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EXISTENCE SOLUTIONS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS AND L^1 DATA

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Abstract. In this article, we study the problem

$$\begin{cases} \frac{\partial b(x,u)}{\partial t} - \operatorname{div}\, a(x,t,u,\nabla u) + \operatorname{div}\, \phi(u) = f & \text{in } \Omega\times]0,T[, \\ u = 0 & \text{on } \partial\Omega\times]0,T[\\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega, \end{cases}$$

in the framework of generalized Sobolev spaces, with $b(x,u)$ unbounded function on u. The main contribution of our work is to prove the existence of renormalized solutions when the second term f belongs to $L^1(Q_T)$.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, $p : \bar{\Omega} \to [2, +\infty)$ be a continuous, real-valued function and let $p^- = \min_{x\in\bar{\Omega}} p(x)$ and $p^+ = \max_{x\in\bar{\Omega}} p(x)$ such that $2 < p^- < p^+ < \infty$, $Q = \Omega \times [0, T]$
And let $Au = -div(a(x, t, u, Du))$ be a Leray-Lions operator defined from the generalized Sobolev space V into its dual $V*$
Now, we consider the parabolic problem associated for the differential equation

$$\begin{gathered} \frac{\partial b(u)}{\partial t} + Au + \operatorname{div}\, \phi(u) = f \quad \text{in } Q, \\ u = 0 \quad \text{on } \partial\Omega\times]0,T[, \\ b(u)(t=0) = b(u_0) \quad \text{on } \Omega \end{gathered} \tag{1.1}$$

where the function b is assumed to be strictly increasing C^1-function, $b(u_0)$ lie in $L^1(\Omega)$, and a is the Caratheodory function .
For the parabolic equation (1.1) the existence of weak solution has been proved by J-M Rakotoson [22] with the strict monotonicity and a measure data, the existence and uniqueness of a renormalized solution has been proved by D. Blanchard and F. Murat [15] in the case where $a(x, t, s, \xi)$ is independent of s, and by D. Blanchard, F. Murat and H. Redwane [13] with the large monotonicity on a. For the degenerated parabolic equations the existence of weak solutions have

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been proved by proved by L. Aharouch and al [1] in the case where a is strictly monotone and $f \in L^{p'}(0,T,W^{-1,p'}(\Omega))$. See also the existence of renormalized solution by Y. Akdim et al [11] in the case $a(x,t,s,\xi)$ is independent of s.
Our paper can be see as a continuous of [20] in the case where $b(u) = u$, $a(x,t,s,\xi) = |\xi|^{p(x)-2}\xi$, and [4] if we replace $g(x,s,\xi)$ by $div(\phi(u))$.
The plan of our paper is as follow: In section 2 we give some preliminaries. In section 3 we make precise all the assumptions on a, and b. In section 4 we establish some technical lemmas. In section 5 we prove our main result: the existence of renormalized solutions of problem (1.1).

2. Preliminaries

For each open bounded subset Ω of $I\!R^N$ $(N \geq 2)$, we denote

$C_+(\overline{\Omega}) = \{$continuous function $\quad p(\cdot) : \overline{\Omega} \longmapsto I\!R^+ \quad$ such that $\quad 1 < p_- \leq p_+ < \infty\}$,

where

$$p_- = \min_{x\in\Omega} p(x) \qquad \text{and} \qquad p_+ = \max_{x\in\Omega} p(x).$$

We define the variable exponent Lebesgue space for $p(\cdot) \in C_+(\overline{\Omega})$ by:

$$L^{p(x)}(\Omega) = \{u : \Omega \longmapsto I\!R \quad \text{measurable} \ / \ \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

The space $L^{p(x)}(\Omega)$ under the norm:

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0, \quad \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \leq 1\right\}$$

is a uniformly convex Banach space, then reflexive. and we denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$.

Proposition 2.1. *(cf[17]) **(i)** For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Hölder type inequality:*

$$\left|\int_\Omega u\,v\,dx\right| \leq \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(x)}\,\|v\|_{p'(x)}\,.$$

***(ii)** For all $p_1,\ p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2. *(cf[17]) If we denote*

$$\rho(u) = \int_\Omega |u|^{p(x)}\,dx \quad \forall u \in L^{p(x)}(\Omega),$$

then the following assertions holds:

- **(i):** $\|u\|_{p(x)} < 1 \quad (resp, = 1, > 1) \quad \Longleftrightarrow \quad \rho(u) < 1 \quad (resp, = 1, > 1)$,
- **(ii):** $\|u\|_{p(x)} > 1 \quad \Rightarrow \quad \|u\|^{p_-}_{p(x)} \leq \rho(u) \leq \|u\|^{p_+}_{p(x)}$ *and* $\|u\|_{p(x)} < 1 \quad \Rightarrow \quad \|u\|^{p_+}_{p(x)} \leq \rho(u) \leq \|u\|^{p_-}_{p(x)}$,
- **(iii):** $\|u\|_{p(x)} \to 0 \quad \Leftrightarrow \quad \rho(u) \to 0,$ *and* $\|u\|_{p(x)} \to \infty \quad \Leftrightarrow \quad \rho(u) \to \infty.$

Let

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega)\quad\text{and}\quad|\nabla u|\in L^{p(x)}(\Omega)\},$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(x)}=\|u\|_{p(x)}+\|\nabla u\|_{p(x)}.$$

The space $(W^{1,p(x)}(\Omega),\|\cdot\|_{1,p(x)})$ is a separable and reflexive Banach space.
Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and

$$p^*(x)=\begin{cases}\dfrac{Np(x)}{N-p(x)} & \text{for}\quad p(x)<N,\\ \infty & \text{for}\quad p(x)\geq N.\end{cases}$$

Proposition 2.3. ***(see*** [18]***)***

- **(i):** *Assuming $1<p_-\leq p_+<\infty$, the spaces $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*
- **(ii):** *If $q(x)\in\mathcal{C}_+(\bar\Omega)$ and $q(x)<p^*(x)$ for a.e. $x\in\Omega$, then the embedding $W_0^{1,p(x)}(\Omega)\hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.*
- **(iii):** *Poincaré type inequality : there exists a constant $C>0$, such that*

$$\|u\|_{p(x)}\leq C\,\|\nabla u\|_{p(x)}\qquad\forall u\in W_0^{1,p(x)}(\Omega).$$

