Gulf Journal of Mathematics
Vol 6, Issue 4 (2018) 14-36

ENTROPY SOLUTIONS FOR A NONLINEAR UNILATERAL $p(x)$ PARABOLIC PROBLEMS WITH $f-\mathbf{div}F$ DATA

MOUNIR MEKKOUR*

Abstract. We study the nonlinear $p(x)$ parabolic problems with obstacle and $f-\mathrm{div}F$ data.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N\geq 2$, T is a positive real number and $Q=\Omega\times(0,T)$, while the variable exponent $p\ \ \bar{\Omega}\to(1,\infty)$ is a continuous function, the data $f\in L^1(Q)$, $u_0\in L^1(\Omega)$ and $F\in(L^{p'(x)}(Q))^N$. The objective of this paper is to study the existence of an entropy solution for the obstacle parabolic problems of the type:

$$(\mathcal{P})\begin{cases} u\geq\psi & \text{a.e. in } \ \Omega\times(0,T),\\ \frac{\partial u}{\partial t}-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)+=g(u)|\nabla u|^{p(x)} & \\ \qquad\qquad\qquad +f-\mathrm{div}F & \text{in } \Omega\times(0,T)\\ u(x,0)=u_0 & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\times(0,T).\end{cases}$$

The function g is a bounded and continuous positive function that belongs to $L^1(\mathbb{R})$. We recall that the notion of solutions for the problem $(\mathcal{P})$ without obstacle has been proved by Dall' Algio - Orsina [19] and Porretta [25] with F=0 and $-g(u)|\nabla u|^{p(x)}=H(x,t,u,\nabla u)$ and the nonlinearity g satisfying the following 'natural' growth condition (of order p):

$$g(x,t,s,\xi)\leq b(s)(|\xi|^p+c(x,t))$$

and the classical sign condition

$$g(x,t,s,\xi)\cdot s\geq 0.$$

In the case $F=0$ the existence of solutions of some unilateral problems in the framework of Orlicz spaces has been established by M. Kbiri Alaoui , D. Meskine, A. Souissi in [22] with the penalization methods. Recently, M. Bendahmane,P.

Date: Received: Dec 2, 2018.
*Corresponding author.
2010 *Mathematics Subject Classification.* 35J15, 35J70, 35J85.
Key words and phrases. Unilateral, Entropy solutions, Nonlinear parabolic problems, Penalized equations.

Wittbold and A. Zimmermann in [14] proved the existence of renormalized solutions, and the existence of renormalized solution in Orlicz spaces has been proved in E. Azroul, H. Redwane and M. Rhoudaf [13] .
The plan of the paper is as follows. Section 2 presents the mathematical preliminaries. In Section 3 we make precise all the assumptions on A, g, f and u_0, the definition of an entropy solution of $(\mathcal{P})$. In Section 4 we establish the existence of such a solution in (Theorem 3.1).

2. Mathematical preliminaries

Some definitions and basic properties of the generalised Lebegue–Sobolev spaces with variable exponent $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and
$W_0^{1,p(x)}(\Omega)$.
Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, we say that a real-valued continuous function $p(.)$ is log-Hölder continuous in Ω if

$$|p(x) - p(y)| \leq \frac{C}{|\log|x-y||} \tag{1}$$

$\forall x, y \in \bar{\Omega}$ such that $|x-y| < \frac{1}{2}$, with possible different constant C.
Let us set $C_+(\bar{\Omega})$=$\{p \in C(\bar{\Omega}) : \min_{x\in\bar{\Omega}} p(x) > 1\}$. For any $p \in C_+(\bar{\Omega})$, we define

$$p^- = \inf_{x\in\Omega} p(x) \quad and \quad p^+ = \sup_{x\in\Omega} p(x).$$

For any $p \in C_+(\bar{\Omega})$, we introduce the variable exponent Lebesgue space by:

$$L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R}; u \; is \; measurable \; with \; : \int_\Omega |u(x)|^{p(x)} dx < \infty\Big\},$$

endowed with the Luxembourg norm

$$||u||_{L^{p(x)}(\Omega)} = ||u||_{p(x)} = \inf\Big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\Big\},$$

which is a separable and reflexive Banach space. The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, (see[21, 27]). If $p(x)$ is a constant function, then the variable exponent Lebesque space coincides with the classical Lebesque space. We define also the variable Sobolev spaces by

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) \;\text{ and }\; |\nabla u| \in L^{p(x)}(\Omega)\Big\}.$$

where the norm is

$$||u||_{1,p(x)} = ||u||_{p(x)} + ||\nabla u||_{p(x)} \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$, i.e.,

$$W_0^{1,p(.)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(.)}(\Omega)}$$

and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ for $p(x) < N$.

Proposition 2.1. *(Young's Inequality) Let* $p,\ p' \in P^{+}(\Omega)$*, where* p' *the conjugate for all* $a, b > 0$*, we have:*

$$ab \leq \frac{a^{p(x)}}{p(x)} + \frac{b^{p'(x)}}{p'(x)}.$$

Proposition 2.2. *(Generalised Hölder inequality) (see[20, 23]).*

- **i)** *For any functions* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$*, we have:* $|\int_\Omega uvdx| \leq (\frac{1}{p^-} + \frac{1}{p'^-})||u||_{p(x)}||v||_{p'(x)} \leq 2||u||_{p(x)}||v||_{p'(x)}$.
- **ii)** *For all* $p,\ q \in C_+(\bar{\Omega})$ *such that* $p(x) \leq q(x)$ *a.e. in* Ω*, we have* $L^{q(x)} \hookrightarrow L^{p(x)}$ *and the embedding is continuous.*

The modular of the space $L^{p(x)}(\Omega)$, which is the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ is defined by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx \text{ for all } u \in L^{p(x)}(\Omega).$$

The next proposition shows that there is a gap between the modular and the norm in $L^{p(x)}(\Omega)$.

Proposition 2.3. *See([21, 27]). For* $u \in L^{p(x)}(\Omega)$ *and* $\{u_k\}_{k\in N} \subset L^{p(x)}(\Omega)$ *then, the following assertions hold*

$$u \neq 0 \Rightarrow [||u||_{p(x)} = \lambda \Leftrightarrow \rho(\frac{u}{\lambda}) = 1] \tag{2}$$

$$||u||_{p(x)} > 1 \Rightarrow ||u||_{p(x)}^{p^-} \leq \rho(u) \leq ||u||_{p(x)}^{p^+} \tag{3}$$

$$||u||_{p(x)} < 1 \Rightarrow ||u||_{p(x)}^{p^+} \leq \rho(u) \leq ||u||_{p(x)}^{p^-} \tag{4}$$

$$\lim_{k\to\infty} ||u_k||_{p(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = 0 \tag{5}$$

$$\lim_{k\to\infty} ||u_k||_{L^{p(x)}(\Omega)} = \infty \Leftrightarrow \lim_{k\to\infty} \rho(u_k) = \infty. \tag{6}$$

Proposition 2.4. *(See[21].)*

- **i)** *Assuming* $1 < p^- \leq p^+ < \infty$ *the spaces* $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*
- **ii)** *If* $q \in C_+(\bar{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \bar{\Omega}$*, then* $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *is continuous and compact. In particular, we have* $W_0^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ *is continuous and compact.*
- **iii)** *(See [23].) Let* $\Omega \subset \mathbb{R}^N$ *be an open, bounded set and* $p \in C_+(\bar{\Omega})$ *satisfy the log-Hölder continuity condition (1). Then, for* $u \in W_0^{1,p(x)}(\Omega)$*, the* $p(x)$*-Poincaré inequality*

$$||u||_{p(x)} \leq C||\nabla u||_{p(x)}$$

holds, where the positive constant C *depends on* $p(x)$ *and* Ω*.*

Remark 2.1. *By (iii) of Proposition 2.4 we know that* $||\nabla u||_{p(x)}$ *and* $||u||_{1,p(x)}$ *are equivalent norms on* $W_0^{1,p(x)}(\Omega)$*.*

