \implies \|u\|_{1,p(x)}^{p^+} \leq \Lambda_{p(x),V}(u) \leq \|u\|_{1,p(x)}^{p^-}$;
(*iii*) $\lim_{n\to\infty} \|u_n\|_{1,p(x)} = 0\,(\to\infty) \Leftrightarrow \lim_{n\to\infty} \Lambda_{p(x)}(u_n) = 0\,(\to\infty)$.

Proposition 2.3 ([7, 10, 13]). *Let $m, h \in C_+\left(\overline{\Omega}\right)$ with $m(x) \leq h(x)$ for all $x \in \mathbb{R}^N$ and $u \in L^{h(x)}(\Omega)$. Then, $|u|^{m(x)} \in L^{\frac{h(x)}{m(x)}}(\Omega)$ and with*

$$\left||u|^{m(x)}\right|_{\frac{h(x)}{m(x)}} \leq |u|_{h(x)}^{m^-} + |u|_{h(x)}^{m^+}.$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ which is equipped with the norm for $u \in W_0^{1,p(x)}(\Omega)$

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}, \ \forall u \in W_0^{1,p(x)}(\Omega).$$

If $p^- > 1$, then the spaces $\left(|\cdot|_{p(x)}, L^{p(x)}(\Omega)\right)$ and $\left(\|\cdot\|_{p(x)}, W_0^{1,p(x)}(\Omega)\right)$ are separable and reflexive Banach spaces (see [10, 13]).

Proposition 2.4 ([7, 10, 13]). *Let $\Omega \subset \mathbb{R}^N$ be bounded and $p \in C(\overline{\Omega})$. If $q \in C(\overline{\Omega})$ and $1 \leq q(x) < p^*(x) = \frac{Np(x)}{N-p(x)}$ for any $x \in \overline{\Omega}$, then the embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is compact.

Proposition 2.5 (Hölder-type ineq.) ([4, 7, 10, 13]). *If $p \in L^\infty(\Omega)$, the conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'}\right) |u|_{p(x)} |v|_{p'(x)} \leq 2\,|u|_{p(x)} |v|_{p'(x)}.$$

The dual space $\left(W_0^{1,p(x)}(\Omega)\right)^*$ will be denoted by $W_0^{-1,p'(x)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$.

Proposition 2.6 ([9]). *The operator*

$$Lu = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} dx$$

satisfies the following propositions:

(i) $L : W_0^{1,p(x)}(\Omega) \to \left(W_0^{1,p(x)}(\Omega)\right)^*$ *is a continuos, bounded and strictly monotone operator;*

(ii) L *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *(weakly) in* $W_0^{1,p(x)}(\Omega)$, *and*

$$\overline{\lim_{n\to\infty}} (L(u_n) - L(u), u_n - u) \le 0,$$

then $u_n \to u$ *(strongly) in* $W_0^{1,p(x)}(\Omega)$.

We deal with the $p(x)$-Laplacian Dirichlet problem

$$\begin{cases} -\triangle_{p(x)} u = f(x,u), \text{ in } \Omega, \\ u(x) > 0, \text{ in } \Omega, \\ u(x) = 0, \text{ on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with $C^2(\Omega)$ boundary,

$$\triangle_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$$

which is so-called $p(x)$-Laplacian and $\nabla u = (\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u)$. Let $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is a Carathéodory function satisfies conditions:

(f_1) :

$$|f(x,t)| \le a_1 + b_1 |t|^{q(x)+1}, \forall (x,t) \in \Omega \times \mathbb{R}$$

provided that $a_1 + b_1 > 0$, $a_1, b_1 \ge 0$ are constants and $q \in C_+\left(\overline{\Omega}\right)$ with

$$q(x) + 2 < p^*(x) = \frac{3p(x)}{3 - p(x)}, 2 < p(x) < 3, \forall x \in \Omega.$$

(f_2) :

$$f_t'(x,t) \ge 0$$

or there is an $a \ge 0$ constant such that

$$|f_t'(x,t)| \le a$$

for all $x \in \Omega, t \in \mathbb{R}$.

For $u, v \in \Re(\Omega)$ we write $u \le v$ if $u(x) \le v(x)$ for a.e. $x \in \Omega$. Define $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{-u(x), 0\}$.

