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SOME RESULTS INVOLVING AUTOMORPHISMS IN PRIME RINGS

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Abstract. In this paper, we study the commutativity of prime rings admitting automorphisms which satisfy certain identities.

Throughout this paper, $\mathcal{R}$ will represent an associative ring with centre $Z(\mathcal{R})$. The symbol $[x, y]$ will denote the commutator $xy - yx$ for all $x, y \in \mathcal{R}$, while the symbol will stand for the anti-commutator $xy + yx$ for all $x, y \in \mathcal{R}$. $\mathcal{R}$ is prime if $a\mathcal{R}b = \{0\}$ implies $a = 0$ or $b = 0$. We say that a mapping $f : \mathcal{R} \to \mathcal{R}$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in mathcalR$. The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory, and ring theory [5], [10]. A mapping $f : \mathcal{R} \to \mathcal{R}$ is said to be strong commutativity preserving (SCP) on a subset S of $\mathcal{R}$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. In [4], Bell and Daif investigated the commutativity in rings admitting a derivation which is SCP on nonzero right ideal. Indeed, they proved that if a semiprime ring $\mathcal{R}$ admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of $\mathcal{R}$, then $I \subseteq Z(\mathcal{R})$. In particular, $\mathcal{R}$ is commutative if $I = \mathcal{R}$. Later, Deng and Ashraf [7] proved that if there exists a derivation d of a semiprime ring $\mathcal{R}$ and a mapping $f : I \to \mathcal{R}$ defined on a nonzero ideal I of $\mathcal{R}$ such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then $\mathcal{R}$ contains a nonzero central ideal. In particular, they showed that $\mathcal{R}$ is commutative if $I = \mathcal{R}$. Further, Ali and Huang [2] showed that if $\mathcal{R}$ is a 2-torsion free semiprime ring and d is a derivation of $\mathcal{R}$ satisfying $[d(x), d(y)] \pm [x, y] = 0$ for all x, y in nonzero ideal I of $\mathcal{R}$ or $d(x) \circ d(y)] \pm x \circ y = 0$ for all x, y in nonzero ideal I of $\mathcal{R}$, then $\mathcal{R}$ contains a nonzero central ideal. Many related generalizations of these results can be found in the literature (see for example [6], [8] etc). Motivated by these results, we will investigate some commutativity theorems for a nonzero ideal and automorphisms α and β of rings by examining the following properties: $[\alpha(x), \beta(y)] \pm [x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{U}$ and $\alpha(x) \circ \beta(y) \pm x \circ y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{U}$.

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1. Prime rings and Automorphisms

We shall do a great deal of calculations with commutators and anti-commutators, routinely using the following basic identities: $[xy,z]=x[y,z]+[x,z]y$, $[x,yz]=y[x,z]+[x,y]z$, $x\circ(yz)=(x\circ y)z-y[x,z]=y(x\circ z)+[x,y]z$ and $(xy)\circ z=x(y\circ z)-[x,z]y=(x\circ z)y+x[y,z]$ for all $x,y,z\in\mathcal{R}$.

Theorem 1.1. *Let $\mathcal{R}$ be a prime ring with char$\mathcal{R}\neq 2$, $\mathcal{U}$ a nonzero ideal of $\mathcal{R}$ and α, β automorphisms of $\mathcal{R}$, then the following assertions are equivalent:*

(i) $[\alpha(x),\beta(y)]-[x,y]\in Z(\mathcal{R})$ *for all* $x,y\in\mathcal{U}$;

(ii) $\mathcal{R}$ *is commutative or* $\alpha=\beta=id_{\mathcal{R}}$.