We will also use the standard notations for Bochner spaces, i.e. if $q \geq 1$ and X is a Banach space then $L^q(0,T;X)$ denotes the space of strongly measurable functions $u:(0,T)\to X$ for which $t\mapsto \|u(t)\|_X \in L^q(0,T)$ Morever, $C([0,T];X)$ denotes the space of continuous functions $u:[0,T]\to X$ endowed with the norm $\|u\|_{C([0,T];X)} = \max_{t\in[0,T]}\|u(t)\|_X$. Set

$$L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)) = \Big\{u:(0,T)\to W_0^{1,p(x)}(\Omega)\ measurable; \Big(\int_0^T \|u(t)\|^{p^-}_{W_0^{1,p(x)}(\Omega)}dt\Big)^{\frac{1}{p^-}} < \infty\Big\}.$$

We introduce the functional space see [14]

$$V = \Big\{f \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)); |\nabla f| \in L^{p(x)}(Q)\Big\}, \tag{7}$$

which endowed with the norm

$$\|f\|_V = \|\nabla f\|_{L^{p(x)}(Q)},$$

or, the equivalent norm

$$|||f|||_V = \|f\|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} + \|\nabla f\|_{L^{p(x)}(Q)}.$$

The space V is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(x)}(Q) \hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega))$ and the Poincaré inequality. We state some further properties of V in the following lemma.

Lemma 2.1. *(See [14]). Let V be defined as in (7) and let its dual space be denoted by V^*. Then, we have the following continuous dense embeddings:*

$$L^{p^+}(0,T;W_0^{1,p(x)}(\Omega)) \hookrightarrow V \hookrightarrow L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)).$$

In particular, since $D(Q)$ is dense in $L^{p^+}(0,T;W_0^{1,p(x)}(\Omega))$, it is dense in V and for the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T;(W_0^{1,p(x)}(\Omega))^*) \hookrightarrow V^* \hookrightarrow L^{(p^+)'}(0,T;(W_0^{1,p(x)}(\Omega))^*).$$

Note that, we have the following continuous dense embeddings

$$L^{p^+}(0,T;L^{p(x)}(\Omega)) \hookrightarrow L^{p(x)}(Q) \hookrightarrow L^{p^-}(0,T;L^{p(x)}(\Omega)).$$

3. Assumptions and definition

Throughout this paper, we assume that the following assumptions hold true.

3.1. **Basic assumptions.** Let $p \in C_+(\bar{\Omega})$ and assume that p(x) satisfies the log-Hölder condition (1) with $1 < p^- \leq p(x) \leq p^+ < \infty$. The differential operator $A : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is defined by

$$Au = -\text{div}(|\nabla u|^{p(x)-2}\nabla u), \tag{8}$$

is a Leray-Lions operator which is coercive and

$$g : \quad \mathbb{R} \to \mathbb{R}^+ \tag{9}$$

is a bounded and continuous positive function that belongs to $L^1(\mathbb{R})$

$$\phi : \quad \mathbb{R} \to \mathbb{R}^N \text{ is a continuous function,} \tag{10}$$

$$f \text{ is an element of } L^1(Q), u_0 \in L^1(\Omega) \cap K, \quad u_0 \geq 0 \textit{ and } p \in C_+(\bar{\Omega}). \tag{11}$$

Let ψ be a measurable function with values in $\overline{\mathbb{R}}$ such that $\psi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, (see [22]), K is defined by: $K = \Big\{ u \in W_0^{1,p(x)}(\Omega); \;\; u(x) \geq \psi(x) \;\; a.e. \;\; in \;\; \Omega \Big\}$ and consider the convex set

$$K_\psi = \Big\{ u \in V, \; u(t) \in K \Big\}.$$

Theorem 3.1. *Under assumptions (8)-(11), there exists at least one entropy solution of problem* $(\mathcal{P})$*: Let* $f \in L^1(Q)$ *and* $u_0 \in L^1(\Omega)$*. A measurable function* u *defined on* Q *is a unilateral entropy solution of problem* $(\mathcal{P})$ *if*

$$u \geq \psi \textit{ a.e. in } Q, \tag{12}$$

$$T_k(u) \in V, \textit{ for all } k \geq 0 \textit{ and } u \in C(0,T;L^1(\Omega)), \tag{13}$$

$$\begin{gathered} \int_\Omega S_k(u-v)(T)dx + \int_Q \frac{\partial v}{\partial t} T_k(u-v)dxdt \\ + \int_Q |\nabla u|^{p(x)-2}\nabla u \nabla T_k(u-v)dxdt \leq \int_Q g(u)|\nabla u|^{p(x)} T_k(u-v)dxdt \\ + \int_Q f T_k(u-v)dxdt + \int_Q F\nabla T_k(u-v) + \int_\Omega S_k(u_0 - v(0))dx, \end{gathered} \tag{14}$$

for all $v \in K_\psi \cap \; L^\infty(Q)$, $\frac{\partial v}{\partial t} \in V^* + L^1(Q)$ *and* $\forall k > 0$,
where $S_k(s) = \int_0^s T_k(r)dr$.

Proof. The proof is divided into 4 steps.
In Step 1, we introduce an approximate problem. In Step 2, we establish a few a priori estimates which allow us to prove that the approximate solutions u_n converge to u a.e. in Q. In Step 3, we define a time regularisation of the field $T_k(u)$ and prove that u_n satisfies (24). In this step using some techniques, we also prove the modular convergence of $T_k(u_n)$ to $T_k(u)$ in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. In Step 4, we pass to the limit which is the final step to prove Theorem 3.1.

Step 1: The approximate problem. Let us introduce the following regularization of the data:

$$f_n \in L^{p'(x)}(Q),\ f_n \to f \text{ a.e. in Q, and strongly } in\ L^1(Q)\ as\ n\ \to \infty, \tag{15}$$

$$u_{0n} \in D(\Omega) : \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}\ and\ u_{0n} \to u_0 \text{ in } L^1(\Omega)\ as\ n\ \to \infty, \tag{16}$$

Let us now consider the following regularized approximate problem

$$(\mathcal{P}_n)\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(|\nabla u_n|^{p(x)-2}\nabla u_n) - nT_n((u_n-\psi)^-) & \\ \qquad\qquad = g(u_n)|\nabla u_n|^{p(x)} + f_n - \operatorname{div}F & \text{in } D'(Q), \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(t=0) = u_{0n} & \text{in } \Omega. \end{cases}$$

Moreover, since $f_n \in V^*$, proving the existence of weak solution $u_n \in V$ of $(\mathcal{P}_n)$ is an easy task (see [24]).

Step 2: A Priori estimates. The estimate derived in this step rely on standard techniques for problems of the type $(\mathcal{P}_n)$.

Proposition 3.1. *Assume that (8)-(11) hold true and let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then for all $k>0$, we have*

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} \leq C\ k \quad \text{for all } n \in \mathbb{N},$$

where C is a constant independent of n.