Definition 2.1. (i). Let $u^-, u^+ \in W_0^{1,p(x)}(\Omega) \cap C\left(\overline{\Omega}\right)$ satisfy $u^-, u^+ > 0$ in a.e. $x \in \Omega$. We say u^- and u^+ are a subsolution and a supersolution of (P) respectively, if

$$\int_\Omega \left|\nabla u^-\right|^{p(x)-2} \nabla u^- \nabla \upsilon dx \le \int_\Omega f(x, u^-) \upsilon dx, \tag{2.1}$$

and

$$\int_\Omega \left|\nabla u^+\right|^{p(x)-2} \nabla u^+ \nabla \upsilon dx \ge \int_\Omega f(x, u^+) \upsilon dx, \tag{2.2}$$

for all $\upsilon \in W_0^{1,p(x)}(\Omega)$ with $\upsilon \geq 0$ and supp$\upsilon \subset\subset \Omega$. We say u is a solution of (P), if it is both a subsolution and a supersolution of (P).

(ii). A function $u \in W_0^{1,p(x)}(\Omega) \cap C\left(\overline{\Omega}\right)$ is called a (weak) solution of (P) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \upsilon dx = \int_\Omega f(x,u)\upsilon dx,\ \forall \upsilon \in W_0^{1,p(x)}(\Omega).$$

Lemma 2.1. *Let $a, b \in \mathbb{R}^N$, then we have*

$$\begin{aligned} &\left(|b|^{p(x)-2} b - |a|^{p(x)-2} a, b-a\right) \\ =\ & \frac{1}{2}\left(|b|^{p(x)-2} + |a|^{p(x)-2}\right)|b-a|^2 + \frac{1}{2}\left(|b|^{p(x)-2} + |a|^{p(x)-2}\right)\left(|b|^2 - |a|^2\right). \end{aligned}$$

Proof. We have

$$\begin{aligned} I_1 &= \left(|b|^{p(x)-2} b - |a|^{p(x)-2} a, b-a\right) \\ &= |b|^{p(x)} + |a|^{p(x)} - \left(|b|^{p(x)-2} b + |a|^{p(x)-2} a\right).(a,b) \end{aligned}$$

on the other hand

$$\begin{aligned} &I_2 \\ =\ & \frac{1}{2}\left(|b|^{p(x)-2} + |a|^{p(x)-2}\right)|b-a|^2 + \frac{1}{2}\left(|b|^{p(x)-2} + |a|^{p(x)-2}\right)\left(|b|^2 - |a|^2\right) \\ =\ & \frac{1}{2}\left(|b|^{p(x)-2} + |a|^{p(x)-2}\right)\left(|b|^2 + |a|^2 - 2(a,b)\right) \\ & + \frac{1}{2}\left(|b|^{p(x)-2} + |a|^{p(x)-2}\right)\left(|b|^2 - |a|^2\right) \\ =\ & |b|^{p(x)} + |a|^{p(x)} - \left(|b|^{p(x)-2} b + |a|^{p(x)-2} a\right).(a,b). \end{aligned}$$

Thus $I_1 = I_2$. This completes the proof. ■

Lemma 2.2. *Let $p^- \geq 2$, then we have*

$$\begin{aligned} &\left(|b|^{p(x)-2} b - |a|^{p(x)-2} a, b-a\right) \\ \geq\ & \frac{1}{2}\left(|b|^{p(x)-2} + |a|^{p(x)-2}\right)|b-a|^2 \geq 2^{2-p^+}|b-a|^{p(x)}. \end{aligned}$$

Proof. Let $f(x,t) = t^{p(x)-2}$, $p(x) \geq 2$, $t \in \mathbb{R}_+$, $x \in \Omega$. Then by Lemma 2.1, it is easy to see that the f function is strictly increasing. Therefore

$$\left(|t_1|^{p(x)-2} t_1 - |t_2|^{p(x)-2} t_2\right)(t_1 - t_2) \geq 0 \text{ for all } t_1, t_2 \in \mathbb{R}_+. \tag{2.3}$$

Thus by using following elementary ineguality

$$(|a| + |b|)^s \leq 2^{s-1}\left(|a|^s + |b|^s\right)$$

we get

$$\begin{aligned} |b-a|^{p(x)} &= |b-a|^2 |b-a|^{p(x)-2} \\ &= |b-a|^2 \left(|b-a|^2\right)^{\frac{p(x)-2}{2}} \\ &= |b-a|^2 \left(|b|^2 - 2(a,b) + |a|^2\right)^{\frac{p(x)-2}{2}} \\ &\le |b-a|^2 2^{\frac{p(x)-2}{2}} \left(|b|^2 + |a|^2\right)^{\frac{p(x)-2}{2}} \\ &\le |b-a|^2 2^{\frac{p^+-2}{2}} 2^{\frac{p^+-2}{2}-1} \left(|b|^{p(x)-2} + |a|^{p(x)-2}\right) \\ &\le 2^{p^+-3} |b-a|^2 \left(|b|^{p(x)-2} + |a|^{p(x)-2}\right). \end{aligned}$$

