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PRIME IDEAL FACTORIZATION AND p-INTEGRAL BASIS OF QUINTIC NUMBER FIELDS DEFINED BY $X^5 + aX + b$

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Abstract. Based on Newton polygon techniques, for every prime integer p, a p-integral basis of $\mathbb{Z}_K$, and the factorization of the principal ideal $p\mathbb{Z}_K$ into prime ideals of $\mathbb{Z}_K$ are given, where K is a quintic number field defined by an irreducible trinomial $X^5 + aX + b \in \mathbb{Z}[X]$.

1. Introduction

Let K be a quintic number field defined by an irreducible trinomial $f(X) = X^5 + aX + b \in \mathbb{Z}[X]$, $\mathbb{Z}_K$ its ring of integers, α a complex root of $f(X)$, $\triangle$ the discriminant of $f(X)$ and $ind(f) = [\mathbb{Z}_K : \mathbb{Z}[\alpha]]$ the index of the abelian group $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_K$.

In [1], for every prime integer p, a p-triangular integral basis $(1, w_1, \cdots, w_4)$ of $\mathbb{Z}_K$ has been calculated. In a such paper, the used method was based on the coefficients of the minimal polynomial of w_i. In order to calculate an element w_i of a such basis, two problems are to be solved : For every $i := 2, 3, 4$, it is needed to determine the maximum power h_i such that $w_i = \frac{H_i(\alpha)}{p^{h_i}} \in \mathbb{Z}_K$, where $H_i(X) \in \mathbb{Z}[X]$ is a monic polynomial of degree i, and also to determine the coefficients of $H_i(X)$ such that $w_i = \frac{H_i(\alpha)}{p^{h_i}}$. These make the computation cost very expensive. In this paper, a new method based on Newton polygon techniques, which provides in a simple way the elements of a p-integral basis is proposed. This new method provides also the factorization of $p\mathbb{Z}_K$ into prime ideals of $\mathbb{Z}_K$: The form $p\mathbb{Z}_K = \prod_{i=1}^{r} \mathfrak{p}_i^{e_i}$ and for every prime factor $\mathfrak{p}_i$, an integral element β_i such that $\mathfrak{p}_i = p\mathbb{Z}_K + \beta_i\mathbb{Z}_K$ are given. a such element β_i is called a p-generator of $\mathfrak{p}_i$ and characterized by $v_{\mathfrak{p}_i}(\beta_i) = 1$ and for every $\mathfrak{p} \neq \mathfrak{p}_i$, $v_{\mathfrak{p}}(\beta_i) = 0$. If the ramification index $e_i = 1$, then the first condition could be replaced by $v_{\mathfrak{p}_i}(\beta_i) \geq 1$. Let p be a prime integer. It is well known that if p does not divide $ind(f)$, then thanks to a theorem of Kummer the factorization of $p\mathbb{Z}_K$ can be derived directly from the factorization of of $\bar{f}(X)$ modulo p; $p\mathbb{Z}_K = \prod_{i=1}^{r} \mathfrak{p}_i^{e_i}$, where every $\mathfrak{p}_i = (p, g_i(\alpha))^{e_i}$ and $\bar{f}(X) = \prod_{i=1}^{r} \bar{g}_i^{e_i}$ is the factorization of $\bar{f}(X)$ in $\mathbb{F}_p[X]$ (see, for example [2, page 257]). In this case, we said that the factorization of $p\mathbb{Z}_K$ is p-analogous

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to the factorization of $\bar{f}(X)$ modulo p. Dedekind gave a criterion to detect if p does not divide $ind(f)$ or not (see [3]). If p divides $ind(f)$, then the previous theorem of Kummer failed. In that case, using Newton polygon techniques, the index $v_p(ind(f))$ is calculated, a $p-$integral basis, the form of $p\mathbb{Z}_K = \prod_{i=1}^{r} \mathfrak{p}_i^{e_i}$, and for every prime factor $\mathfrak{p}_i$, a p-generator of $\mathfrak{p}_i$ are given.

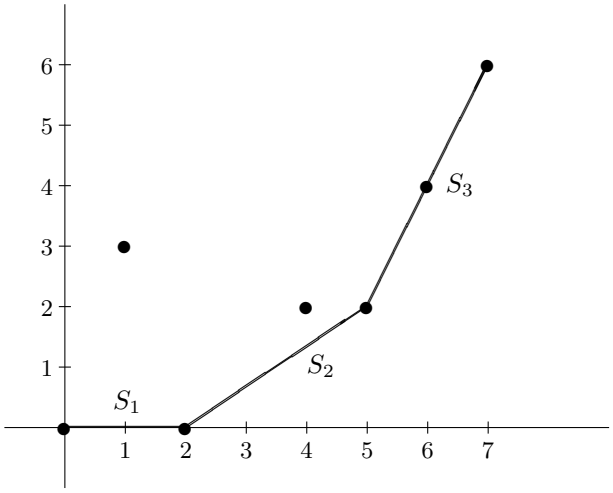
Throughout this paper, the following notations are used: For every prime p, v_p is the p-adic discrete valuation defined in $\mathbb{Q}_p$ by $v_p(p) = 1$, extended to $\mathbb{Q}_p[X]$ by $v_p(A(X)) := Min\{v_p(a_i)\,,\, 0 \leq i \leq r\}$, where $A(X) = \sum_{i=0}^{r} a_i X^i$. For every $(x, m) \in \mathbb{Z}^2$, denote $x_p = \frac{x}{p^{v_p(x)}}$, $x \pmod m$ the remainder of the Euclidean division of x by m and $(\frac{x}{p})$ the Legendre symbol. If p does not divide x and there exists $t \in \mathbb{Z}$ such that $t^n \equiv x \pmod p$, then let $(\frac{x}{p})_n = 1$. Otherwise, $(\frac{x}{p})_n \neq 1$.

2. Newton polygons

In this section, we briefly review the notions of Newton polygons.

Let p be a prime integer such that p^2 divides $\triangle$. Let $\phi(X) \in \mathbb{Z}[X]$ be a monic polynomial of degree m such that $\bar{\phi}(X)$ is an irreducible divisor of $\bar{f}(X)$ modulo p. Let $f(X) = a_0(X)\phi(X)^r + a_1(X)\phi(X)^{r-1} + \cdots + a_r(X)$ be the $\phi(X)$-adic development of $f(X)$ (every $a_i(X) \in \mathbb{Z}[X]$ and $\deg a_i(X) < m$). To any coefficient $a_i(X)$, we attach $u_i = v_p(a_i(X))$. If $u_i < \infty$, then let P_i be the point in the Cartesian plane defined by (i, u_i). The ϕ-Newton polygon of $f(X)$ is the lower convex envelope of the set of points $P_i = (i, u_i)$, $u_i < \infty$, in the Cartesian plane. This open polygon is denoted by $\mathrm{N}_\phi f$.

For instance, for a ϕ-development of degree 7 with $u_i = 0, 3, 0, \infty, 2, 2, 4, 6$ for $i = 0, 1, \ldots, 7$, the polygon is let N be the $\phi(X)$-Newton polygon of $f(X)$.



The *length* $\ell(\mathrm{N}_\phi f)$ and the *height* $h(\mathrm{N}_\phi f)$ of the polygon are the respective lengths of the projection to the horizontal and vertical axis. Clearly, $\deg f(X) = m\ell(\mathrm{N}_\phi f) + \deg a_0(X)$, where $m = deg\phi$.

The ϕ-Newton polygon is the union of different adjacent *sides* $S_1, \ldots, S_t$ with increasing slopes $\lambda_1 < \lambda_2 < \cdots < \lambda_t$. We shall write $\mathrm{N}_\phi f = S_1 + \cdots + S_t$. The points joining two different sides are called the *vertices* of the polygon. The polygon determined by the sides of positive slopes of $\mathrm{N}_\phi f$ is called the *principal*

ϕ-polygon of $f(X)$ and denoted by $\mathrm{N}_\phi^+ f$. The length and the height of $\mathrm{N}_\phi^+ f$ are respectively the lengths of its orthogonal projections to the horizontal and vertical axis. For instance, the polygon of the figure has three sides S_1, S_2, S_3 with slopes $0 < 2/3 < 2$ and $\mathrm{N}_\phi^+ f = S_2 + S_3$. For every side S of the principal part $\mathrm{N}_\phi^+ f$, the *length* $\ell(S)$ and the *height* $h(S)$, of S, are the respective lengths of the projection to the horizontal and vertical axis. The *slope* of S is the quotient $h(S)/\ell(S)$. The positive integer $d(S) := \gcd(h(S), \ell(S))$ is called the *degree* of S. Denote $d := d(S)$ the degree of S, $h := h(S)/d$ and $e := \ell(S)/d$. $\lambda := h/e$ is the slope of S and e is its ramification index. Let $l = \ell(\mathrm{N}_\phi f)$ be the length of $\mathrm{N}_\phi f$. For any $j := 1, \cdots, l$, let $H_j \in \mathbb{Q}$ be the ordinate of the point of $\mathrm{N}_\phi(f)$ of abscissa j, h_j its integral part and $t_j = \mathrm{red}\left(\frac{a_j(X)}{p^{h_j}}\right)$, where *red* is the canonical map defined on $\mathbb{Z}[X]$ by reduction modulo p. Note that If $P_j \notin S$, then $t_j = 0$ and if $P_j \in S$, then $t_j \neq 0$. If i is the abscissa of the initial point of S, let $f_S(Y)$ be the residual polynomial attached to S: $f_S(Y) := t_i Y^d + t_{i+e} Y^{d-1} + \cdots + t_{i+(d-1)e} Y + t_{i+de} \in \mathbb{F}_\phi[Y]$.