Proof. Let $h>k>0$ and consider the test function $\varphi = T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n)) \in V \cap L^\infty(Q)$ in the approximate problem $(\mathcal{P}_n)$, where $G(s) = \int_0^s g(r)dr$, we have

$$\Big\langle \frac{\partial u_n}{\partial t}, T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big\rangle + \int_{\{k\leq|u_n|\leq k+h\}} \Big(|\nabla u_n|^{p(x)-2}\nabla u_n\Big)\nabla u_n \exp(G(u_n))dxdt$$

$$+ \int_Q \Big(|\nabla u_n|^{p(x)-2}\nabla u_n\Big)\nabla u_n T_h\Big(u_n - T_k(u_n)\Big)g(u_n)\exp(G(u_n))dxdt$$

$$-T_k(u_n)\Big)\exp(G(u_n))\Big)dxdt$$

$$-\int_Q nT_n((u_n-\psi)^-)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$= \int_Q g(u_n)|\nabla u_n|^{p(x)}T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$+ \int_Q f_n T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$+ \int_Q F\nabla\Big(T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big)dxdt.$$

Then,

$$\Big\langle\Big\langle \frac{\partial u_n}{\partial t}, T_h(u_n - T_k(u_n))\exp(G(u_n))\Big\rangle\Big\rangle + \int_{\{k\leq |u_n|\leq k+h\}} |\nabla u_n|^{p(x)}\exp(G(u_n))dxdt$$

$$- \int_Q nT_n((u_n-\psi)^-)T_h(u_n - T_k(u_n))\exp(G(u_n))dxdt$$

$$= \int_Q f_n T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$+ \int_Q F\nabla\Big(T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big)dxdt.$$

On the one hand, we have

$$\Big\langle\Big\langle \frac{\partial u_n}{\partial t}, T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))\Big\rangle\Big\rangle = \int_\Omega S_h^k(u_n(T)dx - \int_\Omega S_h^k(u_{0n})dx,$$

where $S_h^k(s) = \int_0^s T_h\Big(q - T_k(q)\Big)\exp(G(q))dq$ and by using the fact that $\int_\Omega S_h^k(u_n(T)dx \geq 0$ and $\int_\Omega S_h^k(u_{0n})dx \leq h\exp(\|g\|_{L^1(\mathbb{R})})\|u_{0n}\|_{L^1(\Omega)}$, we get

$$\int_{\{k\leq |u_n|\leq k+h\}} |\nabla u_n|^{p(x)}\exp(G(u_n))dxdt$$

$$- \int_Q nT_n((u_n-\psi)^-)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$\leq h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}\Big]$$

$$+ \int_Q FT_h\Big(u_n - T_k(u_n)\Big)\nabla u_n g(u_n)\exp(G(u_n))dxdt$$

$$+ \int_{\{k\leq |u_n|\leq k+h\}} F\Big[\exp(G(u_n))\Big]^{1-\frac{1}{p(x)}}\Big[\exp(G(u_n))\Big]^{\frac{1}{p(x)}}|\nabla u_n|dxdt,$$

hence

$$\int_{\{k\leq |u_n|\leq k+h\}} |\nabla u_n|^{p(x)}\exp(G(u_n))dxdt$$

$$- \int_Q nT_n((u_n-\psi)^-)T_h\Big(u_n - T_k(u_n)\Big)\exp(G(u_n))dxdt$$

$$\leq h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)} + \|u_{0n}\|_{L^1(\Omega)}\Big]$$

$$+ \int_Q FT_h\Big(u_n - T_k(u_n)\Big)\nabla u_n g(u_n)\exp(G(u_n))dxdt$$

$$+ \int_{\{k\leq |u_n|\leq k+h\}} \frac{F\Big[\exp(G(u_n))\Big]^{\frac{1}{p'(x)}}}{\Big[\frac{1}{2}p(x)\Big]^{\frac{1}{p(x)}}}\Big[\frac{1}{2}p(x)\Big]^{\frac{1}{p(x)}}\Big|\nabla u_n\Big|\Big[\exp(G(u_n))\Big]^{\frac{1}{p(x)}}dxdt$$

and by Young's inequality, we have

$$\begin{aligned}
&\int_{\{k\leq|u_n|\leq k+h\}}|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt\\
&-\int_Q nT_n((u_n-\psi)^-)T_h(u_n-T_k(u_n))\exp(G(u_n))dxdt\\
&\quad\leq h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\Big]\\
&\quad+\int_Q F\Big(T_h\Big(u_n-T_k(u_n)\Big)\nabla u_n g(u_n)\exp(G(u_n))dxdt\\
&\quad+\frac{1}{2}\int_{\{k\leq|u_n|\leq k+h\}}|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt\\
&\quad+\int_{\{k\leq|u_n|\leq k+h\}}\frac{|F|^{p'(x)}\exp(G(u_n))}{p'(x)(\frac{1}{2}p(x))^{\frac{p'(x)}{p(x)}}}dxdt\\
&\quad\leq h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\Big]\\
&\quad+\int_Q F\Big(T_h\Big(u_n-T_k(u_n)\Big)\nabla u_n g(u_n)\exp(G(u_n))dxdt\\
&\quad+\frac{1}{2}\int_{\{k\leq|u_n|\leq k+h\}}|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt\\
&\quad+C\int_{\{k\leq|u_n|\leq k+h\}}|F|^{p'(x)}\exp(G(u_n))dxdt,
\end{aligned}$$

then

$$\begin{aligned}
&\frac{1}{2}\int_{\{k\leq|u_n|\leq k+h\}}|\nabla u_n|^{p(x)}\exp(G(u_n))dxdt\\
&\quad-\int_Q nT_n((u_n-\psi)^-)T_h\Big(u_n-T_k(u_n)\Big)\exp(G(u_n))dxdt\\
&\quad\leq h\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)}+\|u_{0n}\|_{L^1(\Omega)}\Big]+C_1 \qquad (17)\\
&\quad+\int_Q FT_h\Big(u_n-T_k(u_n)\Big)\nabla u_n g(u_n)\exp(G(u_n))dxdt.
\end{aligned}$$

Let us observe that if we take $\varphi = \rho(u_n) = \int_0^{u_n} g(s)\chi_{\{k\leq|s|\leq k+h\}}ds \exp(G(u_n))$ a test function in the approximate $(\mathcal{P}_n)$, we obtain

$$
\begin{aligned}
\Big[\int_\Omega \varphi_1(u_n)dx\Big]_0^T + \int_{\{k\leq|u_n|\leq k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
- \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt \\
\leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)} \\
+ \int_Q F\nabla u_n g(u_n)\chi_{\{k\leq|u_n|\leq k+h\}}\exp(G(u_n))dxdt \\
+\Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n| g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt,
\end{aligned}
$$

where $\varphi_1(r) = \int_0^r \rho(s)ds$, which implies, using $\varphi_1(r)\geq 0$, we obtain

$$
\begin{aligned}
\int_{\{k\leq|u_n|\leq k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\
- \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt \\
\leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)} \\
+ \int_Q F\nabla u_n g(u_n)\chi_{\{k\leq|u_n|\leq k+h\}}\exp(G(u_n))dxdt \\
+\Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n| g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt,
\end{aligned}
$$

then

$$
\int_{\{k\leq|u_n|\leq k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt - \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt
$$

$$\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big] + \int_Q F\frac{\Big[g(u_n)\exp(G(u_n))\Big]^{\frac{1}{p'(x)}}}{\Big[\frac{1}{2}p(x)\Big]^{\frac{1}{p(x)}}}$$

$$\times\Big[\frac{1}{2}p(x)\Big]^{\frac{1}{p(x)}}\Big[g(u_n)\exp(G(u_n))\Big]^{\frac{1}{p(x)}}\nabla u_n\chi_{\{k\leq|u_n|\leq k+h\}}dxdt$$

$$+\|g\|_\infty\int_Q|F|\frac{\Big[g(u_n)\exp(G(u_n))\Big]^{\frac{1}{p'(x)}}}{\Big[\frac{1}{2}p(x)\Big]^{\frac{1}{p(x)}}}\Big[\frac{1}{2}p(x)\Big]^{\frac{1}{p(x)}}$$

$$\times\Big[g(u_n)\exp(G(u_n))\Big]^{\frac{1}{p(x)}}\nabla u_n\chi_{\{k\leq|u_n|\leq k+h\}}dxdt,$$

and by using Young's Inequality , we have

$$\int_{\{k\leq|u_n|\leq k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt - \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt$$

$$\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big]$$

$$+\int_Q\frac{|F|^{p'(x)}g(u_n)\exp(G(u_n))}{\Big[\frac{p'(x)}{2}p(x)\Big]^{\frac{p'(x)}{p(x)}}}dxdt$$

$$+\frac{1}{2}\int_Q|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt$$

$$+\|g\|_\infty\int_Q\frac{|F|^{p'(x)}g(u_n)\exp(G(u_n))}{\Big[\frac{p'(x)}{2}p(x)\Big]^{\frac{p'(x)}{p(x)}}}dxdt$$

$$+\frac{1}{2}\|g\|_\infty\int_Q|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt,$$