Finally if we apply (2.3), we get

$$\left(|b|^{p(x)-2} b - |a|^{p(x)-2} a, b-a\right) \ge 2^{2-p^+} |b-a|^{p(x)}.$$

This completes the proof. ■

3. Main results

The basic principle of the sub-supersolution method for (P) is the following:

Theorem 3.1. *Let $p \in C_+\left(\overline{\Omega}\right)$ with $2 < p^- \le p(x) \le p^+ < 3$, $\forall x \in \Omega$. Assume that f satisfies (f_1) and (f_2) conditions. Let u^-, $u^+ \in W_0^{1,p(x)}(\Omega)$ are subsolution and supsolution respectively of (P). Let*

$$u^+ \ge 0, u^- \le 0 \text{ on } \partial\Omega, \tag{3.1}$$

and

$$u^- \le u^+ \text{ for a.e. } x \in \Omega. \tag{3.2}$$

If u is any solution of (P), then

$$u^- \le u \le u^+ \text{ for a.e. } x \in \Omega.$$

Proof. Proof consists of several steps.

Step 1. Let's choose a $\lambda > 0$ such that the mapping

$$z \to f(x,z) + \lambda z \tag{3.3}$$

is not decreasing for a.e. $x \in \Omega$. This mapping can be created by virtue of (f_2). Namely it is enough to choose $a \ge 0$ such that

$$a \ge \max\left\{-f'_u(x,u) : x \in \overline{\Omega},\ u \in [u^-(x), u^+(x)]\right\}. \tag{3.4}$$

Remark 3.1. In the special case, $f(x,t) = |t|^{q(x)-1}t$, $q(x) > 2$, $\forall (x,t) \in \Omega \times \mathbb{R}$ function provides the condition (3.4) with $|f'(t)| \le b$.

Now, suppose that $u_0 = u$ and show that the boundary value problem

$$-\triangle_{p(x)} u_{k+1} + \lambda u_{k+1} = f(x,u_k) + \lambda u_k \text{ in } \Omega, u_{k+1} = 0 \text{ on } \partial\Omega \tag{3.5}$$

has a unique weak soluation $u_{k+1} \in W_0^{1,p(x)}(\Omega)$ by using induction method through u_k $(k = 0, 1, 2, ...)$.

Step 2. (i). Firstly, let us show (notice) that

$$u^- = u_0 \le u_1 \le \cdots \le u_k \le u_{k+1} \cdots \text{ for a.e. } x \in \Omega. \tag{3.6}$$

For $k = 0$, (3.5) equation satisfies

$$\int_\Omega \left(|\nabla u_1|^{p(x)-2} \nabla u_1 \nabla \upsilon + \lambda u_1 \upsilon \right) dx = \int_\Omega \left(f(x, u_0) + \lambda u_0 \right) \upsilon dx \tag{3.7}$$

for all $\upsilon \in W_0^{1,p(x)}(\Omega)$. By subtracting the (3.7) from the (2.2) we obtain

$$\int_\Omega \left[\left(|\nabla u_0|^{p(x)-2} \nabla u_0 - |\nabla u_1|^{p(x)-2} \nabla u_1, \nabla \upsilon \right) + \lambda \left(u_0 - u_1, \upsilon \right) \right] dx \le 0,$$

where $u_0 = u^-$. Provided that $\upsilon \ge 0$, let υ be

$$\upsilon = (u_0 - u_1)^+ \in W_0^{1,p(x)}(\Omega).$$

We conclude that

$$\int_\Omega \left(|\nabla u_0|^{p(x)-2} \nabla u_0 - |\nabla u_1|^{p(x)-2} \nabla u_1, \nabla (u_0 - u_1)^+ \right) dx$$

$$+\lambda \int_\Omega \left(u_0 - u_1, (u_0 - u_1)^+ \right) dx \le 0.$$

On the other hand

$$\nabla (u_0 - u_1)^+ = \begin{cases} \nabla (u_0 - u_1) & \text{for a.e. on } \{u_0 \ge u_1\}, \\ 0 & \text{for a.e. on } \{u_1 \ge u_0\}. \end{cases}$$