Extending the definition of variable exponent $p(\cdot):\bar\Omega\to[1,\infty)$ to $\bar Q_T=\bar\Omega\times[0,T]$ by setting $p(x,t):=p(x)$ for all $(x,t)\in\bar Q$, we may also consider the generalized Lebesgue space

$$L^{p(x)}(Q_T)=\{u:Q_T\longmapsto I\!R\quad\text{measurable}\ /\ \int_{Q_T}|u(x,t)|^{p(x)}\,dx\,dt<\infty\},$$

endowed with the norm

$$\|u\|_{L^{p(x)}(Q_T)}=\inf\left\{\lambda>0,\quad\int_{Q_T}\left|\frac{u(x,t)}{\lambda}\right|^{p(x)}dx\,dt\leq 1\right\}$$

which, of course, shares the same type of properties as $L^{p(x)}(\Omega)$.
Now, we introduce the functional space

$$V=\{u\in L^{p^-}(0,T,W_0^{1,p(x)}(\Omega)):|\nabla u|\in L^{p(x)}(Q_T)\}$$

which, endowed with the norm

$$\|u\|_V:=\|\nabla u\|_{L^{p(x)}(Q_T)}$$

or, the equivalent norm

$$|||u|||_V:=\|u\|_{L^{p_-}(0,T,W_0^{1,p(x)}(\Omega))}+\|\nabla u\|_{L^{p(x)}(Q_T)}$$

is separable and reflexive Banach space.
We have used the standard notations for Bochner spaces, i.e. if X is a Banach space and $q\geq 1$, then $L^q(0,T;X)$ denotes the space of strongly measurable function

$$\begin{aligned}u:(0,T)&\longmapsto X\\ t&\longmapsto\|u(t)\|_X\in L^q(0,T).\end{aligned}$$

Moreover, $C([0,T];X)$ denotes the space of continuous functions $u:[0,T]\to X$ endowed with the norm

$$||u||_{C([0,T];X)} := \max_{t\in[0,T]} ||u(t)||_X.$$

Lemma 2.4. *The dual of the Banach space V is denoted by V^*, then we have the following continuous embedding:*

$$L^{p_+}(0,T,W_0^{1,p(x)}(\Omega)) \hookrightarrow V \hookrightarrow L^{p_-}(0,T,W_0^{1,p(x)}(\Omega))$$

In particular, since $\mathcal{D}(Q_T)$ is dense in $L^{p_+}(0,T,W_0^{1,p(x)}(\Omega))$, it is dense in V and for the corresponding dual spaces we have

$$L^{(p_-)'}(0,T,W^{-1,p'(x)}(\Omega)) \hookrightarrow V^* \hookrightarrow L^{(p_+)'}(0,T,W^{-1,p'(x)}(\Omega))$$

Proposition 2.5. *One can represent the elements of V^* as follow: if $T\in V^*$, then there exists $F=(f_1,...,f_N)\in(L^{p'(x)}(Q))^N$ such that $T=divF$ and*

$$< T,\zeta >_{V^*,V} = \int_0^T\int_\Omega F.\nabla\zeta dxdt \quad \textit{for any}\zeta\in V$$

Moreover, we have

$$\|T\|_{V^*} = \max\{\|f_i\|_{L^{p'(x)}(Q)}, i=1,...,N\}$$

3. Basic assumptions

Let Ω be a bounded open subset of $I\!R^N$ $(N\geq 2)$, $T>0$ and $p_->2$.

We consider a Leray-Lions operator A acted from V into its dual $V^\star$, defined by the formula

$$Au = -\text{div } a(x,t,u,\nabla u) \tag{3.1}$$

where $a: Q_T\times I\!R\times I\!R^N \longmapsto I\!R^N$ is a Carathéodory function (measurable with respect to (x,t) in Q_T for every (s,ξ) in $I\!R\times I\!R^N$, and continuous with respect to (s,ξ) in $I\!R\times I\!R^N$ for almost every (x,t) in Q_T) which satisfies the following conditions

$$|a(x,t,s,\xi)| \leq \beta(K(x,t)+|s|^{p(x)-1}+|\xi|^{p(x)-1}), \tag{3.2}$$

$$(a(x,t,s,\xi)-a(x,t,s,\eta))\cdot(\xi-\eta)>0 \quad \text{for} \quad \xi\neq\eta, \tag{3.3}$$

$$a(x,t,s,\xi)\cdot\xi \geq \alpha|\xi|^{p(x)}, \tag{3.4}$$

for a.e. $(x,t)\in Q_T$, all $s\in I\!R$ and $\xi,\eta\in I\!R^N$, where $K(x,t)$ is a positive function lying in $L^{p'(x)}(Q_T)$ and $\alpha,\beta>0$.

We assume that

$$b(\cdot,\cdot):\Omega\times I\!R\longmapsto I\!R \quad \text{is a Carathéodory function such that,} \quad b(x,0)=0 \quad \text{and}$$
$$b(x,\cdot) \text{ is a strictly increasing function in } C^1(I\!R) \text{ for a.e. } x\in\Omega. \tag{3.5}$$

Moreover, For any $K>0$, there exists $\lambda_K>0$, a function $A_K\in L^\infty(\Omega)$ and a function $B_K\in L^{p(x)}(\Omega)$ such that

$$\lambda_K \leq \frac{\partial b(x,s)}{\partial s} \leq A_K(x) \quad \text{and} \quad \Big|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big| \leq B_K(x), \tag{3.6}$$

for almost every $x \in \Omega$, for every s such that $|s| \leq K$.

We consider the quasilinear $p(x)$-parabolic problem

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} - \text{div } a(x,t,u,\nabla u) + \text{div } \phi(u) = f & \text{in } \Omega\times]0,T[, \\ u = 0 & \text{on } \partial\Omega\times]0,T[\\ b(x,u)(t=0) = b(x,u_0) & \text{in } \Omega, \end{cases} \tag{3.7}$$

with $\phi(\cdot) \in C(I\!R, I\!R^N)$, $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$ such that $b(x,u_0) \in L^1(\Omega)$.

Remark 3.1. The problem (3.7) does not admit a weak solution under assumptions (3.2) − (3.6) (even when $b(x,u) = u$) since the growths of $a(x,t,u,\nabla u)$ and $\phi(u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs to V.