Since g is bounded function, then we have

$$\int_{\{k\leq|u_n|\leq k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt - \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt$$

$$\leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big]$$

$$+C'\|g\|_\infty\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\int_Q|F|^{p'(x)}dxdt$$

$$+\frac{1}{2}\int_Q|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt$$

$$+\|g\|_\infty C'\|g\|_{L^1(\mathbb{R})}\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\int_Q|F|^{p'(x)}dxdt$$

$$+\frac{1}{2}\|g\|_\infty\int_Q|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))\chi_{\{k\leq|u_n|\leq k+h\}}dxdt,$$

then

$$\begin{aligned}
\int_{\{k\le|u_n|\le k+h\}} &|\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt - \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt\\
&\le \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}\Big]\\
&\quad +C'\|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[1+\|g\|_\infty\Big]\int_Q |F|^{p'(x)}dxdt\\
&\quad +\frac{1}{2}\int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{k\le|u_n|\le k+h\}}dxdt\\
&\quad +\frac{1}{2}\|g\|_\infty\int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{k\le|u_n|\le k+h\}}dxdt,
\end{aligned}$$

and since

$$\int_Q |F|^{p'(x)}dxdt = \rho(F)\le \max\Big\{||F||^{p^-}_{(L^{P'(.)}(Q))^N}, ||F||^{p^+}_{(L^{P'(.)}(Q))^N}\Big\} = C''$$

then, we have

$$\begin{aligned}
\int_{\{k\le|u_n|\le k+h\}} &|\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt\\
&-\int_Q nT_n(u_n-\psi)^-\rho(u_n)dxdt\\
&\le \|g\|_\infty\Big(\exp(\|g\|_{L^1(\mathbb{R})})\Big)\Big[\|f_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}\Big]\\
&\quad +C_2+\frac{1}{2}\int_{\{k\le|u_n|\le k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt\\
&\quad +\frac{1}{2}\|g\|_\infty\int_{\{k\le|u_n|\le k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt
\end{aligned}$$

where $C_2 = C'\|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[1+\|g\|_\infty\Big]C''$
which give

$$C_3\int_{\{k\le|u_n|\le k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt - \int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt \le C_4,$$

where $C_3 = 1-\frac{1}{2}(1+\|g\|_\infty)$, then

$$\begin{aligned}
C_3\int_{\{k\le|u_n|\le k+h\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt &\le C_4+\int_Q nT_n((u_n-\psi)^-)\rho(u_n)dxdt\\
&\le C_4+(h+k)\|g\|_\infty(\exp(\|g\|_{L^1(\mathbb{R})})\int_Q nT_n(u_n-\psi)^-dxdt\\
&\le C_4+hC_5\int_Q nT_n((u_n-\psi)^-)dxdt
\end{aligned}$$

with $C_5 = 2\|g\|_\infty(\exp(\|g\|_{L^1(\mathbb{R})})$

so, we obtain

$$\int_{\{k\le|u_n|\le k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt \le hC_6\int_Q nT_n((u_n-\psi)^-)dxdt$$

with $C_6=\max(\frac{C_4}{C_3},\frac{C_5}{C_3})$,

Let us take $\rho_1(u_n)=\displaystyle\int_0^{u_n} g(s)\chi_{\{|s|\le k\}}ds\exp(G(u_n))$ a test function in the approximate $(\mathcal{P}_n)$, we obtain

$$\begin{aligned}\Big[\int_\Omega \varphi_2(u_n)dx\Big]_0^T &+ \int_Q |\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))\chi_{\{|u_n|\le k\}}dxdt\\ &-\int_Q nT_n((u_n-\psi)^-)\rho_1(u_n)dxdt\\ &\le \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)}\\ &\quad+\int_Q F\nabla u_n g(u_n)\chi_{\{|u_n|\le k\}}\exp(G(u_n))dxdt\\ &\quad\quad+\Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n|g(u_n)\exp(G(u_n))\chi_{\{|u_n|\le k\}}dxdt,\end{aligned}$$

where $\varphi_2(r)=\displaystyle\int_0^r\rho_1(s)ds$, which implies, using $\varphi_2(r)\ge 0$, by using Young's Inequality, we obtain

$$\begin{aligned}&\int_{\{|u_n|\le k\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\\ &\le \|g\|_\infty\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)}+\|u_0\|_{L^1(\Omega)}\Big]+C_9\\ &\quad+\frac{1}{2}\int_{\{|u_n|\le k\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\\ &\quad\quad+\frac{1}{2}\|g\|_\infty\int_{\{|u_n|\le k\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt,\end{aligned}$$

then

$$C_{10}\int_{\{|u_n|\le k\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\le hC_{11}\int_Q nT_n((u_n-\psi)^-)dxdt$$

Similarly, taking $\rho_2=\int_{u_n}^0 g(s)\chi_{\{|s|\ge k+h\}}ds\exp(G(u_n))$ as a test function in the approximate $(\mathcal{P}_n)$, we conclude that

$$C_{12}\int_{\{|u_n|\ge k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\le hC_{13}\int_Q nT_n((u_n-\psi)^-)dxdt$$

Conclusion, we have :

$$\begin{cases} C_{14}\int_Q |\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt \leq C_{12}\int_{\{|u_n|\geq k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\\ +C_{10}\int_{\{|u_n|\leq k\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\\ +C_5\int_{\{k\leq|u_n|\leq k+h\}}|\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt\\ \leq hC_{15}\int_Q nT_n((u_n-\psi)^-)dxdt\\ where\ \ C_{14}=Min(C_5,C_{10},C_{12})andC_{15}=Max(C_9,C_{11},C_{13}) \end{cases}$$

using (17), we have

$$\begin{aligned} \int_Q |\nabla u_n|^{p(x)}g(u_n)\exp(G(u_n))dxdt &\leq -\int_Q nT_n((u_n-\psi)^-)T_h\Big(u_n-T_k(u_n)\Big)\exp(G(u_n))dxdt\\ &\leq hC_{15}\int_Q nT_n(u_n-\psi)^-dxdt+hC_7, \end{aligned}$$

we obtain

$$-\int_Q nT_n(u_n-\psi)^-T_h\Big(u_n-T_k(u_n)\Big)\exp(G(u_n))dxdt\leq hC_6\int_Q nT_n(u_n-\psi)^-dxdt+hC_7$$

so, that

$$-\int_Q nT_n(u_n-\psi)^-\frac{T_h\Big(u_n-T_k(u_n)\Big)}{h}\exp(G(u_n))dxdt\leq C_6\int_Q nT_n(u_n-\psi)^-dxdt+C_7.$$