Lemma 2.1. *2 Let $P(X) \equiv \bar{\phi}^l(X) \pmod p \in \mathbb{Z}_p[X]$ such that $\mathrm{N}_\phi P = S$ is only one side and $P_S(Y)$ is irreducible in $\mathbb{F}_\phi[Y]$, where $\mathbb{F}_\phi = \frac{\mathbb{F}_p[X]}{(\phi(X))}$ and let $\lambda = h/e$ be the slope of S such that h and e are positive coprime integers. Define $\gamma = \phi(\alpha)^e/p^h$. Then $P(X)$ is irreducible over $\mathbb{Q}_p$, $v_p(\gamma) = 0$, and $P_S(\gamma) = 0$.*

Proof. First by Theorem of the residual polynomial, $P(X)$ is irreducible over $\mathbb{Q}_p$. Let $P(X) = \sum_{k=0}^{l} a_k(X)\phi(X)^k$ be the ϕ-adic development of $P(X)$. Since $\mathrm{N}_\phi P$ is one side of slope λ, we have for every i, $v_p(a_i(X)) \geq (l-i)\lambda$ and $v_p(a_0(X)) = l\lambda$. Thus, $v_p(\phi(\alpha)) = \lambda$. Moreover, since $\frac{P(\alpha)}{p^l} = 0$ and for every k, if $k \notin e\mathbb{N}$, then $\mathrm{red}\,\frac{a_k(X)}{p^{u_k}} = 0$, it follows that $\overline{(\frac{\phi(\alpha)^e}{p^h})^d + \frac{a_{l-e}(\alpha)}{p^h}\frac{\phi(\alpha)^e}{p^h})^{d-1} + \cdots + \frac{a_e(\alpha)}{p^{h(d-1)}}\frac{\phi(\alpha)^e}{p^h}) + \frac{a_{le}(\alpha)}{p^l}} = 0$ in $\mathbb{F}_\phi$. That means, $P_S(\gamma)) = 0$ in $\mathbb{F}_\phi$. □

Now, let $\bar{f}(X) = \bar{\phi}^l(X)\bar{H}(X)$ modulo p, where $\phi(X) \in \mathbb{Z}[X]$ whose reduction is a monic irreducible polynomial in $\mathbb{F}_p[X]$ and does not divide $\bar{H}(X)$. Let $\mathrm{N}_\phi^+ f = S_1 + \cdots + S_t$. $f(X)$ is said to be ϕ-regular if for every $i := 1, \cdots, t$, $f_{S_i}(Y)$ is square free in $\mathbb{F}_p[Y]$.

Let $N = \mathrm{N}_\phi^+ f$. Define $ind_N(f) = m\sum_{j=i_0+1}^{l-1} h_j$ the N-index of $f(X)$, where $i_0 = deg(H)$, l is the length of N, and $m = deg(\phi)$.
For every $i := i_0+1, \cdots, l-1$, let $q_i(X)$ be the quotient of the Euclidean division of $f(X)$ by $\phi^i(X)$. For every $i := i_0+1, \cdots, l-1$ and for every $j := 0, \cdots, m-1$, let $Q_{i,j}(\alpha) = \alpha^j \frac{q_i(\alpha)}{p^{h_i}}$, where h_i is the integral part of H_i and H_i is the ordinate of the point $P_i \in N$ of abscissa i. Then, we have the following results:

Proposition 2.2. *If $f(X)$ is $\phi(X)$-regular and $\bar{H}(X)$ is square free, then*

1. *$v_p(ind(f)) = m \times ind_N(f)$ and $\{Q_{i,j}(\alpha)\, i := i_0 + 1, \cdots, l-1, j := 0, \cdots, m-1\}$ is a p-integral basis of $\mathbb{Z}_K$.*
2. *For every $i := 1, \cdots, t$, let $f_{S_i}(Y) = \prod_{j=1}^{s_i} \psi_j(Y)$ be the factorization of $f_{S_i}(Y)$ in $\mathbb{F}_\phi[Y]$. Then $p\mathbb{Z}_K = I \times \prod_{i=1}^{t}\prod_{j=1}^{s_i} \mathfrak{p}_{ij}^{e_i}$ is the form of the factorization of $p\mathbb{Z}_K$ into prime ideals of $\mathbb{Z}_K$, where e_i is the ramification*

index of the side S_i, $v_{\mathfrak{p}_i}(\phi(\alpha)) = h_i$, $\lambda_i = \frac{h_i}{e_i}$ its slope and I is p-analogous to $\bar{H}(X)$.

Proof. (1) follows from [4, Pr 2.3, p:6, Th 2.6, p:8].
For (2), it is well known that the prime factors of $p\mathbb{Z}_K$ are in bijection with irreducible factors of $f(X)$ in $\mathbb{Q}_p[X]$, where $\mathbb{Q}_p$ is the p-adic local field [6]. Thus, it suffices to factorize $f(X)$ into irreducible polynomials of $\mathbb{Q}_p[X]$. By Hensel's Lemma, $f(X) = F_0(X) \times F(X)$ in $\mathbb{Q}_p[X]$, where $\bar{F}_0(X) = \bar{H}(X)$, $\bar{F}(X) = \bar{\phi}^r(X)$ in $\mathbb{F}_p[X]$ and $\mathrm{N}_\phi F = \mathrm{N}^+_\phi f$, up to a translation and a non-zero multiplicative constant. Let $\mathcal{P}$ be the set of prime ideals of $\mathbb{Z}_K$ above p. It follows that $\mathcal{P}$ splits as disjoint union $\mathcal{P}_0 \cup \mathcal{P}_1$, where every $\mathfrak{p} \in \mathcal{P}_0$ is attached to $F_0(X)$; $F_0(\alpha) \equiv 0 \pmod{\mathfrak{p}}$ and every $\mathfrak{p} \in \mathcal{P}_1$ is attached to $F(X)$. Since $\bar{F}_0(X)$ is square free, $\mathcal{P}_0 = \{\mathfrak{p}_{01}, \cdots, \mathfrak{p}_{0i_0}\}$, where $i_0 = deg(H)$, every $\mathfrak{p}_{0k} = (p, g_k(\alpha))$ and $\bar{H}(X) = \prod_{k=1}^{i} g_k(X)$. For $\mathcal{P}_1$, by the Theorem of the polygon $F(X) = F_1 \times \cdots \times F_t(X)$ in $\mathbb{Q}_p[X]$, where for every $i := 1, \cdots, t$, $\mathrm{N}_\phi F_i = S_i$ up to a translation and $F_i S_i(Y) = f_{S_i}(Y)$, up to a non-zero multiplicative constant. By Theorem of the residual polynomial, for every $i := 1, \cdots, t$, $F_i(X) = \prod_{j=1}^{s_i} F_{ij}(X)$ in $\mathbb{Q}_p[X]$, where for every $i := 1, \cdots, t$ and $j := 1, \cdots, s_i$, $\bar{F}_{ij}(X)$ is a power of $\bar{\phi}$, $\mathrm{N}_\phi F_{ij} = S_{ij}$ is only one side of slope λ_i such that $(F_{ij})_{S_{ij}}(Y)$ is irreducible. By Theorem of the product, if $f_S(Y)$ is irreducible in $\mathbb{F}_\phi[Y]$, then $f(X)$ is irreducible in $\mathbb{Q}_p[X]$. By Hensel's Theorem, $p\mathbb{Z}_K = I \prod_{i=1}^{t} \prod_{j=1}^{s_i} \mathfrak{p}_{ij}^{e_{ij}}$, where $I = \prod_{k=1}^{i_0} \mathfrak{p}_{0k}$ and for every i, j, e_{ij} is the ramification index of $\mathfrak{p}_{ij}$.
Now, assume that $F(X) \in \mathbb{Z}_p[X]$ is a monic irreducible polynomial in $\mathbb{Q}_p[X]$ such that $F(X) \equiv \phi^l(X) \pmod{p}$, $\mathrm{N}_\phi f = S$ is only one side and $F_S(Y)$ is irreducible in $\mathbb{F}_\phi[Y]$ of degree $d(S)$. Denote $\mathbb{K} = \mathbb{Q}_p[\alpha]$, $\mathbb{Z}_\mathbb{K}$ its ring of integers over $\mathbb{Z}_p$ and $\mathfrak{p}$ the maximal ideal of $\mathbb{Z}_\mathbb{K}$. Then $p\mathbb{Z}_\mathbb{K} = \mathfrak{p}^{e(\mathfrak{p})}$, where $e(\mathfrak{p}) = \frac{deg(F)}{f(\mathfrak{p})}$ and $f(\mathfrak{p})$ is the residue degree of $\mathfrak{p}$. Let $\lambda = \frac{e}{h}$ be the slope of S such that e and h are positive coprime. Let $\nu = v_p(a_l(X))$. As N_i is only one side of slope λ, $\nu = l\lambda$. Since $\overline{\phi(X)}$ is the minimal polynomial of $\overline{\alpha} \pmod{\mathfrak{p}}$ over $\mathbb{F}_p$ and $\overline{\phi(X)}$ does not divides $\overline{(\frac{a_l(X)}{p^\nu})}$, we have $\overline{(\frac{a_l(\alpha)}{p^\nu})} \neq \overline{0}$ modulo $\mathfrak{p}$. So $v_\mathfrak{p}(a_l(\alpha)) = e(\mathfrak{p})v_p(a_l(X)) = e(\mathfrak{p})l\lambda$. We now show that $v_\mathfrak{p}(\phi(\alpha)) = e(\mathfrak{p})\lambda$. Note that as N is only one side, $v_p(a_j(X)) \geq j\lambda$ for every $j = 1, \ldots, l-1$. So $v_\mathfrak{p}(a_j(\alpha)) \geq j\lambda e(\mathfrak{p})$ since α is integral over $\mathbb{Z}_p$. Thus, for every $j = 1, \ldots, l-1$, $v_\mathfrak{p}(a_j(\alpha))\phi(\alpha)^{l-j}) \geq j\lambda e(\mathfrak{p}) + (l-j)u$, where $u = v_\mathfrak{p}(\phi(\alpha))$. If $u \neq e(\mathfrak{p})\lambda$, then from the ϕ-adic expansion of $f(X)$ that $v_\mathfrak{p}(f(\alpha)) = \min\{lu, le(\mathfrak{p})\lambda)$, which is impossible since $v_\mathfrak{p}(f(\alpha)) = \infty$. Thus, $u = e(\mathfrak{p})\lambda$ and, therefore, $v_\mathfrak{p}(a_j(\alpha))\phi(\alpha)^{l-j}) \geq lu$. Let $d = gcd(l, v_p(a_l(X)))$, $e = \dfrac{l}{d}$, and $h = \dfrac{v_p(a_l(X))}{d}$. Let show that $e(\mathfrak{p}) = e$. As $v_\mathfrak{p}(\phi(\alpha) = e(\mathfrak{p})\lambda = \frac{e(\mathfrak{p})h}{e}$, e and h are coprime, we conclude that e divides $e(\mathfrak{p})$. Moreover, since $v_\mathfrak{p}(\frac{\phi(\alpha)^e)}{p^h}) = 0$, γ is integral over $\mathbb{Z}_p$. Since $\bar{\phi}(X)$ is the minimal polynomial of $\bar{\alpha} \pmod{\mathfrak{p}}$, we have the following field extensions $\mathbb{F}_p \hookrightarrow \mathbb{F}_\phi \hookrightarrow \mathbb{F}_\phi[\gamma] \hookrightarrow \mathbb{Z}_\mathbb{K}/\mathfrak{p}$. Consider the homomorphism of $\mathbb{F}_p$-algebras: $\iota \colon \mathbb{F}_\phi[Y] \longrightarrow \mathbb{Z}_\mathbb{K}/\mathfrak{p}$, $\quad \bar{a}(Y) \mapsto \overline{a(\gamma)}$. Its kernel is the maximal ideal of $\mathbb{F}_\phi[Y]$ generated by a monic irreducible polynomial $\psi(Y)$ such that $\psi(\gamma) = 0$. Let show that $\psi(Y) = f_S(Y)$ up t o a non zero multiplicative