4. Some technical results

Firstly, we establish some embedding and compactness results in generalized Sobolev spaces.

Let $p_- > 2$, we set $X = W_0^{1,p(x)}(\Omega)$, $H = L^2(\Omega)$ and $X^* = W^{-1,p'(x)}(\Omega)$.
Denoting the space $W^1_{p(x)}(0,T,X,H) = \{v \in V \quad \text{and} \quad v_t \in V^*\}$ endowed with the norm

$$\|u\|_{W^1_{p(x)}} = \|u\|_V + \|u_t\|_{V^*}$$

is a Banach space. Here u_t stands for the generalized derivative of u; i.e.,

$$\int_0^T u_t(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \quad \text{for all } \varphi \in C_0^\infty(0,T).$$

Lemma 4.1. *Let B_0, B and B_1 be Banach spaces with $B_0 \subset B \subset B_1$. Let us set*

$$Y = \left\{u \ : \ u \in L^{p_0}(0,T;B_0) \quad \textit{and} \quad u_t \in L^{p_1}(0,T;B_1)\right\}$$

where $p_0 > 1$ and $p_1 > 1$ are reals numbers.
Assuming that the embedding $B_0 \hookrightarrow\hookrightarrow B$ be compact, then

$$Y \hookrightarrow\hookrightarrow L^{p_0}(0,T;B)$$

and this imbedding is compact.

Remark 4.2. Let $p_- > 2$, we set

$$B_0 = W_0^{1,p(x)}(\Omega), \qquad B = L^2(\Omega) \qquad \text{and} \qquad B_1 = W^{-1,p'(x)}(\Omega),$$

with $p_0 = p_-$ and $p_1 = p'_-$. In view of the Lemma 4.1, we obtain

$$W^1_{p(x)}(0,T,X,H) \subseteq Y \hookrightarrow\hookrightarrow L^2(Q_T). \tag{4.1}$$

Moreover, in view of [9], we have

$$W^1_{p(x)}(0,T,X,H) \subseteq C([0,T];L^2(\Omega)). \tag{4.2}$$

Now, To deal with time derivative in the Sobolev space with variable exponents, we introduce a time mollification of a function $u \in V$ as follows

Proposition 4.3. [9, 27] *The time mollification of a function $u \in V$ for any $\mu \geq 0$ is introduced by*

$$u_\mu(x,t) = \mu \int_{-\infty}^{t} \overline{u}(x,s) exp(\mu(s-t))\, ds \qquad where \quad \overline{u}(x,s) = u(x,s)\chi_{(0,T)}(s),$$

the following assertions holds

(i): *If $u \in L^{p(x)}(Q_T)$, then u_μ is measurable in Q_T, $\dfrac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and*

$$\int_{Q_T} |u_\mu|^{p(x)}\, dx\, dt \leq \int_{Q_T} |u|^{p(x)}\, dx\, dt.$$

(ii): *If $u \in V$, then $u_\mu \to u$ in V as $\mu \to +\infty$.*
(iii): *If $u_n \to u$ in V, then $(u_n)_\mu \to u_\mu$ in V.*
(iv): *We have $|(T_k(u))_\mu| \leq k$ for any $u \in V$.*

Lemma 4.4. *(see. [3]) Let $g \in L^{r(x)}(Q_T)$ and $g_n \in L^{r(x)}(Q_T)$, with $\|g_n\|_{L^{r(x)}(Q_T)} \leq C$, $1 < r < \infty$. If $g_n(x) \to g(x)$ a.e in Q_T, then $g_n \rightharpoonup g$ in $L^{r(x)}(Q_T)$ where $n \to \infty$.*

Lemma 4.5 (see. [1]). *Assume that*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \quad in\ D'(Q_T)$$

where α_n and β_n are bounded respectively in V^* and in $L^1(Q_T)$. If v_n is bounded in V, then $v_n \to v$ in $L^{p(x)}_{\text{loc}}(Q_T)$. Further $v_n \to v$ strongly in $L^1(Q_T)$ where $n \to \infty$.

Lemma 4.6. *Assuming that (3.2) – (3.4) holds, and let $(u_n)_n$ be a sequence in V such that $u_n \rightharpoonup u$ in V and*

$$\int_{Q_T} \Big(a(x,t,\nabla u_n) - a(x,t,\nabla u) \Big) \cdot (\nabla u_n - \nabla u)\, dx \longrightarrow 0, \tag{4.3}$$

then $u_n \to u$ in V for a subsequence.

The proof of this Lemma is the same as in the case of constant exponent p (see. [6]).

5. Main results

Definition 5.1. A measurable function u is an renormalized solution of the Dirichlet problem (3.7) if $T_k(u) \in V \quad \forall k \geq 0$,

$$b(x,u) \in L^\infty(0,T;L^1(\Omega)) \quad \text{ and } \quad \lim_{m\to\infty} \int_{\{m<|u|\leq m+1\}} a(x,t,u,\nabla u) \cdot \nabla u\, dx\, dt = 0, \tag{5.1}$$

$$\begin{aligned} \frac{\partial B_S(x,u)}{\partial t} &- \text{div } (S'(u)a(x,t,u,\nabla u)) + S''(u)a(x,t,u,\nabla u) \cdot \nabla u \\ &-\text{div } (S'(u)\phi(u)) + S\text{"}(u)\phi(u) \cdot \nabla u = fS'(u) \qquad \text{in} \quad D'(Q_T), \end{aligned} \tag{5.2}$$

for any functions $S(\cdot) \in W^{2,\infty}(I\!R)$ such that $S'(\cdot)$ has a compact support in $I\!R$, with

$$B_S(x,z) = \int_0^z \frac{\partial b(x,r)}{\partial r} S'(r)dr \qquad \text{and} \qquad B_S(x,u)(t=0) = B_S(x,u_0) \quad \text{in } \Omega. \tag{5.3}$$

Theorem 5.2. *Let $f \in L^1(Q_T)$ and $b(x,u_0) \in L^1(\Omega)$. Assume that (3.2)–(3.6) hold. Then, there exists at least one renormalized solution u of problem (3.7).*

Proof of the Theorem 5.2.