Let us now fix $k>\|\psi\|_\infty$, by the fact that $nT_n(u_n-\psi)(u_n-k)\chi_{\{u_n\leq\psi;k\leq u_n\leq k+h\}}\geq 0$ and letting $h\to 0$, one has

$$\int_Q nT_n(u_n-\psi)^-dxdt\leq C_8. \tag{18}$$

Let use $v=T_k(u_n)^+\exp(G(u_n))\chi(0,\tau)$ as a test function in $(\mathcal{P}_n)$

$$\begin{aligned} \Big[\int_\Omega \varphi_4(u_n)dx\Big]_0^T &+\int_{Q^\tau}(|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_k(u_n)^+\exp(G(u_n))dxdt\\ &+\int_{Q^\tau}|\nabla u_n|^{p(x)}T_k(u_n)^+g(u_n)\exp(G(u_n))dxdt\\ &-\int_{Q^\tau}nT_n((u_n-\psi)^-)T_k(u_n)^+\exp(G(u_n))dxdt\\ &=\int_{Q^\tau}g(u_n)|\nabla u_n|^{p(x)}T_k(u_n)^+\exp(G(u_n))dxdt\\ &+\int_{Q^\tau}f_nT_k(u_n)^+\exp(G(u_n))dxdt\\ &+\int_{Q^\tau}FT_k(u_n)^+\nabla u_n g(u_n)\exp(G(u_n))dxdt\\ &+\int_{Q^\tau}F\nabla T_k(u_n)^+\exp(G(u_n))dxdt \end{aligned}$$

where $\varphi_4(r) = \int_0^r T_k(s)^+ \exp(G(s))ds$.
Due the definition of φ_4 and $|G(u_n)| \leq \exp(\|g\|_{L^1(\mathbb{R})})\|u_{0n}\|_{L^1(\Omega)}$, we have $0 \leq \int_\Omega \varphi_4(u_{0n})dx \leq k\exp(\|g\|_{L^1(\mathbb{R})})\|u_{0n}\|_{L^1(\Omega)}$. By using (18) then,

$$\begin{aligned}\int_{Q^\tau} |\nabla T_k(u_n)^+|^{p(x)} \exp(G(u_n))dxdt \\ \leq k\exp(\|g\|_{L^1(\mathbb{R})})\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)} + C_8\Big] + C_{16} \\ +\frac{1}{2}\int_{Q^\tau} |\nabla T_k(u_n)^+|^{p(x)} \exp(G(u_n))dxdt \\ +\frac{1}{2}\|g\|_\infty \int_{Q^\tau} |\nabla T_k(u_n)^+|^{p(x)} \exp(G(u_n))dxdt.\end{aligned}$$

Let us take $\rho_5(u_n) = \int_0^{u_n} g(s)\chi_{\{s\geq 0\}}ds \exp(G(u_n))$ a test function in the approximate $(\mathcal{P}_n)$, we obtain

$$\begin{aligned}\Big[\int_\Omega \varphi_5(u_n)dx\Big]_0^T + \int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))\chi_{\{u_n\geq 0\}}dxdt \\ -\int_Q nT_n((u_n-\psi)^-)\rho_5(u_n)dxdt \\ \leq \Big(\int_0^\infty g(s)ds\Big)\exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\|f_n\|_{L^1(Q)} \\ +\int_Q F\nabla u_n g(u_n)\chi_{\{u_n\geq 0\}}\exp(G(u_n))dxdt \\ +\Big(\int_0^\infty g(s)ds\Big)\int_Q |F\nabla u_n| g(u_n)\exp(G(u_n))\chi_{\{u_n\geq 0\}}dxdt,\end{aligned}$$

where $\varphi_5(r) = \int_0^r \rho_5(s)ds$, which implies, using $\varphi_5(r) \geq 0$, by using Young's Inequality, we obtain

$$\begin{aligned}\int_{\{u_n\geq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\ \leq \|g\|_\infty \exp\Big(\|g\|_{L^1(\mathbb{R})}\Big)\Big[\|f_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}\Big] + C_{16} \\ +\frac{1}{2}\int_{\{u_n\geq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \\ +\frac{1}{2}\|g\|_\infty \int_{\{u_n\geq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt,\end{aligned}$$

then

$$\int_{\{u_n\geq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq C_{17}$$

Similarly, taking $\rho_6 = \int_{u_n}^0 g(s)\chi_{\{s\leq 0\}}ds \exp(G(u_n))$ as a test function in the approximate $(\mathcal{P}_n)$, we conclude that

$$\int_{\{u_n\leq 0\}} |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq C_{18}$$

Consequently,

$$\int_Q |\nabla u_n|^{p(x)} g(u_n)\exp(G(u_n))dxdt \leq C_{19}.$$

Above, $C_1, ..., C_{19}$ are constants independent of n, we deduce that

$$\int_Q |\nabla T_k(u_n)^+|^{p(x)} dxdt \leq k\ C_{20}. \tag{19}$$

Similarly to (19), we take $\varphi = T_k(u_n)^-\chi(0,\tau)$ in $(\mathcal{P}_n)$ to deduce that

$$\int_Q |\nabla T_k(u_n)^-|^{p(x)} dxdt \leq k\ C_{21}. \tag{20}$$

Combining (19), (20) and Proposition 2.3, we conclude that

$$\int_0^T \min\Big\{\|\nabla T_k(u_n)\|_{p(.)}^{p^+}, \|\nabla T_k(u_n)\|_{p(.)}|^{p^-}\Big\}dt \leq \rho(\nabla T_k(u_n)) \leq kC_{22}.$$

$$\|T_k(u_n)\|_{L^{p^-}(0,T;W_0^{1,p(.)}(\Omega))} \leq\ k\ C_{22}.$$

Then, we conclude that $T_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, independently of n for any $k > 0$. Now we turn to proving the almost everywhere convergence of u_n.
Consider a non decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s) = \begin{cases} s & if\ |s| \leq \frac{k}{2} \\ k & if\ |s| \geq k. \end{cases}$$

Multiplying the approximate equation by $g_k'(u_n)$, we get

$$\begin{aligned} &\frac{\partial g_k(u_n)}{\partial t} - \operatorname{div}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n)g_k'(u_n)\Big) + |\nabla u_n|^{p(x)}g_k''(u_n) \\ &\qquad -nT_n(u_n-\psi)^- g_k'(u_n) \\ &\qquad = g(u_n)|\nabla u_n|^{p(x)}g_k'(u_n) + f_n g_k'(u_n) - \operatorname{div}(Fg_k'(u_n)) + F\nabla u_n g_k''(u_n) \end{aligned} \tag{21}$$

in the sense of distributions. This implies, thanks to the fact that g_k' has compact support, that $g_k(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, while it's time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q)+V^*$ Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q, which implies that the sequence u_n converge almost everywhere to some measurable function v in Q. Thus by using the same argument as in [16], [17], [18], we can show the following lemma.

Lemma 3.2. *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then,*

$$u_n \rightarrow u \quad a.e. \quad in \quad Q. \tag{22}$$

We can deduce from (19) that

$$T_k(u_n) \rightharpoonup T_k(u) \qquad in \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)). \tag{23}$$

Lemma 3.3. [12] *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla(u_n))\nabla u_n dxdt = 0. \tag{24}$$

Step 3: **Almost every convergence of the gradients** :
This step is devoted to introducing for a fixed $k \geq 0$ fixed, a time regularization of the function $T_k(u)$ in order to perform the monotonicity method.
This specific time regularization of $T_k(u)$ (for fixed $k \geq 0$) is defined as follows. Let $(v_0^\mu)_\mu$ be a sequence of functions defined on Ω such that

$$v_0^\mu \in L^\infty(\Omega)\cap W_0^{1,p(x)}(\Omega) \quad for\ all\ \mu > 0, \tag{25}$$

$$\|v_0^\mu\|_{L^\infty(\Omega)} \le k \quad for\ all\ \mu > 0, \tag{26}$$

$$v_0^\mu \rightarrow T_k(u_0)\ a.e.\ in\ \Omega\ and\ \frac{1}{\mu}\|v_0^\mu\|_{L^{p(x)}(\Omega)} \rightarrow 0,\ as\ \mu\rightarrow\infty. \tag{27}$$

For fixed k, $\mu > 0$, let us consider the unique solution $(T_k(u))_\mu \in L^\infty(Q)\cap L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ of the monotone problem:

$$\frac{\partial (T_k(u))_\mu}{\partial t} + \mu\Big((T_k(u))_\mu - T_k(u)\Big) = 0 \quad in\ D'(Q), \tag{28}$$

$$(T_k(u))_\mu(t=0) = v_0^\mu \quad in \quad \Omega. \tag{29}$$

Note that due to (28), we have for $\mu > 0$ and $k \geq 0$,

$$\frac{\partial (T_k(u))_\mu}{\partial t} \in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)). \tag{30}$$