constant. Since by assumption $f_S(Y)$ is irreducible in $\mathbb{F}_\phi[Y]$, it suffices to show that $f_S(\gamma) = 0$. But by Lemma , $f_S(\gamma) = 0$ and $\psi(Y) = f_\phi(Y)$ up to a non zero multiplicative constant in $\mathbb{F}_\phi[Y]$. Thus $[\mathbb{F}_\phi[\gamma] : \mathbb{F}_p]$ divides $f(\mathfrak{p}) = [\frac{\mathbb{Z}_\mathbb{K}}{\mathfrak{p}} : \mathbb{F}_p]$. As $[\mathbb{F}_\phi[\gamma] : \mathbb{F}_p] = [\mathbb{F}_\phi[\gamma] : \mathbb{F}_\phi][\mathbb{F}_\phi : \mathbb{F}_p] = \deg(f_S(Y)).\deg(\phi(X)) = d.m$, $d.m$ divides $f(\mathfrak{p})$. We have then $e.d.m = \deg(f(X)) = e(\mathfrak{p}).f(\mathfrak{p})$. As $d.m$ divides $f(\mathfrak{p})$ and e divides $e(\mathfrak{p})$, $e(\mathfrak{p}) = e$ and $f(\mathfrak{p}) = d.m$.
For p-generators, if $e_i = 1$, as $v_{\mathfrak{p}_i}(\phi_i(\alpha)) \geq 1$ and $\phi_i(\alpha) \in \mathbb{Z}_K$, then $\mathfrak{p}_i = (p, \phi_i(\alpha))$. If $h_i = 1$, then $v_{\mathfrak{p}_i}(\phi_i(\alpha)) = 1$ and $\mathfrak{p}_i = (p, \phi_i(\alpha))$ too. □

3. p-Integral bases and prime ideal factorization in quintic number fields defined by $X^5 + aX + b$

In this section, based on Newton polygon techniques, for every prime integer p, a p-integral basis of $\mathbb{Z}_K$, the factorization of $p\mathbb{Z}_K$ into prime ideals of $\mathbb{Z}_K$ and for every prime factor $\mathfrak{p}$ of $p\mathbb{Z}_K$, a p-generator integral element are given.

Case : $v_p(a) \geq 1$ and $v_p(b) \geq 1$

Let p be a prime integer such that $v_p(a) \geq 1$ and $v_p(b) \geq 1$. In $TableA1$, for every case, a p-integral basis of $\mathbb{Z}_K$ is given. In $TableB1$, the form of $p\mathbb{Z}_K$ as a product of prime ideals of $\mathbb{Z}_K$, and for every prime factor $\mathfrak{p}$ of $p\mathbb{Z}_K$ a p-generator integral element is given.

$TableA1$

case	$v_p(b)$	$v_p(a)$	p	p-integral basis
1	4	≥ 3		$(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p^2}, \frac{\alpha^4}{p^3})$
2	≥ 4	3		$(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p^2}, \frac{\alpha^4}{p^3})$
3	3	≥ 3		$(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p}, \frac{\alpha^4}{p^2})$
4	2	≥ 2		$(1, \alpha, \alpha^2, \frac{\alpha^3}{p}, \frac{\alpha^4}{p})$
5	≥ 3	2	$\neq 2$	$(1, \alpha, \frac{\alpha^2}{p}, \frac{\alpha^3}{p}, \frac{\alpha^4}{p^2})$
6	≥ 3	2	2	go to Table $A1.6$
7	≥ 2	1		$(1, \alpha, \alpha^2, \alpha^3, \frac{\alpha^4}{p})$
8	1			$(1, \alpha, \alpha^2, \alpha^3, \alpha^4)$

$TableB1$

case	$v_p(b)$	$v_p(a)$	p		$p\mathbb{Z}_K$	p-generators
1	4	≥ 4			$\mathfrak{p}^5$	$\frac{\alpha^4}{p^3}$
2	4	3			$\mathfrak{p}_1^4\mathfrak{p}_2$	$\beta_1 = \frac{\alpha^4+a}{p^3}$, $\beta_2 = \frac{\alpha^4}{p^3}$
3	≥ 5	3			$\mathfrak{p}_1^4\mathfrak{p}_2$	$\beta_1 = \frac{\alpha^3}{p^2} + \frac{\alpha^4+a}{p^3}$, $\beta_2 = \frac{\alpha^4}{p^3}$
4	3	≥ 3			$\mathfrak{p}^5$	$\frac{\alpha^2}{p}$
5	≥ 3	2	2			go to Table $B1.5$
6	≥ 4	2	$\neq 2$	$(\frac{-a_p}{p}) = -1$	$\mathfrak{p}_1^2\mathfrak{p}_2$	$\beta_1 = \frac{\alpha^3}{p} + \frac{\alpha^4+a}{p^2}$, $\beta_2 = \frac{\alpha^4}{p^2}$
7	3	2	$\neq 2$	$(\frac{-a_p}{p}) = -1$	$\mathfrak{p}_1^4\mathfrak{p}_2$	$\beta_1 = \frac{\alpha^4+a}{p^2}$, $\beta_2 = \frac{\alpha^4}{p^2}$
8	≥ 3	2	$\neq 2$	$(\frac{-a_p}{p}) = -1$	$\mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}_3$	$\beta_1 = \alpha + \frac{\alpha(\alpha+pu)}{p}$, $\beta_2 = \alpha + \frac{\alpha(\alpha-pu)}{p}$ $\beta_3 = \frac{\alpha^4}{p^2}$, $(u^2 + a_p \equiv 0 \pmod{p^2})$
9	2	≥ 2			$\mathfrak{p}^5$	$\frac{\alpha^3}{p}$
10	2	1			$\mathfrak{p}_1^4\mathfrak{p}_2$	$\beta_1 = \frac{\alpha^4+a}{p}$, $\beta_2 = \frac{\alpha^4}{p}$
11	≥ 3	1			$\mathfrak{p}_1^4\mathfrak{p}_2$	$\beta_1 = \alpha + \frac{\alpha^4+a}{p}$, $\beta_2 = \frac{\alpha^4}{p}$
12	1				$\mathfrak{p}^5$	α