Step 1: Approximate problems. Let $(f_n)_n$ be a sequence in $V^\star \cap L^1(Q_T)$ such that $f_n \longrightarrow f$ in $L^1(Q_T)$ with $|f_n| \le |f|$, and let $(u_{0,n})_n$ be a sequence in $C_0^\infty(\Omega)$ such that $b(x,u_{0,n}) \to b(x,u_0)$ in $L^1(\Omega)$ and $|b(x,u_{0,n})| \le |b(x,u_0)|$. We define the following approximation

$$b_n(x,r) = b(x,T_n(r)) + \frac{1}{n} r, \quad a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \quad \text{and} \quad \phi_n(s) = \phi(T_n(s)). \tag{5.4}$$

We consider the approximate problem:

$$\begin{cases} \frac{\partial b_n(x,u_n)}{\partial t} - \text{div } a(x,t,T_n(u_n),\nabla u_n) - \text{div } \phi_n(u_n) = f_n & \text{in} \quad Q_T, \\ u_n = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ b_n(x,u_n)(t=0) = b_n(x,u_{0,n}) & \text{in} \quad \Omega. \end{cases} \tag{5.5}$$

In view of [21], there exists at least one weak solution $u_n \in V$ of the problem (5.5).

Step 2: Weak convergence of truncations. Let $\tau \in [0,T]$, taking $T_k(u_n)\chi_{(0,\tau)}$ as a test function in (5.5), we obtain

$$\int_\Omega \langle \frac{\partial b_n(x,u_n)}{\partial t}, T_k(u_n)\chi_{(0,\tau)} \rangle \, dx + \int_0^\tau \int_\Omega a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n) \, dx \, dt \\ + \int_0^\tau \int_\Omega \phi_n(u_n) \cdot \nabla T_k(u_n) \, dx \, dt = \int_0^\tau \int_\Omega f_n T_k(u_n) \, dx \, dt,$$

it follows that

$$\int_\Omega B_k^n(x,u_n(\tau)) \, dx + \int_0^\tau \int_\Omega a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n) \, dx \, dt \\ + \int_0^\tau \int_\Omega \phi_n(u_n) \cdot \nabla T_k(u_n) \, dx \, dt = \int_0^\tau \int_\Omega f_n T_k(u_n) \, dx \, dt + \int_\Omega B_k^n(x,u_{0,n}) \, dx \tag{5.6}$$

with $B_k^n(x,r) = \int_0^r T_k(s) b_n'(x,s) \, ds$.

Taking $\Phi_n(s) = \int_0^s \phi_n(\sigma) \, d\sigma$, then $\Phi_n(0) = 0_{I\!R^N}$ and $\Phi_n(\cdot) \in C^1(I\!R, I\!R^N)$, in view of the Divergence Theorem, we obtain

$$\begin{aligned}\int_0^\tau \int_\Omega \phi_n(u_n)\cdot\nabla T_k(u_n)\,dx\,dt &= \int_0^\tau \int_\Omega \text{div } \Phi_n(T_k(u_n))\,dx\,dt \\ &= \int_0^\tau \int_{\partial\Omega} \Phi_n(T_k(u_n)).\vec{n}\,d\sigma\,dt = 0,\end{aligned} \tag{5.7}$$

since $u_n = 0$ on $\partial\Omega$ and $\vec{n} = (n_1, n_2, \dots, n_N)$ the exterior normal vector on the boundary $\partial\Omega$.
On the other hand, since $b_n(s)$ have the same sign as s, then

$$0 \le \int_\Omega B_k^n(x, u_{0,n})\,dx \le k\int_\Omega |b_n(x, u_{0,n})|dx \le k\|b(x,u_0)\|_{L^1(\Omega)}. \tag{5.8}$$

By using (5.6) $-$ (5.8), we deduce that

$$\begin{aligned}0 \;&\le \int_\Omega B_k^n(x, u_n(\tau))\,dx + \int_0^\tau \int_\Omega a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,dx\,dt \\ &\le k(\|f_n\|_{L^1(Q_T)} + \|b(u_0)\|_{L^1(\Omega)}) \\ &\le C_1 k \qquad\qquad\qquad\qquad \text{for any } \tau\in[0,T].\end{aligned} \tag{5.9}$$

It's clear that $\displaystyle\int_\Omega B_k^n(x,u_n(\tau))\,dx \ge 0$, Thanks to (3.4) we obtain

$$\|\nabla T_k(u_n)\|^{p_-}_{L^{p(x)}(Q_T)} \le \int_{Q_T} |\nabla T_k(u_n)|^{p(x)}\,dx\,dt + 1 \le C_2 k \qquad \forall k\ge 1. \tag{5.10}$$

We have $T_k(u_n)$ is bounded in V. In view of the Remark 4.2, there exists a subsequence still denoted $T_k(u_n)$ and $v_k \in V$ such that

$$T_k(u_n) \rightharpoonup v_k \quad \text{in} \quad V \qquad \text{and} \qquad T_k(u_n)\to v_k \quad \text{a.e. in} \quad Q_T. \tag{5.11}$$

Let $k > 0$ large enough, by virtue of Holder's and Poincaré inequality, we have:

$$\begin{aligned}k.\text{ meas }\{|u_n|>k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|\,dx\,dt \le \int_{Q_T} |T_k(u_n)|\,dx\,dt \\ &\le (\frac{1}{p_-} + \frac{1}{p'_-})\|T_k(u_n)\|_{L^{p(x)}(Q_T)}\|1\|_{L^{p'(x)}(Q_T)} \\ &\le C_3\|\nabla T_k(u_n)\|_{L^{p(x)}(Q_T)} \\ &\le C_4 k^{\frac{1}{p_-}},\end{aligned}$$

which implies that

$$\text{meas}\big\{|u_n|>k\big\} \le C_5\frac{1}{k^{1-\frac{1}{p_-}}} \longrightarrow 0 \quad \text{as} \quad k\to+\infty, \tag{5.12}$$

since for all $\delta > 0$ that

$$\begin{aligned}\text{meas}\big\{|u_n - u_m| > \delta\big\} \le\; & \text{meas}\big\{|u_n|>k\big\} + \text{meas}\big\{|u_m|>k\big\} \\ +\; & \text{meas}\big\{|T_k(u_n) - T_k(u_m)| > \delta\big\}.\end{aligned} \tag{5.13}$$

Using (5.12) we get that for all $\varepsilon > 0$, that exists $k_0 > 0$ such that

$$\text{meas}\big\{|u_n|>k\big\} \le \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\big\{|u_m|>k\big\} \le \frac{\varepsilon}{3} \qquad \forall k\ge k_0(\varepsilon). \tag{5.14}$$