We just recall here that (28)–(29) imply that

$$(T_k(u))_\mu \rightarrow T_k(u)\ a.e.\ in\ Q, \tag{31}$$

as well as weakly in $L^\infty(Q)$ and strongly in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ as $\mu\rightarrow\infty$. Note that for any μ and any $k \geq 0$, we have

$$\|(T_k(u))_\mu\|_{L^\infty(Q)} \le \max\Big(\|T_k(u)\|_{L^\infty(Q)};\ \|v_0^\mu\|_{L^\infty(\Omega)}\Big) \le k. \tag{32}$$

We introduce a sequence of increasing $C^\infty(\mathbb{R})$–functions S_m such that

$$S_m(r) = r\ for\ |r|\le m,\ supp(S_m') \subset [-(m+1), m+1], \|S_m''\|_{L^\infty(\mathbb{R})} \le 1,$$

for any $m \geq 1$, and we denote by $\omega(n,\mu,\eta,m)$ the quantities such that

$$\lim_{m\to\infty}\lim_{\eta\to 0}\lim_{\mu\to\infty}\lim_{n\to\infty}\omega(n,\mu,\eta,m) = 0.$$

Lemma 3.4. *([12, 18]). We have*

$$\int_0^T \left\langle \frac{\partial u_n}{\partial t}, T_\eta\Big(u_n-(T_k(u))_\mu\Big)^+ \exp(G(u_n))S'_m(u_n)\right\rangle dt \geq \omega(n,\mu,\eta)\ \forall m\geq 1. \tag{33}$$

Taking now $v = T_\eta\Big(u_n-(T_k(u))_\mu\Big)^+ S'_m(u_n)\exp(G(u_n))$, in $(\mathcal{P}_n)$ and by adapting the same way as in [1, 11, 10, 9, 8, 6], we obtain

$$\int_Q \Big(\Big[|\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n)) - |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u))\Big]\times \\ \times\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]\Big)^\theta\, dx\, dt = w(n) \tag{34}$$

which implies that

$$T_k(u_n)\to T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))\ \forall\, k\geq 0. \tag{35}$$

By using [16, 17], there exist a subsequence also denoted by u_n such that

$$\nabla u_n \to \nabla u \ \text{ a.e. in } \ Q. \tag{36}$$

Proposition 3.2. *Let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then $u\geq\psi$ a.e. in Q.*

Proof. Thanks to (18), we can write $\int_Q T_n(u_n-\psi)^- dxdt \leq \frac{C}{n}$. And by using Fatou's Lemma as $n\to\infty$, we have that $\int_Q (u-\psi)^- dxdt$ converges to zeros, we get $(u-\psi)^- = 0$ a.e. in Q. Consequently we conclude that $u\geq\psi$ a.e. in Q.

Step 4:**Passing to the limit**

a) We claim that $u\in C(0,T;L^1(\Omega))$ .We will show that

$$u_n\to u \quad in \quad C(0,T;L^1(\Omega)).$$

Since $T_k(u)\in K_\psi$, for every $k\geq \|\psi\|_{L^\infty}$ there exists a sequence $v_j\in K_\psi\cap D(\bar{Q})$ such that

$$v_j\to T_k(u) \quad in \quad L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$$

for the modular convergence.

Let $\omega^{i,l}_{j,\mu} = (T_l(v_j))_\mu + e^{-\mu t}T_l(\eta_i)$ with $\eta_i\geq 0$ converge to u_0 in $L^1(\Omega)$, where $(T_l(v_j))_\mu$ is the mollification of $T_l(v_j)$ with respect to time. Choosing now $v = T_k(u_n-\omega^{i,l}_{j,\mu})\chi_{(0,\tau)}$ as test function in $(\mathcal{P}_n)$, we get

$$\begin{aligned}\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n-\omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau} &+ \int_{Q^\tau}|\nabla u_n|^{p(x)-2}\nabla u_n\nabla T_k(u_n-\omega^{i,l}_{j,\mu})dxdt \\ &- \int_{Q^\tau} nT_n(u_n-\psi)^- T_k(u_n-\omega^{i,l}_{j,\mu})dxdt \\ &= \int_{Q^\tau} g(u_n)|\nabla u_n|^{p(x)}T_k(u_n-\omega^{i,l}_{j,\mu})dxdt + \int_{Q^\tau} f_nT_k(u_n-\omega^{i,l}_{j,\mu})dxdt \\ &\quad + \int_{Q^\tau} F\nabla T_k(u_n-\omega^{i,l}_{j,\mu})dxdt.\end{aligned} \tag{37}$$

By using the fact that

$$-\int_{Q^\tau} nT_n(u_n-\psi)^- T_k(u_n-\omega_{j,\mu}^{i,l})dxdt \geq 0$$

, we deduce that:

$$\Big\langle \frac{\partial u_n}{\partial t}, T_k(u_n-\omega_{j,\mu}^{i,l})\Big\rangle_{Q^\tau} + \int_{Q^\tau} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla T_k(u_n-\omega_{j,\mu}^{i,l})dxdt$$
$$= \int_{Q^\tau} g(u_n)|\nabla u_n|^{p(x)} T_k(u_n-\omega_{j,\mu}^{i,l})dxdt + \int_{Q^\tau} f_n T_k(u_n-\omega_{j,\mu}^{i,l})dxdt$$
$$+ \int_{Q^\tau} F\nabla T_k(u_n-\omega_{j,\mu}^{i,l})dxdt.$$

On the one hand, we have

$$\begin{aligned} I &= \int_{Q^\tau} |\nabla u_n|^{p(x)-2}\nabla u_n \nabla T_k(u_n-\omega_{j,\mu}^{i,l})dxdt \\ &= \int_{\{|T_k(u_n)-\omega_{j,\mu}^{i,l}|\leq k\}} |\nabla u_n|^{p(x)-2}\nabla u_n [\nabla T_k(u_n)-\nabla\omega_{j,\mu}^{i,l}]dxdt. \end{aligned} \tag{38}$$

In the following, we pass to the limit in (38): first we let n tend to ∞,

$$\begin{aligned} I &= \int_{Q^\tau} |\nabla T_k(u_n)|^{p(x)-2}\nabla T_k(u_n) \nabla T_k(u_n-\omega_{j,\mu}^{i,l})dxdt \\ &= \int_{\{|T_k(u)-\omega_{j,\mu}^{i,l}|\leq k\}} |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\Big[\nabla T_k(u)-\nabla\omega_{j,\mu}^{i,l}\Big]dxdt + \epsilon(n) \end{aligned}$$

and let μ tend to ∞. Then,

$$I = \int_{\{|T_k(u)-T_l(v_j)|\leq k\}} |\nabla T_k(u)|^{p(x)-2}\nabla T_k(u)\Big[\nabla T_k(u)-\nabla T_l(v_j)\Big]dxdt + \epsilon(n,\mu),$$

finally j tend to ∞, we have

$$I = \epsilon(n,\mu,j,l).$$

On the other hand, we have

$$J = \int_{Q^\tau} g(u_n)|\nabla u_n|^{p(x)} T_k(u_n-\omega_{j,\mu}^{i,l})dxdt. \tag{39}$$

In the following, we pass to the limit in (39): first we let n tend to ∞, since

$$g(u_n)|\nabla u_n|^{p(x)} \to g(u)|\nabla u|^{p(x)} \ \text{ in } L^1(Q),$$

we obtain $J = \int_{Q^\tau} g(u)|\nabla u|^{p(x)} T_k(u-\omega_{j,\mu}^{i,l})dxdt + \epsilon(n)$ and let μ tend to ∞ and $j \to \infty$, we have

$$J = \epsilon(n,\mu,j,l).$$

Similarly to (39) and by using (15), we have

$$\int_{Q^\tau} f_n\Big[T_k(u_n-\omega_{j,\mu}^{i,l})\Big]dxdt = \epsilon(n,\mu,j,l)$$

and we have

$$\int_{Q^\tau} F\nabla\Big[T_k(u_n-\omega^{i,l}_{j,\mu})\Big]dxdt=\epsilon(n,\mu,j,l)$$

and by using Vitali's theorem, we get

$$\limsup_{k\to\infty}\limsup_{i\to 0}\limsup_{j\to\infty}\limsup_{\mu\to\infty}\lim_{n\to\infty}\Big\langle\frac{\partial u_n}{\partial t},T_k(u_n-\omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau}\leq 0. \tag{40}$$