Proof. It is obvious that (ii) gives easily (i). For $(i)\Rightarrow(ii)$, we start treating the case $Z(\mathcal{R})=\{0\}$, so we have

$$[\alpha(x),\beta(y)]=[x,y] \quad \text{for all} \quad x,y\in\mathcal{U}. \tag{1.1}$$

Replacing x by ux in (1.1) and expanding it, yields that

$$\alpha(u)[\alpha(x),\beta(y)]+[\alpha(u),\beta(y)]\alpha(x)=u[x,y]+[u,y]x \quad \text{for all} \quad x,y,u\in\mathcal{U}. \tag{1.2}$$

From (1.1) and (1.2), we can easily arrive at

$$(\alpha(u)-u)[x,y]=[u,y](x-\alpha(x)) \quad \text{for all} \quad x,y,u\in\mathcal{U}. \tag{1.3}$$

For $y=u$, we get $(\alpha(u)-u)[x,u]=0$ for $x,u\in\mathcal{U}$. Replacing x by xt, where $t\in\mathcal{R}$ in the last equation and using it, we arrive at $(\alpha(u)-u)\mathcal{U}[u,t]=\{0\}$ for $u\in\mathcal{U}$, $t\in\mathcal{R}$, which implies that $(\alpha(u)-u)\mathcal{R}\mathcal{U}\mathcal{R}[u,t]=\{0\}$ for $u\in\mathcal{U}$, $t\in\mathcal{R}$. By primeness of $\mathcal{R}$, we get either $\alpha(u)=u$ or $u\in Z(\mathcal{R})$ for all $u\in\mathcal{U}$. The sets $H=\{u\in\mathcal{U}\mid u\in Z(\mathcal{R})\}$ and $K=\{u\in\mathcal{U}\mid \alpha(u)=u\}$ are additive subgroups of $\mathcal{U}$. But a group cannot be the union of proper subgroups. Hence we get $H=\mathcal{U}$ or $K=\mathcal{U}$ which force that $\mathcal{U}\subseteq Z(\mathcal{R})$ in this case $\mathcal{R}$ is commutative or $\alpha(u)=u$ for all $u\in\mathcal{U}$. The second case implies that $\alpha=id_{\mathcal{R}}$ by [3, Lemma 1 (i)]. In this case, equation (1.1) becomes

$$[x,\beta(y)]=[x,y] \quad \text{for all} \quad x,y\in\mathcal{U}. \tag{1.4}$$

Taking yt in place of y in (1.4) and expanding it, we arrive at

$$[x,y](t-\beta(t))=(\beta(y)-y)[x,t] \quad \text{for all} \quad x,y,t\in\mathcal{U}. \tag{1.5}$$

This equation is the same as (1.3). By the same arguments after (1.3), we conclude that $\mathcal{R}$ is commutative or $\beta=id_{\mathcal{R}}$, so finally we arrive at $\mathcal{R}$ is commutative or $\alpha=\beta=id_{\mathcal{R}}$.
Now suppose that $Z(\mathcal{R})\neq\{0\}$ and

$$[\alpha(x),\beta(y)]-[x,y]\in Z(\mathcal{R}) \quad \text{for all} \quad x,y\in\mathcal{U}. \tag{1.6}$$

Replacing x by x^2 in (1.6) and expanding it, we obtain

$$\alpha(x)[\alpha(x),\beta(y)]+[\alpha(x),\beta(y)]\alpha(x)-x[x,y]-[x,y]x\in Z(\mathcal{R}) \text{ for all } x,y\in\mathcal{U}. \tag{1.7}$$

It follows that

$$2\alpha(x)([\alpha(x),\beta(y)]-[x,y])+(\alpha(x)-x)[x,y]-[x,y](\alpha(x)-x)\in Z(\mathcal{R}). \tag{1.8}$$

Using equation (1.6), then (1.8) becomes

$$\alpha(x)(\alpha(x)-x)[x,y] = [x,y](\alpha(x)-x)\alpha(x) \text{ for all } x,y\in\mathcal{U}. \tag{1.9}$$

Taking $y[x,z]$ in place of y in (1.9), we can reformulate it to

$$\alpha(x)(\alpha(x)-x)y[x,[x,z]] = y[x,[x,z]](\alpha(x)-x)\alpha(x) \text{ for all } x,y,z\in\mathcal{U}. \tag{1.10}$$