If we use the Lemma 2.2 we obtain

$$\int\limits_{u_0 \ge u_1} \left(|\nabla u_0|^{p(x)-2} \nabla u_0 - |\nabla u_1|^{p(x)-2} \nabla u_1, \nabla u_0 - \nabla u_1 \right) dx$$

$$\ge \quad 2^{2-p^+} \int\limits_{u_0 \ge u_1} |\nabla u_0 - \nabla u_1|^{p(x)} dx.$$

Consequently,

$$\int\limits_{u_0 \ge u_1} \left[|\nabla u_0 - \nabla u_1|^{p(x)} + \lambda \left(u_0 - u_1 \right)^2 \right] dx \le 0,$$

and then we have

$$u_0(x) \le u_1(x) \text{ for a.e. } x \in \Omega.$$

(ii). Now by using induction method assume that

$$u_{k-1}(x) \le u_k(x) \text{ for a.e. } x \in \Omega. \tag{3.8}$$

From (3.5) we obtain

$$\int_\Omega \left(|\nabla u_{k+1}|^{p(x)-2} \nabla u_{k+1} \nabla \upsilon + \lambda u_{k+1} \upsilon \right) dx = \int_\Omega \left(f(x, u_k) + \lambda u_k \right) \upsilon dx, \tag{3.9}$$

and

$$\int_{\Omega}\left(|\nabla u_k|^{p(x)-2}\nabla u_k\nabla \upsilon+\lambda u_k\upsilon\right)dx=\int_{\Omega}\left(f(x,u_{k-1})+\lambda u_{k-1}\right)\upsilon dx$$

for all $\upsilon\in W_0^{1,p(x)}(\Omega)$. Provided that $\upsilon\geq 0$, it can be shown as

$$\upsilon\equiv(u_k-u_{k+1})^+$$

then we have

$$\begin{aligned}&2^{2-p^+}\int_{u_k\geq u_{k+1}}\left(|\nabla(u_k-u_{k+1})|^{p(x)}\lambda(u_k-u_{k+1})^2\right)dx\\ \leq\ &\int_{\Omega}[f(x,u_{k-1})+\lambda u_{k-1}-f(x,u_k)+\lambda u_k](u_k-u_{k+1})^+dx\leq 0.\end{aligned}$$

The last inequality is obtained from equations (3.3) and (3.8). Therefore we obtain $u_k\leq u_{k+1}$ for a.e. $x\in\Omega$, and so the result we want is proven.

Step 3. Now, we show that for a.e. $x\in\Omega$

$$u_k\leq u^+,k=0,1,2.... \tag{3.10}$$

We obtain that the (3.10) is satisfied from (3.1) and (3.2) when $k=0$. For any k, $u_k\leq u^+$ is provided a.e. on Ω. Let us subtract the (3.9) from (2.1) equation we say that

$$\upsilon\equiv\left(u_{k+1}-u^+\right)^+.$$

Then, by using (3.10) and (3.3), we get

$$\begin{aligned}&2^{2-p^+}\int_{u_{k+1}\geq u^+}\left(\left|\nabla\left(u_{k+1}-u^+\right)\right|^{p(x)}\lambda\left(u_{k+1}-u^+\right)^2\right)dx\\ \leq\ &\int_{\Omega}\left[f(x,u_k)+\lambda u_k-f(x,u^+)+\lambda u^+\right]\left(u_{k+1}-u^+\right)^+dx\leq 0.\end{aligned}$$

So we proved that $u_k\leq u^+$.

Step 4.

1. From (3.6) and (3.10), for a.e. $x\in\Omega$ we have

$$u^-\leq ...\leq u_k\leq u_{k+1}\leq ...\leq u^+. \tag{3.11}$$

Therefore

$$u\equiv\lim_{k\to\infty}u_k. \tag{3.12}$$

On the other hand, because of the following condition

$$u_k\to u\text{ strongly in }L^{q(x)+2}(\Omega). \tag{3.13}$$

Thus, (3.11) can be proven by Majorant Theorem. Indeed we obtain

$$\int_{\Omega}|(u_k(x)-u(x))|^{q(x)+2}dx\leq c(q^-,q^+)\int_{\Omega}|V(x)|^{q(x)+2}dx<+\infty,$$

where $V(x):=\max\{|u^-(x)|,|u^+(x)|\}\in C(\Omega)$. In addition to these by using (3.12), (3.13) we get the result we want.