Proof. All cases except $p = 2$, $v_2(b) \geq 3$ and $v_2(a) = 2$, correspond to a situation where $f(X)$ is X-regular. The case : $p = 2$, $v_2(a) = 2$ and $v_2(b) \geq 3$ is handled in $TableA1.6$ and $TableB1.5$ by using techniques of Newton polygons of second order.
If $p \neq 2$, $v_p(b) \geq 3$ and $v_p(a) = 2$, then $\mathrm{N}_X(f) = S_1+S_2$ such that $f_{S_1}(Y) = Y^2+a_p$ and $f_{S_2}(Y) = Y + b_p$. Thus, if $(\frac{-a_p}{p}) = -1$, then $p\mathbb{Z}_K = \mathfrak{p}_1^2\mathfrak{p}_2$. If $(\frac{-a_p}{p}) = 1$, then $p\mathbb{Z}_K = \mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}_3$, where $v_{\mathfrak{p}_1}(\alpha) = v_{\mathfrak{p}_2}(\alpha) = 1$ and $v_{\mathfrak{p}_3}(\alpha) = v_p(b) - 2$. Let $u \in \mathbb{Z}$ such that $u^2 + a_p \equiv 0 \pmod{p^2}$. Then $v_{\mathfrak{p}_1}(\alpha + pu) \geq 2$, $v_{\mathfrak{p}_2}(\alpha - pu) \geq 2$, and $v_{\mathfrak{p}_3}(\alpha \mp pu) = 1$. Thus, $\mathfrak{p}_1 = (p, \alpha + \frac{\alpha(\alpha+pu)}{p})$, $\mathfrak{p}_2 = (p, \frac{\alpha(\alpha-pu)}{p})$, and $\mathfrak{p}_3 = (p, \frac{\alpha^4}{p^2})$ (because for $i = 1, 2$, $v_{\mathfrak{p}_i}(\alpha) = 1$ and $v_{\mathfrak{p}_i}(\frac{\alpha^4}{p^2}) = 0$).

□

Newton polygons of second order

Let $f(X) = X^5 + aX + b \in \mathbb{Z}[x]$ be an irreducible polynomial such that $v_2(a) = 2$ and $v_2(b) \geq 3$. Let $\phi(X) = X + 2u$, where $u \in \mathbb{Z}$. Let $f(X) = \phi^5 - 10u\phi^4 + 40u^2\phi^3 - 80u^3\phi^2 + (a+80u^4)\phi + (b - 2ua - 32u^5)$. Then $\mathrm{N}_\phi f = S_1+S_2$ is two sides of slopes $1/2$ and $v_2(b) - 2$ such that $F_{S_1}(Y) = Y^2 + 1 = (Y + 1)^2$ in $\mathbb{F}_2[Y]$ and $F_{S_2}(Y) = Y + 1$. It follows that for every $\phi(X) = X + 2u \in \mathbb{Z}[X]$, $f(X)$ is not $\phi(X)$-regular. Thus, we shall pass to use the second order Newton polygon. Let $\phi_2(X) = X^2 + 2uX + 2v \in \mathbb{Z}[X]$, where $v_2(v) = 0$. Let $v_2^{(2)}$ be the 2-*adic valuation of second order* as defined in [5, Sec.2.2]. Recall the following properties of $v_2^{(2)}$:

(1) $v_2^{(2)}(m) = 2v_2(m)$, for all $m \in \mathbb{Z}$,
(2) $v_2^{(2)}(X) = 1$,
(3) Extended to $\mathbb{Q}_p(X)$, by $v_2^{(2)}(\sum_{i=0}^n a_i X^{n-i} n) = \min\{2v_2(a_i) + (n - i), i = 0, \cdots, n\}$ and so, $v_2^{(2)}(\phi(X)) = 2$.

Now, let $f(X) = A_0(X)\phi_2^2 + A_1(X)\phi_2(X) + A_2(X)$ be the ϕ_2-adic development of $f(x)$. For every $i := 0, 1, 2$, let $r_i = v_2^{(2)}(A_i(X)\phi_2(X)^{2-i})$. Then the $\phi_2(X)$-*Newton polygon of the second order*, $\mathrm{N}_2(f)$, is the lower convex envelope of the set of points (i, r_i), $i = 0, 1, 2$, of the Euclidean plane. The *second order index* is $ind_2(f)$ defined by $ind_2(f) = \lfloor r-5 \rfloor$, where r is the ordinate of the point on $\mathrm{N}_2(f)$ of abscissa 1. If $f(X)$ is $\phi_2(X)$-regular; all second order residual polynomials are square free, then from [5, Thm.4.18, p: 47], we have $v_2(ind(f)) = ind_2(f) + ind_1(f) = \lfloor r-5 \rfloor + 4 = \lfloor r-1 \rfloor$.

Theorem 3.1. *With the above notations, let $f(X) = A_0(X)\phi_2^2 + A_1(X)\phi_2(X) + A_2(X)$ be the ϕ_2-adic development of $f(X)$ and let $Q(X) = A_0(X)\phi_2(X)$. Assume that $f(X)$ is ϕ_2-regular in second order. Then*

1. *If $r_2 = 2r_1 - 5$, then $2\mathbb{Z}_K = \mathfrak{p}_1^2\mathfrak{p}$, $v_{\mathfrak{p}_1}(\alpha)) = 1$ and $v_{\mathfrak{p}_1}(\phi_2(\alpha)) = r_1 - 3$.*
2. *If $2r_1 > r_2 + 5$ and $r_2 = 2k$, $2\mathbb{Z}_K = \mathfrak{p}_1^4\mathfrak{p}$, $\mathfrak{p} = (2, 2 + \frac{\alpha^2}{2})$, $v_{\mathfrak{p}_1}(\alpha)) = 2$ and $v_{\mathfrak{p}_1}(\phi_2(\alpha)) = r_2 - 1$.*
3. *If $2r_1 < r_2 + 5$, then $2\mathbb{Z}_K = \mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}$, where for every $i := 1, 2$, $v_{\mathfrak{p}_i}(\alpha)) = 1$, $v_{\mathfrak{p}_1}(\phi_2(\alpha)) = r_1 - 3$ and $v_{\mathfrak{p}_2}(\phi_2(\alpha)) = r_2 - r_1 + 2$.*
4. *$(1, \alpha, \frac{\alpha^2}{2}, \frac{Q(\alpha)}{2^{\lfloor \nu \rfloor}}, \frac{\alpha Q(\alpha)}{2^{\lfloor \nu + (1/2) \rfloor}})$ is a 2-integral basis of $\mathbb{Z}_K$, where $\nu = \frac{r}{2} - 1$.*

Proof. Note that in all cases, $v_{\mathfrak{p}}(\alpha) = v_2(b) - 2$. Moreover, since $\mathrm{N}_X(f) = S_1 + S_2$ and $f_{S_2}(Y) = Y + 1$, by Theorem of the polygon $2\mathbb{Z}_K = I\mathfrak{p}$, where $\mathfrak{p}$ is the prime ideal attached to the side S_2, $v_{\mathfrak{p}}(\alpha) = v_2(b) - 2$ and for every prime factor $\mathfrak{p}_i$ of I, $v_{\mathfrak{p}_i}(\alpha) = \frac{e(\mathfrak{p}_i)}{2}$. It follows that:

1. If $r_2 = 2r_1 - 5$, then $\mathrm{N}_2(f) = S$ is one side of slope $r_1 - 5$. So, $2\mathbb{Z}_K = \mathfrak{p}_1^2\mathfrak{p}$, where $v_{\mathfrak{p}_1}(\phi_2(\alpha)) = r_1 - 3$.
2. If $2r_1 > r_2 + 5$ and $r_2 = 2k$, then $\mathrm{N}_2(f) = S$ is one side of slope $\frac{r_2 - 5}{2}$. Thus $2\mathbb{Z}_K = \mathfrak{p}_1^4\mathfrak{p}$, $v_{\mathfrak{p}_1}(\alpha)) = 2$ and $v_{\mathfrak{p}_1}(\phi_2(\alpha)) = r_2 - 1$.
3. If $2r_1 < r_2 + 5$, then $\mathrm{N}_2(f) = S_1^2 + S_2^2$ is two sides. It follows that : $p\mathbb{Z}_K = \mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}$, where for every $i := 1, 2$, $v_{\mathfrak{p}_i}(\alpha)) = 1$, $v_{\mathfrak{p}_1}(\phi_2(\alpha)) = r_1 - 3$, and $v_{\mathfrak{p}_2}(\phi_2(\alpha)) = r_2 - r_1 + 2$.
4. Since $v_p(ind(f)) = \lfloor r-1 \rfloor$ and the set $(1, \alpha, \frac{\alpha^2}{2}, \frac{Q(\alpha)}{2^{\lfloor \nu \rfloor}}, \frac{\alpha Q(\alpha)}{2^{\lfloor \nu + (1/2) \rfloor}})$ is free over $\mathbb{Z}$, it suffices to check that all these elements are integral. So we have only to check if for every prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ lying above p, $v_{\mathfrak{p}}(Q(\alpha)) \geq e(\mathfrak{p})\nu$ and $v_{\mathfrak{p}}(\alpha Q(\alpha)) \geq e(\mathfrak{p})(\nu + \frac{1}{2})$, where $e(\mathfrak{p})$ is the ramification index of $\mathfrak{p}$. But as $\mathrm{N}_2(f)$ is convex, $v_2^{(2)}(Q(X)) \geq 2\nu$ and $v_2^{(2)}(XQ(X)) \geq 2\nu + 1$. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying above p.
   (a) If $e(\mathfrak{p}) = 1$, then $v_{\mathfrak{p}}(\alpha) = v_2(b) - 2$ and $v_{\mathfrak{p}}(\phi_2(\alpha)) \geq 1$. Since $A_1(X) = (12u^2 - 4v)X - 8u^3 + 16uv$, if $u \equiv 0 \pmod 2$, then $r_1 = 7$. If $u \equiv 1 \pmod 2$, then $r_1 = 8$. Thus, $\nu \leq 3$. As $A_2(X) = UX + V = (a + 4v^2 - 24vu^2 + 16u^4)X + b + 16vu^3 - 16uv^2$, if $v_2(b) = 3$, then $\nu = 7/4$. If $v_2(b) = 4$, then $\nu \in \{2, 9/4\}$. If $v_2(b) \geq 5$, then $v_{\mathfrak{p}}(\alpha\phi_2(\alpha)) \geq 4$ and $\nu \leq 3$.
   (b) If $e(\mathfrak{p}) = 2$, then $v_{\mathfrak{p}}(Q(\alpha)) \geq e(\mathfrak{p})\nu$ and $v_{\mathfrak{p}}(\alpha Q(\alpha)) \geq e(\mathfrak{p})\nu + 1$.
   (c) If $e(\mathfrak{p}) = 4$, then $v_{\mathfrak{p}}(Q(\alpha)) \geq e(\mathfrak{p})\nu$ and $v_{\mathfrak{p}}(\alpha Q(\alpha)) \geq e(\mathfrak{p})\nu + 2$.

□

$TableA1.6:\ v_2(a)=2,\ v_2(b)\geq 3$

conditions	2-integral basis of $\mathbb{Z}_K$
$v_2(b)=3$	$(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3}{2},\frac{\alpha^4}{4})$
$v_2(b)=4,\ v_2(a-4)=3$	$(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3+2\alpha}{4},\frac{\alpha^4}{4})$
$v_2(b)\geq 5,\ v_2(a-4)=3$	$(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3+2\alpha}{4},\frac{\alpha^4+2\alpha^2}{8})$
$v_2(b)=4,\ v_2(a-4)\geq 4$	$(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3+2\alpha^2+2\alpha}{4},\frac{\alpha^4}{4})$
$v_2(b)=5,\ v_2(a-4)=4$	$(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3+2\alpha^2-2\alpha}{8},\frac{\alpha^4+2\alpha^3-2\alpha^2}{8})$
$v_2(b)\geq 6,\ v_2(a-4)=4$	$(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3+2\alpha^2-2\alpha}{4},\frac{\alpha^4+2\alpha^3-2\alpha^2}{8})$
$v_2(b)=5,\ v_2(a-4)\geq 5$	$(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3+2\alpha^2+2\alpha}{4},\frac{\alpha^4+2\alpha^3+2\alpha^2}{8})$
$v_2(b)\geq 6,\ v_2(a-4)\geq 5$	$(1,\alpha,\frac{\alpha^2}{2},\frac{\alpha^3+2\alpha^2+2\alpha}{8},\frac{\alpha^4+2\alpha^3+2\alpha^2}{8})$

$TableB1.5:\ v_2(a)=2,\ v_2(b)\geq 3$

conditions	$2\mathbb{Z}_K$	2-generators
$v_2(b)=3$	$\mathfrak{p}_1^4\mathfrak{p}$	$\beta_1=\frac{\alpha^2+2}{2},\ \beta=\frac{\alpha^2}{2}$
$v_2(b)=4,\ v_2(a-4)\geq 4$	$\mathfrak{p}_1^4\mathfrak{p}$	$\beta_1=\frac{\alpha^4+a}{4}+\frac{\alpha^3+2\alpha^2+2\alpha}{4},\ \beta=\frac{\alpha^2}{2}$
$v_2(b)=4,\ v_2(a-4)=3$	$\mathfrak{p}_1^4\mathfrak{p}$	$\beta_1=\frac{\alpha^4+a}{4}+\frac{\alpha^+2\alpha}{4},\ \beta=\frac{\alpha^2}{2}$
$v_2(b)\geq 5,\ v_2(a-4)=3$ $v_2(a+4)=4$	$\mathfrak{p}_1^2\mathfrak{p}$	$\beta_1=\alpha+\frac{\alpha^4+a}{4},\ \beta=\frac{\alpha^2}{2}$
$v_2(b)\geq 6,\ v_2(a-4)=4$	$\mathfrak{p}_1^4\mathfrak{p}$	$\beta_1=\frac{\alpha^4+2\alpha^3-2\alpha^2}{8}+\frac{\alpha^4+a}{4},\ \beta=\frac{\alpha^2}{2}$
$v_2(b)=5,\ v_2(a-4)=4$ $v_2(a+44)=5$	$\mathfrak{p}_1^2\mathfrak{p}$	$\beta_1=\alpha+\frac{\alpha^4+a}{4},\ \beta=\frac{\alpha^2}{2}$
$v_2(b)=5,\ v_2(a-4)=4$ $v_2(a+44)\geq 6$	$\mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}$	$\beta_1=\alpha+\frac{\alpha^2+2}{4}+\frac{\alpha^4+a}{4},\ \beta=\frac{\alpha^2}{2}$ $\beta_2=\alpha+\frac{\alpha^3+2\alpha^2-2\alpha}{8}+\frac{\alpha^4+a}{4}$
$v_2(b)\geq 5,\ v_2(a-4)=3$ $v_2(a+4)\geq 5$	$\mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}$	$\beta_1=\alpha+\frac{\alpha^2+2\alpha+2}{2}+\frac{\alpha^4+a}{4},\ \beta=\frac{\alpha^2}{2}$ $\beta_2=\alpha+\frac{\alpha^2+2}{2}+\frac{\alpha^4+a}{4}$
$v_2(b)=5,\ v_2(a-4)\geq 5$	$\mathfrak{p}_1^4\mathfrak{p}$	$\beta_1=\frac{\alpha^4+2\alpha^3+2\alpha^2}{8},\ \beta=\frac{\alpha^2}{2}$
$v_2(b)\geq 6,\ v_2(a-4)=5$	$\mathfrak{p}_1^2\mathfrak{p}$	$\beta_1=\frac{\alpha^4+a}{4}+\alpha,\ \beta=\frac{\alpha^2}{2}$
$v_2(b)\geq 6,\ v_2(a-4)\geq 6$	$\mathfrak{p}_1^2\mathfrak{p}_2^2\mathfrak{p}$	$\beta_1=\alpha+\frac{(\alpha^2+2)}{2}+\frac{\alpha^4+a}{4},\ \beta_2=\frac{(\alpha-12)(\alpha^2+6\alpha+2)}{8},\ \beta=\frac{\alpha^2}{2}$ $\beta_1=\alpha+\frac{(\alpha^2+2\alpha+2)}{2}+\frac{\alpha^4+a}{4}$

Proof. In the following table, for every case, ϕ_2 such that $f(X)$ is ϕ_2-regular, $A_0(X)$, $A_1(X)$, and $A_2(X)$ for every $i=0,1,2$ $A_i(X)$ are given

conditions	ϕ_2	A_0	A_1	A_2
$v_2(b)=3$	X^2+2	X	$-4X$	$(a+4)X+b$
$v_2(b)\geq 4,\ v_2(a-4)=3$	X^2+2	X	$-4X$	$(a+4)X+b$
$v_2(b)=4,\ v_2(a-4)\geq 4$	X^2+2X+2	$X-4$	$8+8X$	$(a-4)X+b$
$v_2(b)\geq 5,\ v_2(a-4)=4$	X^2+2X-2	$X-4$	$-24+16X$	$(a+44)X+(b-32)$
$v_2(b)\geq 5,\ v_2(a-4)\geq 4$	X^2+2X+2	$X-4$	$8+8X$	$(a-4)X+b$

□

Case :$v_p(b) = 0$ and $v_p(a) \geq 1$

If $p \neq 5$, then $v_p(\triangle) = 0$ and so, $v_p(ind(f) = 0$; $(1, \alpha, \alpha^2, \alpha^3, \alpha^4)$ is a p-integral basis of $\mathbb{Z}_K$, and $p\mathbb{Z}_K$ is p-analogeous to $\bar{f}(X)$. For $p = 5$, go to $TableA2$ and $TableB2$. Let $\theta = \alpha - b$, $A = 5b^4 + a$ and $B = -b(b^4 - 1 + a)$.