We have (see([1]))

$$\Big\langle\frac{\partial\omega^{i,l}_{j,\mu}}{\partial t},T_k(u_n-\omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau}=\mu\int_{Q^\tau}(T_k(v_j)-\omega^{i,l}_{j,\mu}))T_k(u_n-\omega^{i,l}_{j,\mu})\geq\epsilon(n,j,\mu,l). \tag{41}$$

uniformly on τ. Therefore, by writing

$$\int_\Omega S_k\Big(u_n(\tau)-\omega^{i,l}_{j,\mu}(\tau)\Big)dx=\Big\langle\frac{\partial u_n}{\partial t},T_k(u_n-\omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau}$$
$$-\Big\langle\frac{\partial\omega^{i,l}_{j,\mu}}{\partial t},T_k(u_n-\omega^{i,l}_{j,\mu})\Big\rangle_{Q^\tau}+\int_\Omega S_k\Big(u_n(0)-T_l(\eta_i)\Big)dx \tag{42}$$

and using (40) and (41) and (42), we see that

$$\int_\Omega S_k\Big(u_n(\tau)-\omega^{i,l}_{j,\mu}(\tau)\Big)dx\leq\epsilon(n,j,\mu,l), \tag{43}$$

which implies, by writing

$$\int_\Omega S_k\Big(\frac{u_n(\tau)-u_m(\tau)}{2}\Big)dx\leq\frac{1}{2}\Big(\int_\Omega S_k(u_n(\tau)-\omega^{i,l}_{j,\mu}(\tau)\Big)dx$$
$$+\int_\Omega S_k\Big(u_m(\tau)-\omega^{i,l}_{j,\mu}(\tau))dx\Big) \tag{44}$$

that

$$\int_\Omega S_k\Big(\frac{u_n(\tau)-u_m(\tau)}{2}\Big)dx\leq\epsilon_1(n,m).$$

We deduce then that

$$\int_\Omega|u_n(\tau)-u_m(\tau)|dx\leq\epsilon_2(n,m),\ \text{not depending on}\ \tau \tag{45}$$

and thus (u_n) is a Cauchy sequence in $C(0,T;L^1(\Omega))$ and since $u_n\to u$, a.e. in Q, we deduce that

$$u_n\to u\quad in\ C(0,T;L^1(\Omega)). \tag{46}$$

b) we prove that u satisfies (14)

Indeed, let $v \in K_\psi \cap L^\infty(Q)$, $\frac{\partial v}{\partial t} \in L^{(p^-)'}(0,T;W^{-1,p(x)}(\Omega))$. By the pointwise multiplication of the approximate problem $(\mathcal{P}_n)$ by $T_k(u_n - v)$, we get

$$\begin{aligned}
&\int_\Omega S_k\Big(u_n(T) - v(T)\Big)dx - \int_\Omega S_k\Big(u_{0n} - v(0)\Big)dx \\
&\quad + \int_Q \frac{\partial v}{\partial t} T_k(u_n - v)dxdt + \int_Q (|\nabla u|^{p(x)-2}\nabla u)\nabla T_k(u_n - v)dxdt \\
&\quad - \int_Q \phi_n(u_n)\nabla T_k(u_n - v)dxdt \\
&\quad - \int_Q nT_n(u_n - \psi)^- T_k(u_n - v)dxdt \\
&\quad = \int_Q g(u_n)|\nabla u_n|^{p(x)} T_k(u_n - v)dxdt \\
&\quad + \int_Q f_n T_k(u_n - v)dxdt + \int_Q F\nabla T_k(u_n - v)dxdt,
\end{aligned}$$

where $S_k(s) = \displaystyle\int_0^s T_k(r)dr$.
Since $v \in K_\psi \cap L^\infty(Q)$, we have $-\int_Q nT_n(u_n - \psi)^- T_k(u_n - v)dxdt \geq 0$, we deduce that

$$\begin{aligned}
&\int_\Omega S_k\Big(u_n(T) - v(T)\Big)dx - \int_\Omega S_k\Big(u_{0n} - v(0)\Big)dx + \int_Q \frac{\partial v}{\partial t} T_k(u_n - v)dxdt \\
&\quad + \int_Q (|\nabla u|^{p(x)-2}\nabla u)\nabla T_k(u_n - v)dxdt - \int_Q \phi_n(u_n)\nabla T_k(u_n - v)dxdt \\
&\quad \leq \int_Q g(u_n)|\nabla u_n|^{p(x)} T_k(u_n - v)dxdt \\
&\quad + \int_Q f_n T_k(u_n - v)dxdt + \int_Q F\nabla T_k(u_n - v)dxdt.
\end{aligned} \tag{47}$$

• Let us pass to the limit with $n \to \infty$ in each term in (47). we saw that $u_n \to u$ in $C(0,T,L^1(\Omega))$. Therefore $u_n(t) \to u(t)$ in $L^1(\Omega)$ for all $t \leq T$.
As S_k is lipschitz of coefficient k, when $n \to \infty$, we have

$$\int_\Omega S_k(u_n - v)(T)dx \to \int_\Omega S_k(u - v)(T)dx$$
$$\textit{and} \int_\Omega S_k(u_n - v)(0)dx = \int_\Omega S_k(u_{0n} - v(0))dx \to \int_\Omega S_k(u_0 - v(0))dx.$$

• Since $\frac{\partial v}{\partial t} \in L^{(p^-)'}(0,T;W^{-1,p'(x)}(\Omega))$, that is

$$\int_0^T \Big\langle \frac{\partial v}{\partial t}, T_k(u_n - v)\Big\rangle dt \to \int_0^T \Big\langle \frac{\partial v}{\partial t} T_k(u - v)\Big\rangle dt.$$

• Let us pass to the limit with $n \to \infty$ for the term $\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_k(u_n - v)dxdt$.
Since $v \in L^\infty(Q)$, we note $M = \|v\|_\infty$, we get

$$\begin{aligned}
&\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_k(u_n - v)dxdt \\
&\qquad = \int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_k(T_{k+M}(u_n) - v)dxdt \\
&\qquad = \int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_{k+M}(u_n))\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt \\
&\qquad - \int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt.
\end{aligned}$$

As $T_{k+M}(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, $\nabla u_n \to \nabla u$ a.e. in Q,

$$\nabla T_{k+M}(u_n) \to \nabla T_{k+M}(u) \text{ almost everywhere}$$

and by using Lebesgue theorem, we deduce that

$$\begin{aligned}
&\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla T_{k+M}(u_n))\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt \\
&\qquad \to \int_Q (|\nabla u|^{p(x)-2}\nabla u)\mathbf{1}_{\{|T_{k+M}(u-v)|\le k\}}dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n)\nabla v\mathbf{1}_{\{|T_{k+M}(u_n)-v|\le k\}}dxdt \to \\
&\qquad \int_0^T \int_\Omega (|\nabla u|^{p(x)-2}\nabla u)\nabla v\mathbf{1}_{\{|T_{k+M}(u-v)|\le k\}}dxdt
\end{aligned}$$

then

$$\int_Q (|\nabla u_n|^{p(x)-2}\nabla u_n)T_k(u_n - v)dxdt \to \int_Q (|\nabla u|^{p(x)-2}\nabla u)T_k(u - v)dxdt.$$