On the other hand, in view of (5.11), we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q_T, then for all $k > 0$ and $\delta,\ \varepsilon > 0$ there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\big\{|T_k(u_n) - T_k(u_m)| > \delta\big\} \leq \frac{\varepsilon}{3} \qquad \forall n, m \geq n_0. \tag{5.15}$$

By combining $(5.13) - (5.15)$, we deduce that for all $\varepsilon,\ \delta > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon \qquad \forall n, m \geq n_0. \tag{5.16}$$

If follows that $(u_n)_n$ is a Cauchy sequence in measure, then there exists a subsequence still denoted $(u_n)_n$ such that

$$u_n \to u \quad \text{a.e. in } Q_T, \tag{5.17}$$

$$b_n(x, u_n) \to b(x, u) \quad \text{a.e. in } Q_T. \tag{5.18}$$

Using (5.11), we deduce that

$$T_k(u_n) \rightharpoonup T_k(u) \qquad \text{in} \qquad V, \tag{5.19}$$

and in the view of the Lebesgue dominated convergence theorem

$$T_k(u_n) \longrightarrow T_k(u) \qquad \text{in} \qquad L^{p(x)}(Q_T). \tag{5.20}$$

We now establish that $b(x, u)$ belongs to $L^\infty(0, T; L^1(\Omega))$. Using (5.9), (5.17) and Fatou's Lemma, we conclude that

$$0 \leq \frac{1}{k}\int_\Omega B_k(x, u(\tau))dx \leq \big(\|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}\big) = C_6,$$

for any $\tau \in (0, T)$. Due to the definition of $B_k(x, s)$ and the fact that $\frac{1}{k}B_k(x, u)$ converges pointwise to $b(x, u)$, as k tends to 0, shows that $b(x, u)$ belong to $L^\infty(0, T; L^1(\Omega))$.

Step 3 : A priori estimates. Taking $T_1(u_n - T_h(u_n))$ as a test function in (5.5), we obtain

$$\int_0^T \langle \frac{\partial b_n(x, u_n)}{\partial t}, T_1(u_n - T_h(u_n))\rangle dt + \int_{\{h<|u_n|\leq h+1\}} a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \, dt$$
$$+ \int_{\{h<|u_n|\leq h+1\}} \phi_n(u_n) \cdot \nabla u_n \, dx \, dt = \int_{Q_T} f_n T_1(u_n - T_h(u_n)) \, dx \, dt.$$

We have

$$\int_{\{h<|u_n|\leq h+1\}} \phi_n(u_n) \cdot \nabla u_n \, dx \, dt = \int_0^T \int_\Omega \text{div } \Phi_n(T_{h+1}(u_n)) \, dx \, dt - \int_0^T \int_\Omega \text{div } \Phi_n(T_h(u_n)) \, dx \, dt = 0,$$

it follows that

$$\int_\Omega B_{n,h}(x, u_n(T)) \, dx + \int_{\{h<|u_n|\leq h+1\}} a(x, t, T_n(u_n), \nabla u_n) \cdot \nabla u_n \, dx \, dt$$
$$\leq \int_{\{|u_n|>h\}} |f_n| \, dx \, dt + \int_{\{|u_0|>h\}} B_{n,h}(x, u_{0,n}) \, dx$$

with $B_{n,h}(x,r) = \int_0^r b'_n(x,s)T_1(s - T_h(s))\,ds$, it's clear that $B_{n,h}(x,u_n(T)) \geq 0$, then

$$\lim_{n\to\infty} \int_{\{h<|u_n|\leq h+1\}} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n dxdt \leq \int_{\{|u|>h\}} |f|dxdt + \int_{\{|u_0|>h\}} b(x,u_0)dx.$$

Since $f \in L^1(Q_T)$ and $b(x,u_0) \in L^1(\Omega)$, we obtain

$$\lim_{h,n\to\infty} \int_{\{h\leq|u_n|\leq h+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt = 0. \tag{5.21}$$

and thanks to (3.4), we deduce that

$$\lim_{h,n\to\infty} \int_{\{h\leq|u_n|\leq h+1\}} |\nabla u_n|^{p(x)}\,dx\,dt = 0. \tag{5.22}$$

Step 4 : Convergence of the gradient. This step is devoted to establish the following limits

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)) \quad \text{weakly in} \quad (L^{p'(x)}(Q_T))^N. \tag{5.23}$$

Let $h \geq k > 0$ and n large enough ($n > h+1$), In the sequel, we denote by $\varepsilon_i(n)\, i = 1, 2, \ldots$ a various functions of real numbers which converges to 0 as n tends to infinity (respectively for $\varepsilon_i(n,\mu)$ and $\varepsilon_i(n,\mu,h)$).
Let $\omega_{n,\mu} = T_k(u_n) - (T_k(u))_\mu$, where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$.
Taking $S_h(\cdot) \in W^{2,\infty}(I\!R)$ an increasing function, such that $S_h(r) = r$ for $|r| \leq h$ and $\text{supp}(S'_h) \subset [-h-1, h+1]$, then $\text{supp}(S''_h) \subset [-h-1,-h] \cup [h,h+1]$.
Using $\omega_{n,\mu}S'_h(u_n)$ as a test function in (5.5), we obtain

$$\begin{aligned}
&\int_0^T \langle \frac{\partial b_n(x,u_n)}{\partial t}, \omega_{n,\mu}S'_h(u_n)\rangle\,dt + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla\omega_{n,\mu}S'_h(u_n)\,dx\,dt \\
&+ \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla u_n S''_h(u_n)\omega_{n,\mu}\,dx\,dt + \int_{Q_T} \phi_n(u_n)\cdot\nabla\omega_{n,\mu}S'_h(u_n)\,dx\,dt \\
&+ \int_{Q_T} \phi_n(u_n)\cdot\nabla u_n S''_h(u_n)\omega_{n,\mu}\,dx\,dt = \int_{Q_T} f_n\omega_{n,\mu}S'_h(u_n)\,dx\,dt.
\end{aligned} \tag{5.24}$$