$TableA2$

conditions	5-integral basis
$v_5(B) = 1$	$(1, \theta, \theta^2, \theta^3, \theta^4)$
$v_5(B) \geq 2$, $v_5(A) = 1$	$(1, \theta, \theta^2, \theta^3, \frac{\theta^4}{5})$
$v_5(B) = 2$, $v_5(A) \geq 2$	$(1, \theta, \theta^2, \frac{\theta^3}{5}, \frac{\theta^4}{5})$
$v_5(B) \geq 3$, $v_5(A) = 2$	$(1, \theta, \frac{\theta^2}{5}, \frac{\theta^3}{5}, \frac{\theta^4 - 5b\theta^3 + 10b^2\theta^2 - 10b^3\theta}{5^2})$
$v_5(B) \geq 3$, $v_5(A) \geq 3$	$(1, \theta, \theta^2, \frac{\theta^3}{5}, \frac{\theta^4 + 5s\theta^3 + 10s^2\theta^2 + 10s^3\theta}{5^r})$ $r = \lfloor \frac{v_5(\triangle}{2} \rfloor - 2$ $4a_5 s \equiv -4b \pmod{5^{v_5(\triangle)+1}}$

$TableB2$

conditions	$5\mathbb{Z}_K$	5-generators
$v_5(B) = 1$	$\mathfrak{p}^5$	θ
$v_5(B) \geq 2$, $v_5(A) = 1$	$\mathfrak{p}_1^4\mathfrak{p}_2$	$\beta_1 = \theta + \frac{\theta^4 + A}{5}$, $\beta_2 = \frac{\theta^4}{5}$
$v_5(B) = 2$,$v_5(A) \geq 2$	$\mathfrak{p}_1^3\mathfrak{p}_2^2$	$\beta_1 = \frac{\theta^3 - 5b\theta^2 + 10b^2\theta}{5}$, $\beta_2 = \frac{\theta^3}{5}$
$v_5(B) \geq 3$, $v_5(A) \geq 3$	go to $TableB2.4$	

$TableB2.4$

Let $r = \lfloor \frac{v_5(\triangle}{2} \rfloor - 2$, $s \in \mathbb{Z}$ such that $4a_5 s \equiv -4b \pmod{5^{v_5(\triangle)+1}}$, $\theta = \alpha - s$, $A = 5s^4 + a$ and $B = s^5 + as + b$.

conditions	$5\mathbb{Z}_K$	5-generators
$v_5(\triangle) = 8$, $(\frac{-B_5}{5}) = -1$	$\mathfrak{p}_1^3\mathfrak{p}_2$	$\beta_1 = \frac{\theta^3 + 5s\theta^2 + 10s^2\theta + 10s^3}{5}$, $\beta_2 = \frac{\theta^3}{5}$
$v_5(\triangle) = 8$, $(\frac{-B_5}{5}) = 1$ $k \geq 5$	$\mathfrak{p}_1^3\mathfrak{p}_2\mathfrak{p}_3$	$\beta_1 = \theta + \frac{\theta^3 + 5s\theta^2 + 10s^2\theta + 10s^3}{5}$, $2s^3u^2 + B_5 \equiv 25 \pmod{5^3}$ $\beta_2 = \frac{\theta^3}{5} + \frac{(\theta^3 + 5s\theta^2 + 10s^2\theta + 10s^3)(\theta + 5u)}{5^2}$ $\beta_3 = \frac{\theta^3}{5} + \frac{(\theta^3 + 5s\theta^2 + 10s^2\theta + 10s^3)(\theta - 5u)}{5^2}$
$v_5(\triangle) = 2k$, $(\frac{-B_5}{5}) = -1$ $k \geq 5$	$\mathfrak{p}_1^3\mathfrak{p}_2$	$\beta_1 = \theta + \frac{\theta^3 + 10s^3}{5}$, $\beta_2 = \frac{\theta^3}{5}$
$v_5(\triangle) = 2k$, $(\frac{-B_5}{5}) = 1$ $k \geq 5$	$\mathfrak{p}_1^3\mathfrak{p}_2\mathfrak{p}_3$	$\beta_1 = \theta + \frac{\theta^3 + 10s^3}{5}$, $2s^3u^2 + B_5 \equiv 25 \pmod{5^3}$ $\beta_2 = \frac{\theta^3}{5} + \frac{(\theta^3 + 5s\theta^2 + 10s^2\theta + 10s^3)(\theta + 5^{k-3}u)}{5^{k-2}}$ $\beta_3 = \frac{\theta^3}{5} + \frac{(\theta^3 + 5s\theta^2 + 10s^2\theta + 10s^3)(\theta - 5^{k-3}u)}{5^{k-2}}$
$v_5(\triangle) = 2k + 1$ $k \geq 5$	$\mathfrak{p}_1^3\mathfrak{p}_2^2$	$\beta_1 = \theta + \frac{\theta^3 + 5s\theta^2 + 10s^2\theta + 10s^3}{5}$ $\beta_2 = \frac{\theta^3}{5} + \frac{\theta^4 + 5s\theta^3 + 10s^2\theta^2 + 10s^3\theta + A}{5^{k-2}}$

Proof. For $p = 5$, $v_5(b) = 0$ and $v_5(a) \geq 1$, $\bar{f}(X) \equiv (X + b)^5 \pmod 5$. Let $F(X) = f(X - b) = X^5 - 5bX^4 + 10b^2X^3 - 10b^3X^2 + AX + B$ and $\theta = \alpha + b$, where $A = 5b^4 + a$ and $B = -b(b^4 - 1 + a)$. It follows that:

(1) If $v_5(B) = 1$, then $v_5(ind(f)) = 0$ and $5\mathbb{Z}_K = (5, \theta)^5$.

(2) If $v_5(A) = 1$ and $v_5(B) \geq 2$. Thus, $F(X)$ is X-regular, $(1, \theta, \theta^2, \theta^3, \frac{\theta^4}{5})$ is a 5-integral basis of $\mathbb{Z}_K$ and $5\mathbb{Z}_K = \mathfrak{p}_1^4\mathfrak{p}$.

(3) If $v_5(A) \geq 2$ and $v_5(B) = 2$. Thus, $F(X)$ is X-regular, $(1, \theta, \theta^2, \frac{\theta^3}{5}, \frac{\theta^4}{5})$ is a 5-integral basis of $\mathbb{Z}_K$, and since $\mathrm{N}_X(F)$ is two sides with respect degrees 3 and 2, $5\mathbb{Z}_K = \mathfrak{p}_1^3\mathfrak{p}_2^2$.

(4) If $v_5(B) \geq 3$ and $v_5(A) \geq 3$, then $v_5(\triangle) \geq 8$. Let $s \in \mathbb{Z}$ such that $4a_5 s \equiv -b \pmod{5^{v_5(\triangle)+1}}$. Let $F(X) = f(X + s) = X^5 + 5sX^4 + 10s^2X^3 + 10s^3X^2 + AX + B$, $\theta = \alpha - s$, $A = 5s^4 + a$, and $B = s^5 + as + b$. Then $(4a)^4A = 5^4(4a_5s)^4 + 2^8a_5^5) \equiv \triangle \pmod{5^{v_5(\triangle)+1}}$ and $(4a)^5B \equiv 5^5(-b^5 - 2^8ba_5^5) \equiv -b\triangle \pmod{5^{v_5(\triangle)+1}}$. Hence $v_5(A) = v_5(\triangle) - 4$ and $v_5(B) = v_5(\triangle) - 5$, $\mathrm{N}_X(F) = S_1 + S_2$ with respective slopes $1/3$ and $\frac{m-6}{2}$ such that $F_{S_1}(Y) = Y + 2s^3$. If $v_5(\triangle) = 2k + 1$, then $F_{S_2}(Y) = 2s^3Y + B_5$. If $v_5(\triangle) = 2k$, then $F_{S_2}(Y) = 2s^3Y^2 + B_5$. It follows that : $(1, \alpha, \alpha^2, \frac{\theta^3}{5}, \frac{\theta^4+5s\theta^3+10s^2\theta^2+10s^3\theta}{5^r})$ is a 5-integral basis of $\mathbb{Z}_K$, where $r = \lfloor \frac{v_5(\triangle)-4}{2} \rfloor$, and

(a) If $v_5(\triangle) = 2k$ and $(\frac{-B_5}{5}) = 1$, then $5\mathbb{Z}_K = \mathfrak{p}_1^3\mathfrak{p}_2\mathfrak{p}_3$.