Replacing y by ty, where $t\in\mathcal{R}$ in (1.10), we have

$$\begin{aligned}\alpha(x)(\alpha(x)-x)ty[x,[x,z]] &= ty[x,[x,z]](\alpha(x)-x)\alpha(x)\\ &= t\alpha(x)(\alpha(x)-x)y[x,[x,z]] \text{ for all } x,y,z\in\mathcal{U}, t\in\mathcal{R}\end{aligned}$$

which implies

$$[\alpha(x)(\alpha(x)-x),t]\mathcal{U}[x,[x,z]] = \{0\} \text{ for all } x,z\in\mathcal{U}, t\in\mathcal{R}. \tag{1.11}$$

By primeness of $\mathcal{R}$, (1.11) gives

$$\alpha(x)(\alpha(x)-x)\in Z(\mathcal{R}) \text{ or } [x,[x,z]]=0 \text{ for all } x,z\in\mathcal{U}. \tag{1.12}$$

Suppose there exists $x_0\in\mathcal{U}$ such $[x_0,[x_0,z]]=0$ for all $z\in\mathcal{U}$. Replacing z by zt, we obtain $[x_0,z[x_0,t]+[x_0,z]t]=0$ for all $z,t\in\mathcal{U}$ that is, $[x_0,z[x_0,t]]+[x_0,[x_0,z]t]=0$ for all $z,t\in\mathcal{U}$. By expanding the last expression and also using our assumption, we get $2[x_0,z][x_0,t]]=0$ for all $z,t\in\mathcal{U}$. Putting tz instead of t in the last expression and using it with $char\mathcal{R}\neq 2$, we can easily arrive at $[x_0,z]\mathcal{U}[x_0,z]=\{0\}$ for all $z\in\mathcal{U}$. In view of primeness of $\mathcal{R}$, we obtain $[x_0,z]=0$ for all $z\in\mathcal{U}$ and replacing z by zr, where $r\in\mathcal{R}$ and using it again, we conclude that $x_0\in Z(\mathcal{R})$. In this case, (1.12) becomes

$$\alpha(x)(\alpha(x)-x)\in Z(\mathcal{R}) \text{ or } x\in Z(\mathcal{R}) \text{ for all } x\in\mathcal{U}. \tag{1.13}$$

Suppose there exists $x_0\in\mathcal{R}$ such that $\alpha(x_0)(\alpha(x_0)-x_0)\in Z(\mathcal{R})$. Moreover, from the equation (1.9), we obtain $\alpha(x_0)(\alpha(x_0)-x_0)[x_0,y]=[x_0,y](\alpha(x_0)-x_0)\alpha(x_0)$ for all $y\in\mathcal{U}$ by simplifying this expression, we get $[x_0,y][x_0,\alpha(x_0)]=0$ for all $y\in\mathcal{U}$. Substituting yt for y, where $t\in\mathcal{R}$ in the last expression and using it, we find that $[x_0,y]\mathcal{R}[x_0,\alpha(x_0)]=\{0\}$ for all $y\in\mathcal{U}$. By primeness of $\mathcal{R}$, one can easily deduce that $x_0\in Z(\mathcal{R})$ or $[x_0,\alpha(x_0)]=0$, the two last expressions give $[x_0,\alpha(x_0)]=0$. we make a retour to equation (1.13), we conclude that $[\alpha(x),x]=0$ for all $x\in\mathcal{U}$. By [9, Theorem 1], we conclude that $\mathcal{R}$ is commutative. □

To avoid repetition, we use the same proof with minor changes we find the following result:

Theorem 1.2. *Let $\mathcal{R}$ be a prime ring with $char\mathcal{R}\neq 2$, $\mathcal{U}$ a nonzero ideal of $\mathcal{R}$ and α, β automorphisms of $\mathcal{R}$, then the following assertions are equivalent:*

(i) *$[\alpha(x),\beta(y)]+[x,y]\in Z(\mathcal{R})$ for all $x,y\in\mathcal{U}$;*
(ii) *$\mathcal{R}$ is commutative or $\alpha=\beta=-id_{\mathcal{R}}$.*

When $\alpha=\beta$, we have the following corollaries:

Corollary 1.3. *Let $\mathcal{R}$ be a prime ring with $char\mathcal{R}\neq 2$, α a non-identical automorphism of $\mathcal{R}$, and $\mathcal{U}$ a nonzero ideal of $\mathcal{R}$. Suppose that $[\alpha(x),\alpha(y)]-[x,y]\in Z(\mathcal{R})$ for all $x,y\in\mathcal{U}$, then $\mathcal{R}$ is commutative.*