Now, we study each terms of the above inequality.
For the first term on the left hand side of (5.24), we have

$$\int_0^T \langle \frac{\partial b_n(x,u_n)}{\partial t}, \omega_{n,\mu}S'_h(u_n)\rangle\,dt \;\; = \int_\Omega \int_0^T \frac{\partial b_n(x,u_n)}{\partial t}(T_k(u_n) - (T_k(u))_\mu)S'_h(u_n)\,dt\,dx \geq \varepsilon_1(n,\mu),$$

see Lemma 3.2 of [13].
Concerning the third term on the left-hand side of (5.24). Let n large enough,

since supp $(S_h'') \subset [-h-1,-h] \cup [h,h+1]$, and thanks to (5.21), we obtain

$$\begin{aligned} \varepsilon_2(n,h) &\leq \Big| \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot \nabla u_n S_h''(u_n)\omega_{n,\mu}\,dx\,dt \Big| \\ &= \int_{\{h<|u_n|\leq h+1\}} a(x,t,T_n(u_n),\nabla u_n)\cdot \nabla u_n\, |S_h''(u_n)||\omega_{n,\mu}|\,dx\,dt \\ &\leq 2k\|S_h''\|_{L^\infty(I\!R)} \int_{\{h<|u_n|\leq h+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt \longrightarrow 0 \quad \text{as} \quad n,h \longrightarrow \infty. \end{aligned} \tag{5.25}$$

For The fourth and fifth terms on the left-hand side of (5.24), it's clear that $\phi_n(T_{h+1}(u_n)) = \phi(T_{h+1}(u_n)) \to \phi(T_{h+1}(u))$ in $(L^{p'(x)}(Q_T))^N$, and since $\nabla T_k(u_n) - \nabla(T_k(u))_\mu \rightharpoonup 0$ in $(L^{p(x)}(Q_T))^N$, we conclude that

$$\begin{aligned} \varepsilon_3(n,\mu) &= \Big| \int_{Q_T} \phi_n(u_n))\cdot \nabla\omega_{n,\mu} S_h'(u_n)\,dx\,dt \Big| \\ &= \Big| \int_{\{|u_n|\leq h+1\}} \phi_n(T_{h+1}(u_n))\cdot(\nabla T_k(u_n) - \nabla(T_k(u))_\mu)S_h'(u_n)\,dx\,dt \Big| \\ &\leq \|S_h'\|_{L^\infty(I\!R)} \int_{\{|u_n|\leq h+1\}} |\phi(T_{h+1}(u_n))|\,|\nabla T_k(u_n) - \nabla(T_k(u))_\mu|\,dx\,dt \to 0 \ \text{ as } \ n,\ \mu\to\infty, \end{aligned} \tag{5.26}$$

In view of Young's inequality and (5.22), we obtain

$$\begin{aligned} \varepsilon_4(n,\mu,h) &= \Big| \int_{Q_T} \phi_n(u_n)\cdot\nabla u_n S_h''(u_n)\omega_{n,\mu}\,dx\,dt \Big| \\ &\leq \|S_h''\|_{L^\infty(I\!R)} \int_{\{h<|u_n|\leq h+1\}} |\phi_n(T_{h+1}(u_n))||\nabla u_n||\omega_{n,\mu}|\,dx\,dt \\ &\leq \|S_h''\|_{L^\infty(I\!R)} \int_{\{h<|u_n|\leq h+1\}} \frac{|\phi_n(T_{h+1}(u_n))|^{p'(x)}}{p'(x)}|\omega_{n,\mu}|\,dx\,dt \\ &+ 2k\|S_h''\|_{L^\infty(I\!R)} \int_{\{h<|u_n|\leq h+1\}} \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\,dt \to 0 \quad \text{as} \quad n,\ \mu \text{ then } h\to\infty. \end{aligned} \tag{5.27}$$

For the term on the right-hand side of (5.24), we have $f_n \to f$ in $L^1(Q_T)$, and since $T_k(u_n) - (T_k(u))_\mu \rightharpoonup 0$ weak-$\star$ in $L^\infty(Q_T)$, then

$$\begin{aligned} \big|\varepsilon_5(n,\mu)\big| \leq\ & \|S_h'\|_{L^\infty(I\!R)} \int_{Q_T} |f_n|\,|\xi_k(T_k(u_n) - (T_k(u))_\mu)|\,dx\,dt \\ & \longrightarrow 0 \qquad \text{as} \quad n,\ \mu\to\infty. \end{aligned} \tag{5.28}$$

By combining (5.24) – (5.28), we deduce that

$$\limsup_{n,h,\mu\to\infty} \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u_n) - (T_k(u))_\mu)S_h'(u_n)\,dx\,dt \leq 0, \tag{5.29}$$

and since $S_h'(u_n) = 1$ on $\{|u_n|\leq k\}$, then

$$\begin{aligned} &\limsup_{n\to\infty} \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,dx\,dt \\ &\leq \limsup_{n,h,\mu\to\infty} \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u))_\mu S_h'(u_n)\,dx\,dt. \end{aligned} \tag{5.30}$$

We have

$$\begin{aligned}
&\int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot\nabla(T_k(u))_\mu S'_h(u_n)\,dx\,dt\\
&=\int_{\{|u_n|\le k\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla(T_k(u))_\mu\,dx\,dt\\
&\quad+\int_{\{k<|u_n|\le h+1\}} a(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))\cdot\nabla(T_k(u))_\mu S'_h(u_n)\,dx\,dt
\end{aligned}\tag{5.31}$$

we have $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L^{p'(x)}(Q_T))^N$, then there exists $\xi_k\in(L^{p'(x)}(Q_T))^N$ such that $a(x,t,T_k(u_n),\nabla T_k(u_n))\rightharpoonup\xi_k$ in $(L^{p'(x)}(Q_T))^N$, it follows that

$$\lim_{n,\mu\to\infty}\int_{\{|u_n|\le k\}} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla(T_k(u))_\mu\,dx\,dt=\int_{Q_T}\xi_k\cdot\nabla T_k(u)\,dx\,dt.\tag{5.32}$$

Similarly, we have

$$\begin{aligned}
&\lim_{n,\mu\to\infty}\int_{\{k<|u_n|\le h+1\}} a(x,t,T_{h+1}(u_n),\nabla T_{h+1}(u_n))\cdot\nabla(T_k(u))_\mu S'_h(u_n)\,dx\,dt\\
&\qquad=\int_{\{k<|u|\le h+1\}}\xi_{h+1}\cdot\nabla T_k(u)S'_h(u)\,dx\,dt=0.
\end{aligned}\tag{5.33}$$