2. Since the Carathéodory function f satisfies (f_1) and(f_2), the Nemytsckij operator N_f generated by f, $N_f(u)(x) = f(x, u(x))$ is well defined from $L^{q(x)+2}(\Omega)$ into $L^{\frac{q(x)+2}{q(x)+1}}(\Omega)$:

$$N_f(u) : L^{q(x)+2}(\Omega) \to L^{\frac{q(x)+2}{q(x)+1}}(\Omega)$$

continuous and bounded (see [10]). The restriction $q(x) + 2 < p^*(x)$ ensures that the imbedding $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)+2}(\Omega)$ is compact. Hence, the diagram

$$W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)+2}(\Omega) \overset{N_f}{\hookrightarrow} L^{\frac{q(x)+2}{q(x)+1}}(\Omega) \hookrightarrow W_0^{-1,p'(x)}(\Omega)$$

shows that N_f is a compact operator (continuous and maps bounded sets into relatively compact sets) from $W_0^{1,p(x)}(\Omega)$ into $W_0^{-1,p'(x)}(\Omega)$:

$$N_f : W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow W_0^{-1,p'(x)}(\Omega).$$

An element $u \in W_0^{1,p(x)}(\Omega) \cap C\left(\overline{\Omega}\right)$ is said to be solution of (P) problem. If

$$-\triangle_{p(x)} u = N_f(u)$$

in the sense of $W_0^{-1,p'(x)}(\Omega)$ i.e.

$$\left\langle -\triangle_{p(x)} u, \upsilon \right\rangle = \left\langle N_f(u), \upsilon \right\rangle$$

for all $\upsilon \in W_0^{1,p(x)}(\Omega)$, where $\langle ., . \rangle : W_0^{1,p(x)}(\Omega) \to W_0^{-1,p'(x)}(\Omega)$ or

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \upsilon dx = \int_\Omega f(x, u)\upsilon dx, \forall \upsilon \in W_0^{1,p(x)}(\Omega).$$

3. If we multiply both sides of (3.5) by $u_{k+1} \in W_0^{1,p(x)}(\Omega)$, we get

$$\left\langle -\triangle_{p(x)} u_{k+1} + \lambda u_{k+1}, u_{k+1} \right\rangle = \left\langle f(x, u_k) + \lambda u_k, u_{k+1} \right\rangle. \tag{3.14}$$

Then, by applying partial integration to both sides of (3.14) we get

$$\int_\Omega |\nabla u_{k+1}|^{p(x)} dx + \lambda \int_\Omega u_{k+1}^2 dx = \int_\Omega f(x, u_k) u_{k+1} dx + \lambda \int_\Omega u_k u_{k+1} dx, \tag{3.15}$$

for any $u_{k+1} \in W_0^{1,p(x)}(\Omega)$. Notice that

$$u^-(x) \le u_k(x) \le u^+(x) \text{ for a.e. } x \in \Omega,$$

and

$$u^-, u^+ \in W_0^{1,p(x)}(\Omega)$$

and since $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)+2}(\Omega)$ we get

$$|u|_{q(x)+2} \le C_1 \|u\|_{p(x)}, C_1 > 0.$$

Thus

$$|u_k(x)| \le V(x) := \max\left\{\left|u^-(x)\right|, \left|u^+(x)\right|\right\} \in L^{q(x)+2}(\Omega) \hookrightarrow L^2(\Omega) \tag{3.16}$$

and since $L^{q(x)+2}(\Omega) \hookrightarrow L^2(\Omega)$ then

$$|u|_2 \le C_2 |u|_{q(x)+2} \le C_2 C_1 \|u\|_{p(x)}, C_2 > 0 \tag{3.17}$$

and

$$\int_{\Omega} |V(x)|^{q(x)+2}\, dx < +\infty. \tag{3.18}$$

By using (f_1) condition, (3.16), (3.17), (3.18) inequality, Proposition 2.3 and Proposition 2.5, we get