(b) If $v_5(\triangle) = 2k$ and $(\frac{-B_5}{5}) = -1$, then $5\mathbb{Z}_K = \mathfrak{p}_1^3\mathfrak{p}_2$.

(5) If $v_5(\triangle) = 2k + 1$, then $5\mathbb{Z}_K = \mathfrak{p}_1^3\mathfrak{p}_2^2$.

□

Case : $v_p(a) = 0$ and $v_p(b) \geq 1$

If $p \neq 2$, then $v_p(\triangle) = 0$, $v_p(ind(f)) = 0$, and $p\mathbb{Z}_K$ is p-analogous to $\bar{f}(X)$. For $p = 2$, let $\theta = \alpha - 1$, $A = 5 + a$, $B = 1 + a + b$, and go to tables $A3$ and $B3$.

*TableA*3

Conditions	2-integral basis of $\mathbb{Z}_K$
$v_2(B) = 1$	$(1, \theta, \theta^2, \theta^3, \theta^4)$
$b \equiv 2 \pmod 4$, $a \equiv 3 \pmod 4$	$(1, \alpha, \alpha^2, \alpha^3, \alpha^4)$
$v_2(B) \geq 2$, $v_2(A) = 1$	$(1, \theta, \theta^2, \theta^3, \frac{\theta^4+\theta^3}{2})$
$v_2(B) = 2$, $v_2(A) \geq 2$	$(1, \theta, \theta^2, \frac{\theta^3+\theta^2}{2}, \frac{\theta^4+\theta^3}{2})$
$v_2(B) \geq 3$, $v_2(A) = 2$	$(1, \theta, \theta^2, \frac{\theta^3+\theta^2}{2}, \frac{\theta^4+\theta^3}{4})$
$v_2(B) \geq 3$, $v_2(A) \geq 3$	go to Table $A3.6$

*TableA*3.6 : $v_2(B) \geq 3, v_2(A) \geq 3$

Let $s \in \mathbb{Z}$ such that $as \equiv -5b_2 \pmod{2^{v_2(\triangle)+3}}$, $u = s + 2^k$ and $\theta = \alpha - u$, where $k = \lfloor \frac{v_2(\triangle-8}{2} \rfloor$.

conditions	2-integral basis
$v_2(\triangle) = 2k + 8$	$(1, \theta, \theta^2, \frac{\theta^3+\theta^2}{2}, \frac{\theta^4+5s\theta^3+10s^2\theta^2+10s^3\theta}{2^k})$
$v_2(\triangle) = 2k + 9$ $\triangle_2 \equiv 1 \pmod 4$	$(1, \alpha, \alpha^2, \frac{\theta^3+\theta^2}{2}, \frac{\theta^4+5u\theta^3+10u^2\theta^2+10u^3\theta}{2^{k+1}})$
$v_2(\triangle) = 2k + 9$ $\triangle_2 \equiv 3 \pmod 4$	$(1, \alpha, \alpha^2, \frac{\theta^3+\theta^2}{2}, \frac{\theta^4+5u\theta^3+10u^2\theta^2+10u^3\theta}{2^{k+2}})$

*TableB*3

conditions	$2\mathbb{Z}_K$	2-generators
$v_2(B)=1$	$\mathfrak{p}_1\mathfrak{p}_2^4$	$\beta_1=\alpha,\ \beta_2=\theta$
$v_2(B)\geq 2,\ v_2(A)=1$	$\mathfrak{p}_1\mathfrak{p}_2^3\mathfrak{p}_3$	$\beta_1=\alpha,\ \beta_3=\frac{\theta^4+5\theta^3}{2}$ $\beta_2=\theta+\frac{\theta^4+5\theta^3+10\theta^2+10\theta+A}{2}$
$v_2(B)=2,\ v_2(A)\geq 2$	$\mathfrak{p}_1\mathfrak{p}_2^2$	$\beta_1=\alpha,\ \beta_2=\theta$
$v_2(B)=3,\ v_2(A)=2$	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3$	$\beta_1=\alpha,\ \beta_2=\theta+\frac{\theta^4+5\theta^3+10\theta^2+10\theta}{4}$ $\beta_3=\theta+\frac{\theta^3+5\theta^2}{2}$
$v_2(B)\geq 4,\ v_2(A)=2$	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_4$	$\beta_1=\alpha,\ \beta_2=\theta+\frac{\theta^3+5\theta^2+10\theta+10}{2}+\frac{\theta^4+5\theta^3+10\theta^2+10\theta}{4}$ $\beta_3=\theta+\frac{\theta^3+5\theta^2}{2}+\frac{\theta^4+5\theta^3+10\theta^2+10\theta+A}{4}$ $\beta_4=\theta+\frac{\theta^3+5\theta^2}{2}+\frac{(\theta^3+5\theta^2+10\theta+10)(\theta+2u)}{4}$,$(4u+A\equiv 8 \pmod{16})$
$v_2(B)\geq 3,\ v_2(A)\geq 3$	go to $TableB3.6$	

$TableB3.6 : v_2(B)\geq 3, v_2(A)\geq 3$

Let $s\in\mathbb{Z}$ such that $as\equiv -5b_2 \pmod{2^{v_2(\triangle)+3}}$, $u=s+2^k$, $\theta_1=\alpha-u$, $A=5u^4+a$, and $B=u^5+au+b$, where $k=\lfloor\frac{v_2(\triangle-8}{2}\rfloor$.

conditions	$2\mathbb{Z}_K$	Generators
$v_2(\triangle)=2k+8$	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2$	$\beta_1=\alpha,\ \beta_2=\theta+\frac{\theta^3+5s\theta^2+10s^2\theta+10s^3}{2}$ $\beta_3=\theta+\frac{\theta^3+\theta^2}{2}+\frac{\theta^4+5s\theta^3+10s^2\theta^2+10s^3\theta}{2^k}$
$v_2(\triangle)=2k+9$ $\triangle_2\equiv 1 \pmod 4$	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2$	$\beta_1=\alpha,\ \beta_2=\theta+\frac{\theta^3+5s\theta^2+10s^2\theta+10s^3}{2}$ $\beta_3=\theta+\frac{\theta^3+\theta^2}{2}+\frac{\theta^4+5u\theta^3+10u^2\theta^2+10u^3\theta}{2^{k+1}}$
$v_2(\triangle)=2k+9$ $\triangle_2\equiv 3 \pmod 8$	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3$	$\beta_1=\alpha,\ \beta_2=\theta+\frac{\theta^3+5s\theta^2+10s^2\theta+10s^3}{2}$ $\beta_3=\theta+\frac{\theta^3+\theta^2}{2}$
$v_2(\triangle)=2k+9$ $\triangle_2\equiv 7 \pmod 8$	$\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_4$	$\beta_1=\alpha,\ \beta_2=\theta+\frac{\theta^3+5s\theta^2+10s^2\theta+10s^3}{2}$ $\beta_3=\theta+\frac{\theta^3+\theta^2}{2}$ $\beta_4=\theta+\frac{\theta^3+\theta^2}{2}+\frac{\theta^4+5u\theta^3+10u^2\theta^2+10u^3\theta}{2^{k+2}}$

Proof. Let $F(X)=f(X+1)=X^5+5X^4+10X^3+10X^2+AX+B$ and $\theta=\alpha-1$, where $A=5+a$ and $B=1+a+b$.

(1) If $v_2(B)=1$ and $v_2(A)\geq 1$, then $v_2(ind(F))=0$ and $2\mathbb{Z}_K=\mathfrak{p}_1\mathfrak{p}_2^4$, where $\mathfrak{p}_1=(2,\alpha)$ and $\mathfrak{p}_2=(2,\theta)$.
(2) If $v_2(B)\geq 2$ and $v_2(A)=1$, then $v_2(ind(F))=1$ and $2\mathbb{Z}_K=\mathfrak{p}_1\mathfrak{p}_2^3\mathfrak{p}_3$.
(3) If $v_2(B)=2$ and $v_2(A)\geq 2$, then $v_2(ind(F))=2$, $2\mathbb{Z}_K=\mathfrak{p}_1\mathfrak{p}_2^2$,
(4) If $v_2(A)=2$, $v_2(B)=3$, then $v_2(ind(F))=3$ and $2\mathbb{Z}_K=\mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3$. If $v_2(A)=2$, $v_2(B)\geq 4$ and $\mathrm{N}_X(F)=S_1+S_2+S_3+S_4$ such that $F_{S_1}(Y)=F_{S_2}(Y)=F_{S_3}(Y)=F_{S_4}(Y)=Y+1$. Thus, $v_2(ind(F))=3$, $2\mathbb{Z}_K=P_1P_2^2P_3P_4$, $v_{P_1}(\theta+3)=1$, $v_{P_2}(\theta)=1=v_{P_3}(\theta)$ and $v_{P_4}(\theta)=v_2(B)-2\geq 2$.
Let $u\in\mathbb{Z}$, $s=1+2u$ and $F(X)=f(X+s)=X^5+5sX^4+10s^2X^3+10s^3X^2+a_4X+a_5$, where $a_4=5s^4+a$ and $a_5=s^5+as+b$. Then $v_2(a_4)=2$ and $a_5=B-2uA+40u^2-80u^3+80u^4-32u^5$. If $v_2(B)=4$, then for $u\equiv -A_2 \pmod 4$ and if $v_2(B)\geq 5$, then for $u\equiv 2-A_2 \pmod 4$, we have $v_2(a_5)\geq 5$. Thus, $v_{P_3}(\theta-2u)\geq 3$, and then $\beta_3=\theta+\frac{(\theta^3+5\theta^2+10\theta+10)(\theta+2u)}{4}$ and $\beta_4=\theta+\frac{(\theta^3+5\theta^2+10\theta+10)(\theta+4)}{4}$.
(5) If If $v_2(A)\geq 3$ and $v_2(B)\geq 3$, i.e, $a\equiv 3 \pmod 8$ and $b\equiv 4 \pmod 8$, then $v_2(\triangle)\geq 11$, then let $s\in\mathbb{Z}$ such that $as\equiv -5b_2 \pmod{2^{v_2(\triangle)+3}}$, $\theta=\alpha-s$, and $F(X)=f(X-s)=X^5+5sX^4+10s^2X^3+10s^3X^2+AX+B$, where $A=5s^4+a$