Corollary 1.4. *Let $\mathcal{R}$ be a prime ring with with char$\mathcal{R} \neq 2$, α an automorphism of $\mathcal{R}$ such that $\alpha \neq -id$, and $\mathcal{U}$ a nonzero ideal of $\mathcal{R}$. Suppose that $[\alpha(x), \alpha(y)] + [x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{U}$, then $\mathcal{R}$ is commutative.*

Theorem 1.5. *Let $\mathcal{R}$ be a prime ring with char$\mathcal{R} \neq 2$, $\mathcal{U}$ a nonzero ideal of $\mathcal{R}$ and α, β automorphisms of $\mathcal{R}$, then the following assertions are equivalent:*

(i) $\alpha(x) \circ \beta(y) = x \circ y$ *for all* $x, y \in \mathcal{U}$;
(ii) $\alpha = \beta = id_{\mathcal{R}}$.

Proof. It is obvious that (ii) gives easily (i). We need only to prove the nontrivial implications. For $(i) \Rightarrow (ii)$, we first suppose that $Z(\mathcal{R}) = \{0\}$, so we have

$$\alpha(x) \circ \beta(y) = x \circ y \quad \text{for all} \quad x, y \in \mathcal{U}. \tag{1.14}$$

Replacing x by ux in (1.14) and expanding it, we obtain

$$\alpha(u)(\alpha(x) \circ \beta(y)) - [\alpha(u), \beta(y)]\alpha(x) = u(x \circ y) - [u, y]x \text{ for all } x, y, u \in \mathcal{U}. \tag{1.15}$$

From (1.14) and (1.15), we can easily arrive at

$$(\alpha(u) - u)(x \circ y) = [\alpha(u), \beta(y)]\alpha(x) - [u, y]x \quad \text{for all} \quad x, y, u \in \mathcal{U}. \tag{1.16}$$