$$\begin{aligned}
&\left|\int_{\Omega} f(x, u_k)\, u_{k+1} dx\right| \\
\leq\ & a_1 \int_{\Omega} |u_{k+1}|\, dx + b_1 \int_{\Omega} |u_k|^{q(x)+1} |u_{k+1}|\, dx \\
\leq\ & \frac{a_1}{2} \int_{\Omega} u_{k+1}^2 dx + \frac{a_1}{2} \int_{\Omega} dx + 2b_1 \left| |u_k|^{q(x)+1} \right|_{\frac{q(x)+2}{q(x)+1}} |u_{k+1}|_{q(x)+2} \\
\leq\ & \frac{a_1}{2} \int_{\Omega} u_{k+1}^2 dx + \frac{|\Omega|\, a_1}{2} + b_1 \max\left\{ |V|_{q(x)+2}^{q^-+1}, |V|_{q(x)+2}^{q^++1} \right\} |u_{k+1}|_{q(x)+2} \\
\leq\ & \frac{a_1}{2} \int_{\Omega} u_{k+1}^2 dx + \frac{|\Omega|\, a_1}{2} + C_3 \|u_{k+1}\|_{p(x)},
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\lambda \int_{\Omega} u_k u_{k+1} dx \\
\leq\ & \frac{\lambda}{2} \int_{\Omega} |u_k|^2 dx + \frac{\lambda}{2} \int_{\Omega} |u_{k+1}|^2 dx \\
\leq\ & C_4 |V|_{q(x)+2}^2 + \frac{\lambda}{2} \int_{\Omega} |u_{k+1}|^2 dx \leq C_5 + \frac{\lambda}{2} \int_{\Omega} |u_{k+1}|^2 dx.
\end{aligned} \tag{3.20}$$

By using (3.19), (3.20) and Proposition 2.1 for $\|u_{k+1}\|_{p(x)} > 1$ in (3.15), we have

$$\begin{aligned}
&\|u_{k+1}\|_{p(x)}^{p^-} + \lambda \int_{\Omega} u_{k+1}^2 dx \\
\leq\ & \int_{\Omega} |\nabla u_{k+1}|^{p(x)} dx + \lambda \int_{\Omega} u_{k+1}^2 dx = \int_{\Omega} f(x, u_k)\, u_{k+1} dx + \lambda \int_{\Omega} u_k u_{k+1} dx \\
\leq\ & \frac{a_1}{2} \int_{\Omega} |u_{k+1}|^2 dx + \frac{|\Omega|\, a_1}{2} + C_3 \|u_{k+1}\|_{p(x)} + \frac{\lambda}{2} \int_{\Omega} |u_{k+1}|^2 dx + C_5,
\end{aligned}$$

where $0 < \lambda < a_1$. Since $p^- > 2$ we get

$$\|u_{k+1}\|_{p(x)} \leq K. \tag{3.21}$$

Thus (3.21) inequality proves that the $\{u_k\}$ sequence is smooth bounded for all $k \in \mathbb{N}$ in $W_0^{1,p(x)}(\Omega)$ space. On the other hand, because of $W_0^{1,p(x)}(\Omega)$ space is reflective, there is a $\{u_{k_j}\} \subset \{u_k\}$ sequence such that $u_{k_j} \rightharpoonup u$ weak in $W_0^{1,p(x)}(\Omega)$. Since the operator $-\triangle_{p(x)}$ satisfies the (S_+) condition (see Proposition 2.6) then

$$u_{k_j} \to u$$

strongly in $W_0^{1,p(x)}(\Omega)$ as $k_j \to +\infty$ and

$$u_{k_j} \to u$$

strongly in $L^2(\Omega)$ as $k_j \to +\infty$, therefore

$$\triangle_{p(x)} u_{k_j} \to \triangle_{p(x)} u$$

strongly in $W_0^{-1,p'(x)}(\Omega)$ as $k_j \to +\infty$.

Step 5. Finally, let us show that u is a weak solution of (P) problem. For this let us assume $\upsilon \in W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)+2}(\Omega)$. Then, from (3.14) we get

$$\int_\Omega \left(\left|\nabla u_{k_j}\right|^{p(x)-2} \nabla u_{k_j} \nabla \upsilon + \lambda u_{k_j} \upsilon \right) dx = \int_\Omega \left(f(x, u_{k_j-1}) + \lambda u_{k_j-1} \right) \upsilon dx. \quad (3.22)$$

Let $k_j \to +\infty$, then

$$f(x, u_{k_j-1}) \to f(x,u) \text{ strongly in } L^{(q(x)+2)/(q(x)+1)}(\Omega),$$

$$u_{k_j-1} \to u \text{ strongly in } L^2(\Omega).$$

Thus, by using (3.22) equality, we obtain

$$\int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla \upsilon + \lambda u \upsilon \right) dx = \int_\Omega (f(x,u) + \lambda u) \upsilon dx.$$

Finally

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \upsilon dx = \int_\Omega f(x,u) \upsilon dx, \forall \upsilon \in W_0^{1,p(x)}(\Omega).$$

This completes the proof. ■

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