By using (5.30) $-$ (5.33), we conclude that

$$\limsup_{n\to\infty}\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,dx\,dt\le\int_{Q_T}\xi_k\cdot\nabla T_k(u)\,dx\,dt.\tag{5.34}$$

On the other hand, thanks to (3.3) we have

$$\int_{Q_T}\begin{aligned}&\big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(u))\big)\\&\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\,dx\,dt\ge 0\end{aligned}$$

then

$$\begin{aligned}
\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,dx\,dt &\ge\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u)\,dx\,dt\\
&\ge\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u))\cdot\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\,dx
\end{aligned}$$

it follows that

$$\liminf_{n\to\infty}\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,dx\,dt\ge\int_{Q_T}\xi_k\cdot\nabla T_k(u)\,dx\,dt.\tag{5.35}$$

Having in mind (5.34), we obtain

$$\lim_{n\to\infty}\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)\,dx\,dt=\int_{Q_T}\xi_k\cdot\nabla T_k(u)\,dx\,dt.\tag{5.36}$$

Which imply that,

$$\lim_{n\to\infty}\int_{Q_T} \big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\big) \cdot\big(\nabla T_k(u_n) - \nabla T_k(u)\big)\,dx\,dt = 0. \tag{5.37}$$

In view of the Lemma 4.6, we deduce that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in} \quad V \qquad \text{then} \qquad \nabla u_n \longrightarrow \nabla u \quad \text{a.e in } Q_T. \tag{5.38}$$

Then $a(x,t,T_k(u_n),\nabla T_k(u_n)) \to a(x,t,T_k(u),\nabla T_k(u))$ a.e. in Q_T, and since $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L^{p'(x)}Q_T)^N$, we get

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)) \qquad \text{in} \quad (L^{p'(x)}Q_T)^N. \tag{5.39}$$

Moreover, in view of (5.37), we deduce that

$$\lim_{n\to\infty}\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot\nabla T_k(u_n)dxdt = \int_{Q_T} a(x,t,T_k(u),\nabla T_k(u))\cdot\nabla T_k(u)dxdt. \tag{5.40}$$

Step 5 : Passage to the limit. Let $\varphi \in V \cap L^\infty(Q_T)$ and $S(\cdot) \in C^\infty(I\!R)$, with *supp* $S'(\cdot) \subset [-M,M]$ for some $M>0$. Taking $S'(u_n)\varphi$ a test function in (5.5), we obtain

$$\int_0^T \langle \frac{\partial B_S^n(x,u_n)}{\partial t},\varphi\rangle\,dt + \int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot(S''(u_n)\varphi\nabla u_n + S'(u_n)\nabla\varphi)\,dx\,dt + \int_{Q_T}\phi_n(u_n)\cdot(S''(u_n)\varphi\nabla u_n + S'(u_n)\nabla\varphi)\,dx\,dt = \int_{Q_T} f_n S'(u_n)\varphi\,dx\,dt. \tag{5.41}$$

with $B_S^n(x,z) = \int_0^z S'(\tau)\frac{\partial b_n(x,\tau)}{\partial\tau}\,d\tau$.
Firstly, since $S'(\cdot)$ is bounded and $B_S^n(x,u_n)$ converge to $B_S^n(x,u_n)$ a.e. in Q_T and weak$-*$ in $L^\infty(Q_T)$, then

$$\frac{\partial B_S^n(x,u_n)}{\partial t} \rightharpoonup \frac{\partial B_S(x,u_n)}{\partial t} \qquad \text{in} \qquad D'(Q_T). \tag{5.42}$$

Concerning the second term on the left-hand side of (5.41), we have

$$\int_{Q_T} a(x,t,T_n(u_n),\nabla u_n)\cdot(S''(u_n)\varphi\nabla u_n + S'(u_n)\nabla\varphi)\,dx\,dt$$
$$= \int_{Q_T} a(x,t,T_M(u_n),\nabla T_M(u_n))\cdot(S''(u_n)\varphi\nabla T_M(u_n) + S'(u_n)\nabla\varphi)\,dx\,dt,$$

and thanks to (5.39), we have

$$a(x,t,T_M(u_n),\nabla T_M(u_n)) \rightharpoonup a(x,t,T_M(u),\nabla T_M(u)) \quad \text{in} \;\; (L^{p'(x)}(Q_T))^N,$$

and since

$$S''(u_n)\varphi\nabla T_M(u_n) + S'(u_n)\nabla\varphi \longrightarrow S''(u)\varphi\nabla T_M(u) + S'(u)\nabla\varphi \quad \text{in} \;\; (L^{p(x)}(Q_T))^N,$$

hence

$$\begin{aligned}&\lim_{n\to\infty}\int_{Q_T} a(x,t,T_M(u_n),\nabla T_M(u_n))\cdot(S''(u_n)\varphi\nabla T_M(u_n)+S'(u_n)\nabla\varphi)\,dx\,dt\\&=\int_{Q_T} a(x,t,T_M(u),\nabla T_M(u))\cdot(S''(u)\varphi\nabla T_M(u)+S'(u)\nabla\varphi)\,dx\,dt\\&=\int_{Q_T} a(x,t,u,\nabla u)\cdot(S''(u)\varphi\nabla u+S'(u)\nabla\varphi)\,dx\,dt.\end{aligned}\tag{5.43}$$

Similarly, since $\phi_n(u_n)=\phi(T_M(u_n))$ for $n\geq M$, then

$$\begin{aligned}&\lim_{n\to\infty}\int_{Q_T}\phi_n(u_n)\cdot(S''(u_n)\varphi\nabla u_n+S'(u_n)\nabla\varphi)\,dx\,dt\\&=\lim_{n\to\infty}\int_{Q_T}\phi(T_M(u_n))\cdot(S''(u_n)\varphi\nabla T_M(u_n)+S'(u_n)\nabla\varphi)\,dx\,dt\\&=\int_{Q_T}\phi(T_M(u))\cdot(S''(u)\varphi\nabla T_M(u)+S'(u)\nabla\varphi)\,dx\,dt\\&=\int_{Q_T}\phi(u)\cdot(S''(u)\varphi\nabla u+S'(u)\nabla\varphi)\,dx\,dt.\end{aligned}\tag{5.44}$$