For $y = u$, we get $(\alpha(u) - u)(x \circ u) = [\alpha(u), \beta(u)]\alpha(x)$ for $x, u \in \mathcal{U}$. Writing xu instead of x in the last relation and using it, we obtain $[\alpha(u), \beta(u)]\alpha(\mathcal{U})(\alpha(u) - u) = \{0\}$ for $u \in \mathcal{U}$. Invoking the primeness of $\mathcal{R}$, the last relation implies that $\alpha(u) = u$ or $\alpha(u) = \beta(u)$ for all $u \in \mathcal{U}$. The sets $H = \{u \in \mathcal{U} \mid \alpha(u) = u\}$ and $K = \{u \in \mathcal{U} \mid \alpha(u) = \beta(u)\}$ are additive subgroups of $\mathcal{U}$. But a group cannot be the union of proper subgroups. Hence we get $H = \mathcal{U}$ or $K = \mathcal{U}$ which force that $\alpha(u) = u$ for all $u \in \mathcal{U}$ or $\alpha(v) = \beta(v)$ for all $v \in \mathcal{U}$.
If $\alpha(u) = u$ for all $u \in \mathcal{U}$, then $x \circ \beta(y) = x \circ y$ for all $x, y \in \mathcal{U}$ which implies that $x(\beta(y) - y) = (\beta(y) - y)x$ for all $x, y \in \mathcal{U}$. Putting xt in place of x, where $t \in \mathcal{R}$ in the letter expression and using it, it is easy to verify that $\mathcal{U}[\beta(y) - y, t] = \{0\}$ for all $y \in \mathcal{U}$, $t \in \mathcal{R}$. By primeness of $\mathcal{R}$, we get $\beta(y) - y \in Z(\mathcal{R})$ for all $y \in \mathcal{U}$. Replacing y by yz, we find that $(\beta(y) - y)\beta(z) + (\beta(z) - z)y \in Z(\mathcal{R})$ for all $y, z \in \mathcal{U}$ which implies that $(\beta(z) - z)y\beta(z) = \beta(z)(\beta(z) - z)y$ for all $y, z \in \mathcal{U}$. Taking yt in place of y, where $t \in \mathcal{R}$, we find that $(\beta(z) - z)\mathcal{U}[\beta(z), t] = \{0\}$ for all $y \in \mathcal{U}$, $t \in \mathcal{R}$. By primeness of $\mathcal{R}$, we get either $\beta(z) = z$ or $\beta(z) \in Z(\mathcal{R})$ for all $z \in \mathcal{U}$. If there exists $z_0 \in \mathcal{U}$ such that $\beta(z_0) \in Z(\mathcal{R})$, equation $x \circ \beta(z_0) = x \circ z_0$ for all $x \in \mathcal{U}$ becomes $2\mathcal{U}(\beta(z_0) - z_0) = \{0\}$ and primeness with $char\mathcal{R} \neq 2$ force that $\beta(z_0) = z_0$. Consequently, in all cases we have $\beta(z) = z$ for all $z \in \mathcal{U}$. By [3, Lemma 1 (i)], we conclude that $\beta = id_{\mathcal{R}}$.
The second case with (1.16) give $(\alpha(u) - u)(x \circ u) = 0$ for all $x, u \in \mathcal{U}$. Substituting xr for x, where $r \in \mathcal{R}$ in the last expression and using it, it is obvious that $(\alpha(u) - u)x[u, r] = \{0\}$ for all $x, u \in \mathcal{U}$, $r \in \mathcal{R}$ which can be rewritten as $(\alpha(u) - u)\mathcal{U}[u, r] = \{0\}$ for all $u \in \mathcal{U}$, $r \in \mathcal{R}$. According to the primeness of $\mathcal{R}$, we have $\alpha(u) = u$ or $u \in Z(\mathcal{R})$ from what precedes the last expression implies that $\alpha = id_{\mathcal{R}}$ or $\mathcal{R}$ is commutative which gives easily $\alpha = \beta = id_{\mathcal{R}}$ or $\mathcal{R}$ is commutative. Suppose the second case and applying it in our hypotheses, we obtain that $\alpha(x)\beta(y) = xy$ for all $x, y \in \mathcal{U}$. Replacing y by yz in the last relation and using it, we obtain $xy(\beta(z) - z) = 0$ for all $x, y, z \in \mathcal{U}$ ie. $x\mathcal{U}(\beta(z) - z) = \{0\}$

for all $x, z \in \mathcal{U}$. Thus in view of primeness of $\mathcal{R}$, we conclude that $\beta = id_{\mathcal{R}}$ and using same lines as above with minor variations, we can easily arrive at $\alpha = id_{\mathcal{R}}$. □

Proceeding along the same lines with necessary variations, we can prove the following theorem:

Theorem 1.6. *Let $\mathcal{R}$ be a prime ring with char$\mathcal{R} \neq 2$, $\mathcal{U}$ a nonzero ideal of $\mathcal{R}$ and α, β automorphisms of $\mathcal{R}$, then the following assertions are equivalent:*

(i) $\alpha(x) \circ \beta(y) = -x \circ y$ *for all* $x, y \in \mathcal{U}$;
(ii) $\alpha = \beta = -id_{\mathcal{R}}$.

The following Corollaries are immediate consequences of the Theorems 1.5, 1.6.

Corollary 1.7. *Let $\mathcal{R}$ be a prime ring with char$\mathcal{R} \neq 2$, $\mathcal{U}$ a nonzero ideal of $\mathcal{R}$ and α automorphism of $\mathcal{R}$, then the following assertions are equivalent:*

(i) $\alpha(x) \circ \alpha(y) = x \circ y$ *for all* $x, y \in \mathcal{U}$;
(ii) $\alpha = id_{\mathcal{R}}$.

Corollary 1.8. *Let $\mathcal{R}$ be a prime ring with char$\mathcal{R} \neq 2$, $\mathcal{U}$ a nonzero ideal of $\mathcal{R}$ and α automorphism of $\mathcal{R}$, then the following assertions are equivalent:*

(i) $\alpha(x) \circ \alpha(y) = -x \circ y$ *for all* $x, y \in \mathcal{U}$;
(ii) $\alpha = -id_{\mathcal{R}}$.

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