and $B = s^5 + as + b$. It follows that $(4a)^4 A \equiv 5^5 b^4 + 2^8 a^5 \equiv \triangle \pmod{2^{v_2(\triangle)+3}}$ and $(4a)^5 B \equiv -5^5 b^5 - 5b2^8 a^5 + (4a)^5 b \equiv -b\triangle \pmod{2^{v_2(\triangle)+3}}$. Thus, $v_2(A) = v_2(\triangle) - 8$, $v_2(B) = v_2(\triangle) + v_2(b) - 10 = v_2(\triangle) - 8$, and $\mathrm{N}_X(F) = S_1 + S_2 + S_3$ with respective slopes 0, 1/2 and $\lambda_3 = \frac{v_2(\triangle)}{2} - 4$ such that $F_{S_1}(Y) = F_{S_2}(Y) = Y + 1$.

(a) If $v_2(\triangle) = 2k+8$, then $F_{S_3}(Y) = Y+1$, $v_2(ind(f)) = k+1$, $(1, \alpha, \alpha^2, \frac{\theta^3+\theta^2}{2}, \frac{\theta^4+5s\theta^3+10s^2\theta^2+10s^3\theta}{2^k})$ is a 2-integral basis of $\mathbb{Z}_K$ and $2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2$.

(b) If $v_2(\triangle) = 2k + 9$, then $\lambda_3 = k$ and $F_{S_3}(Y) = (Y + 1)^2$. Let $u = s + 2^k$, $F(X) = f(X + u) = X^5 + 5uX^4 + 10u^2X^3 + 10u^3X^2 + a_4X + a_5$ and $\theta = \alpha - u$, where $a_4 = 5u^4 + a = 5.2^{4k} + 5s2^{3k+2} + 15s^2 2^{2k+1} + 5s^3 2^{k+2}) + A$ and $a_5 = f(u) = (2^{5k} + 5s2^{4k} + 5s^2 2^{3k+1}) + (5s^3.2^{2k+1} + B) + 2^k A$. Since $k \geq 2$, $v_2(a_4) = k + 2$. The fact that $(4a)^5 a_5 \equiv -2^{2k+11} b_2(\triangle_2 + 5^4 b_2^2 a^2) + 2^{k+2} a\triangle + 2^{3k+11} 5^3 b_2^2 a^3 - 2^{4k+10} 5^2 b_2 a^4 + 2^{5k+10} a^5 \pmod{2^{v_2(\triangle)+3}}$ implies that $v_2(a_5) \geq 2k + 2$.

(i) If $\triangle_2 1 \equiv \pmod 4$, then $v_2(a_5) = 2k + 2$, $v_2(ind(f)) = k + 2$ and $2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2$.

(ii) If $\triangle_2 \equiv 7 \pmod 8$, then $v_2(a_5) \geq 2k + 4$. Thus, $v_2(ind(f)) = k + 3$ and $2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3\mathfrak{p}_4$.

(iii) If $\triangle_2 \equiv 3 \pmod 8$, then $v_2(a_5) = 2k + 3$, $v_2(ind(f)) = k + 3$ and $2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3$.

□

Case : $v_p(ab) = 0$

In that case, if $p \in \{2, 5\}$, then $v_p(\triangle) = 0$, p does not divide $ind(f)$, and $p\mathbb{Z}_K$ is p-analogous to $\bar{f}(X)$.

If $p \notin \{2, 5\}$ and $v_p(\triangle) \geq 2$, then let $s \in \mathbb{Z}$ such that $4as \equiv -5b \pmod{p^{v_p(\triangle)+1}}$. Let $r = \lfloor \frac{v_p(\triangle)}{2} \rfloor$, $\theta = \alpha - s$, $A = 5s^4 + a$ and $B = s^5 + as + b$. In $TableA4$ and $TableB4$, for any prime integer $p \notin \{2, 5\}$, a p-integral bases of $\mathbb{Z}_K$ and the factorization of $p\mathbb{Z}_k$ are given.

$TableA4$

Conditions	p-integral basis of $\mathbb{Z}_K$
$v_p(ab) = 0$, $v_p(\triangle) \leq 1$	$(1, \theta, \theta^2, \theta^3, \theta^4)$
$v_p(\triangle) \geq 2$	$(1, \theta, \theta^2, \theta^3, \frac{\theta^4+5s\theta^3+10s^2\theta^2+10s^3}{p^r})$

$TableB4$:

Conditions	$p\mathbb{Z}_K$	p-generators
$v_p(\triangle) \leq 1$		p-analogious to $\bar{f}(X)$
$v_p(\triangle) = 2k$ $k \geq 1, (\frac{-B_p}{p}) = -1$	$I\mathfrak{p}$	I is p-analogous to $X^3 + 5sX^2 + 10s^2X + 10s^3$ $\mathfrak{p} = (p, \frac{\theta^4+5s\theta^3+10s^2\theta^2+10s^3\theta}{p^{k-1}})$
$v_p(\triangle) = 2k$ $k \geq 1, (\frac{-B_p}{p}) = 1$	$I\mathfrak{p}_1\mathfrak{p}_2$	I is p-analogous to $X^3 + 5sX^2 + 10s^2X + 10s^3$ $\mathfrak{p}_1 = (p, \frac{(\theta^3+5s\theta^2+10s^2\theta+10s^3)(\theta+p^k u)}{p^k})$, $10s^3u^2 + B_p \equiv p \pmod{p^2}$ $\mathfrak{p}_1 = (p, \frac{(\theta^3+5s\theta^2+10s^2\theta+10s^3)(\theta-p^k u)}{p^k})$
$v_p(\triangle) = 2k + 1$ $k \geq 1$	$I\mathfrak{p}^2$	I is p-analogous to $X^3 + 5sX^2 + 10s^2X + 10s^3$ $\mathfrak{p} = (p, \frac{\theta^4+5s\theta^3+10s^2\theta^2+10s^3\theta}{p^k})$

Proof. If $v_p(\triangle) \leq 1$, then $v_p(ind(f)) = 0$ and $p\mathbb{Z}_K$ p-analogous to $\bar{f}(X)$.

For $p \notin \{2, 5\}$ and $v_p(\triangle) \geq 2$, let $s \in \mathbb{Z}$ such that $4as \equiv -5b \pmod{p^{v_p(\triangle)+1}}$. Let

$F(X) = f(X+s) = X^5 + 5sX^4 + 10s^2X^2 + 10s^3X + AX + B$ and $\theta = \alpha - s$, where $A = 5s^4 + a$ and $B = s^5 + as + b$. Since $(4a)^4A \equiv 5^5b^4 + 2^8a^5 \equiv \triangle \pmod{p^{v_p(\triangle)+1}}$ and $(4a)^5B \equiv -b(5^5b^4 + 2^8a^5) \equiv -b\triangle \pmod{p^{v_p(\triangle)+1}}$, $v_p(A) = v_p(B) = v_p(\triangle)$ and $\mathrm{N}_X(F) = S_1 + S_2$ such that :

(1) If $v_p(\triangle) = 2k$, then $F_{S_1}(Y) = Y^3+5sY^2+10s^2Y+10s^3$ and $F_{S_2}(Y) = Y^2+B_p$ are square free. Thus, $v_p(ind(f)) = r$, where $r = \lfloor \frac{v_p(\triangle)}{2} \rfloor$. Moreover, if $(\frac{-B_p}{p}) = -1$, then $p\mathbb{Z}_K = I\mathfrak{p}$. If $(\frac{-B_p}{p}) = 1$, then $p\mathbb{Z}_K = I\mathfrak{p}_1\mathfrak{p}_2$.

(2) If $v_p(\triangle) = 2k+1$, then $F_{S_1}(Y) = Y^3+5sY^2+10s^2Y+10s^3$ and $F_{S_2}(Y) = Y+B_p$ are square free. Thus, $v_p(ind(f)) = r$ and $p\mathbb{Z}_K = I\mathfrak{p}^2$.

□

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