• Let us pass to the limit for other term

$$\int_Q \phi_n(u_n)\nabla T_k(u_n - v)dxdt = \int_Q \phi_n(u_n)\nabla T_k(T_{k+M}(u_n) - v)dxdt$$

Since $T_{k+M}(u_n)$ is bounded in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ and ϕ_n is a Lipschitz continuous bounded function $\mathbb{R}$ in to $\mathbb{R}^N$, such that ϕ_n uniformly converge to ϕ on any compact subset of $\mathbb{R}$ as n tend to $+\infty$. Then, we have

Due to (11) , $T_k(u_n) \to T_k(u)$ *in* $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$ $\forall\, k \ge 0$ and $u_n \to u$ a.e. in Q, we have

$$f_nT_k(u_n - v) \to fT_k(u - v) \text{ strongly in } L^1(Q)$$

and by Lebesgue theorem, we have

$$\int_Q f_nT_k(u_n - v) \to \int_Q fT_k(u - v) \textit{ strongly in } L^1(Q).$$

• Similarly, since g is a bounded and continuous functions belong to $L^1(\mathbb{R})$ and $u_n \to u$ a.e. in Q, we obtain

$$\int_Q g(u_n)|\nabla u_n|^{p(x)-2}T_k(u_n - v) \to \int_Q g(u)|\nabla u|^{p(x)-2}T_k(u - v) \textit{ strongly in } L^1(Q).$$

• For the second term of the right hand side of (47), we have

$$\int_Q F\nabla T_k(u_n - v) \to \int_Q F\nabla T_k(u - v) \; asn \to +\infty,$$

since $\nabla T_k(u_n - v) \to \nabla T_k(u - v)$ in $(L^{p(x)}(Q))^N$, while $F \in (L^{p'(x)}(Q))^N$ and Lebesgue theorem. Then, we conclude that u satisfies (14).
As a conclusion of Step 1 to Step 4, the proof of Theorem 3.1 is complete.□

References

[1] R. Aboulaich, B. Achchab, D. Meskine, and A. Souissi, *Parabolic inequalities with nonstandard growths and L^1 data*, Boundary Value Problems, Volume **2006**, pp. 1–18.

[2] Y. Akdim, J. Bennouna, M. Mekkour, *Solvability of degenerated parabolic equations without sign condition and three unbounded nonlinearities* Electronic Journal of Differential Equations,Volume **2011**(2011), no. 03, pp. 1–26.

[3] L. Aharouch, E. Azroul and M. Rhoudaf, *Strongly nonlinear variational degenerated parabolic problems in weighted sobolev spaces.* Aust. J. Math. Anal. Appl., Vol. 5, Article 13, pp 1-25, 2008.

[4] Aberqi, A.; Bennouna, J.; Mekkour, M.; Redwane, H. Existence results for a nonlinear parabolic problems with lower order terms. Int. J. Math. Anal. (Ruse) 7 (2013), no. 25-28, 1323-1340.

[5] Akdim, Y.; Bennouna, J.; Bouajaja, A.; Mekkour, M. Renormalized solutions for nonlinear parabolic systems with three unbounded nonlinearities in weighted Sobolev spaces. Anal. Theory Appl. 29 (2013), no. 3, 234-254.

[6] Akdim, Youssef; El Gorch, Nezha; Mekkour, Mounir Nonlinear parabolic inequalities in Sobolev space with variable exponent. J. Nonlinear Evol. Equ. Appl. 2015, 67-90.

[7] Akdim, Y.; Bennouna, J.; Mekkour, M.; Redwane, H. Existence of renormalized solutions for parabolic equations without the sign condition and with three unbounded nonlinearities. Appl. Math. (Warsaw) 39 (2012), no. 1, 1-22.

[8] Akdim, Y.; Bennouna, J.; Bouajaja, A.; Mekkour, M.; Redwane, H. Entropy unilateral solutions for strongly nonlinear parabolic problems without sign condition and via a sequence of penalized equations. Int. J. Math. Stat. 12 (2012), no. 2, 113-134.

[9] Akdim, Y.; Bennouna, J.; Mekkour, M.; Redwane, H. Strongly nonlinear parabolic inequality in Orlicz spaces via a sequence of penalized equations. Afr. Mat. 26 (2015), no. 7-8, 1669-1695.

[10] Aberqi, A., Bennouna, J., Mekkour, M., Redwane, H. Nonlinear parabolic inequalities with lower order terms . Applicable Analysis Volume 96-Issue 12, pp. 1-16(2017).

[11] Bennouna, J.; El hamdaoui, B.; Mekkour, M.; Redwane, H. Nonlinear parabolic inequalities in Lebesgue-Sobolev spaces with variable exponent. Ric. Mat. 65 (2016), no. 1, 93-125.

[12] Y. Akdim, J. Bennouna, M. Mekkour, *Solvability of degenerated parabolic equations without sign condition and three unbounded nonlinearities* Electron. J. Diff. Eqns., Vol. 2011(2011), No. 03, pp. 1-26.

[13] A. Azroul, H. Redwane and M. Rhoudaf, *Existence of a Renormalized solution for a class of nonlinear parabolic equations in Orlicz spaces*, Portugal. Math. 1, Vol. 66,(2009), pp. 29-63.

[14] M. Bendahmane, P. Wittbold, A. Zimmermann, *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1-data* J. Differential equations 249 (2010), pp. 1483-1515.

[15] A.Benkirane, A.Elmahi, *A stongly nonlinear elliptic equation having natural growth terms and L^1 data*, Nonlinear Anal., Theory, Methods and Appl., 39(2000), pp. 403-411.

[16] D. Blanchard, F. Murat and H. Redwane, *Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic Problems*, J. Differential Equations, (177) (2001), pp. 331-374.

[17] D. Blanchard and F. Murat, *Renormalized solutions of nonlinear parabolic problems with L^1-data: existence and uniqueness*, Proc. Roy. Soc. Edinburgh Sect., A127,(1997), pp. 1137-1152.

[18] D. Blanchard, and H. Redwane, *Renormalized solutions of nonlinear parabolic evolution problems*, J. Math. Pure Appl.,77, (1998), pp. 117-151.

[19] A. Dall'Aglio, L. Orsina, *Nonlinear parabolic equations with natural growth conditions and L^1 data*, Nonlinear Anal., Theory Methods Appl., 27 (1996), pp. 59-73.

[20] X. L Fan, D. Zhao, *on the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*. J. Math Anal, Appl 263 (2001), pp. 424-446.

[21] X. L Fan, D. Zhao, *the generalised Orlicz-Sobolev space $W^{k,p(x)}(\Omega)$*, J. Gansu Educ, College 12(1)(1998),pp. 1-6.

[22] M. Kbiri Alaoui , D. Meskine, A. Souissi, *On some class of nonlinear parabolic inequalities in Orlicz spaces*, Nonlinear Anal., Theory Methods Appl. 74 (2011), pp. 5863-5875.

[23] O. Kovacik, J. Rakonsik, *On spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$* Czechoslovak Math J. 41(116)(1991), pp. 592-618.

[24] J. L. Lions, Quelques mèthode de rèsolution des problèmes aux limites non linèaires, Dunod, Paris 1969.

[25] A. Porretta, *Existence results for nonlinear parabolic equations via strong convergence of truncation*, Ann. Mat. Pura Appl. 177(4)(1999), pp.143-172.

[26] J. Simon, *Compact sets in the space $L^p(0,T,B)$*, Ann. Mathematical. Pura and Applications., **146** (1987), pp. 65–96.

[27] D. Zhao, W, J. Qiang, X. L Fan, *On generalised Orlicz-Sobolev spaces $L^{p(x)}(\Omega)$* J. Gansu Sci. 9(2), 1997, pp. 1-7.

Sidi Mohamed Ben Abdellah University, Poly-disciplinary Faculty of Taza, Laboratory LSI, Department MPI, P.O.Box 1223 Taza Gare, Morocco

Email address: mounir.mekkour@usmba.ac.ma