Moreover, since $S(u_n)\ \varphi \rightharpoonup S(u)\ \varphi$ weak$-\star$ in $L^\infty(\Omega)$, then

$$\int_\Omega f_n S'(u_n)\ \varphi\,dx \longrightarrow \int_\Omega f S'(u)\ \varphi\,dx.\tag{5.45}$$

By combining (5.41) $-$ (5.45), we deduce that

$$\begin{aligned}&\int_{Q_T}\frac{\partial B_S^n(x,u_n)}{\partial t}\varphi\,dx\,dt+\int_{Q_T} a(x,t,u,\nabla u)\cdot(S''(u)\varphi\nabla u+S'(u)\nabla\varphi)\,dx\,dt\\&\quad+\int_{Q_T}\phi(u)\cdot(S''(u)\varphi\nabla u+S'(u)\nabla\varphi)\,dx\,dt=\int_{Q_T} fS'(u)\varphi\,dx\,dt,\end{aligned}\tag{5.46}$$

we conclude that u is a renormalized solution to problem (5.30).

It remains to show that $B_S(x,u)$ satisfies the initial condition (5.3). To this end, firstly remark that, $S(\cdot)$ being bounded, $B_S^n(x,u_n)$ is bounded in $L^\infty(Q_T)$. Secondly, (**??**) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial B_S^n(u_n)}{\partial t}$ is bounded in $L^1(Q_T)+V*$. As consequence $B_n^S(u_n)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$. It follows that on the one hand, $B_n^S(u_n)(t=0)=B_n^S(u_n)(u_0^n)$ converges to $B_S(u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S implies that

$$B_S(u)(t=0)=B_S(u_0)\quad in\quad \Omega.$$

As conclusion of step 1 to step 4, the proof of theorem 5.2 is complete.

References

[1] L. Aharouch, E. Azroul and M. Rhoudaf, *Strongly nonlinear variational parabolic problems in weighted sobolev spaces.* The Australian journal of Mathematical Analysis and Applications, Vol.5, Issue 2, Article 13, pp. 1-25,2008 .

[2] E. Azroul, H. Redwane and M. Rhoudaf, *Existence of a Renormalized solution for a class of nonlinear parabolic equations in Orlicz* spaces, *Portugal. Math.* **1**, Vol. 66, (2009), 29-63.

[3] E. Azroul, H. Redwane, C. Yazough *Strongly Nonlinear Nonhomogeneous Elliptic Unilateral Problems With L^1 Data And No S Sign Conditions* Electronic Journal of Differential Equations, Vol. (2012),No.79,pp.1-20.

[4] E. Azroul , M.Benboubker , H. Redwane, C. Yazough.*Renormalized solutions for a class of nonlinear parabolic equations without sign condition involving nonstandard growth* Annals of the University of Craiova, Mathematics and Computer Science Series Volume 41(1), 2014, Pages 1-19.

[5] R.-J. Di Perna, P.-L. Lions, *On the Cauchy problem for Boltzmann equations : Global existence and weak stability.* Ann. Math., **130**, (1989) pp. 321-366.

[6] D. Blanchard and F. Murat, *Renormalized solutions of nonlinear parabolic problems with L^1-data, Existence and Uniqueness*. Proc. Roy. Soc. Edinburgh Sect. **A127**, (1997), pp. 1137-1152.

[7] D. Blanchard and H. Redwane, *Renormalized solutions of nonlinear parabolic evolution problems.* J. Math. Pure Appl, **77**, (1989), pp. 117-151.

[8] Y. Akdim, J. Bennouna and M. Mekkour, *Solvability of degenerated parabolic equations without sign condition and three unbounded nonlinearities,* Vol. 2011 (2011), No. 03, pp. 1-26.

[9] M. Bendahmane; P. Wittbold and A. Zimmermann, *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L1-data.* J. Differential Equations 249 (2010), no. 6, 1483-1515.

[10] D. Blanchard, F. Murat and H. Redwane, *Existence and uniqueness of renormalized solution for a fairly general class of nonlinear parabolic problems.* J. Differential Equations 177 (2001), 331-374.

[11] Y. Akdim, J. Bennouna, M. Mekkour and M. Rhoudaf *Renormalised solutions of nonlinear degenerated parabolic problems with L^1 data: existence and uniqueness*, RECENT DEVELOPMENTS IN NONLINEAR ANALYSIS - Proceedings of the Conference in Mathematics and Mathematical Physics.

[12] H. Redwane, *Solution renormalisèes de problèmes paraboliques et elleptique non linéaires. Ph.D. thesis, Rouen (1997).*

[13] H. Redwane, *Existence of a solution for a class of parabolic equations with three unbounded nonlinearities,* Adv. Dyn. Syst. Appl., 2, (2007), p.p. 241-264.

[14] M. B. Benboubker, E. Azroul, A. Barbara, *Quasilinear elliptic problems with non standard growth.*Electronic Journal of Differential Equations, Vol. (2011), No. **62**, pp. 1-16.

[15] D. Blanchard, F. Murat. *Renormalized solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness. Proceedings of the Royal Society of Edinburgh, 127A, 1137-1152, (1997).*

[16] Y. Chen, S. Levine and R. Rao, *Functionals with p(x)-growth in image processing*, http:// www.mathcs.duq.edu/∼sel/CLR05SIAPfinal.pdf

[17] X. Fan and D. Zhao; *On the spaces $L^{p(x)}(\Omega)$and $W^{k,p(x)}(\Omega)$.* J. Math. Anal. Appl. **263**, (2001), 424-446.

[18] X. L. Fan and D. Zhao, *On the generalised Orlicz-Sobolev Space $W^{k,p(x)}(\Omega)$*, J. Gansu Educ. College12(1)(1998) 1-6.

[19] J.P. Gossez, *Nonlinear elliptic boundary value prolems for equations with rapidly (or slowly) increasing coefficients*, Trans. Am. Math. Soc. 190 (1974), 163-205.

[20] M. Bendahmane, P. Wittbold, A. Zimmermann *Renormalized solutions for a nonlinear parabolic equation with variable exponent and L^1-data* J. Differential Equations **249** 1483-1515 (2010).

[21] J. L. Lions, *Quelques methodes de résolution des problèmes aux limites non linéaires*, Dunod et Gauthiers-Villars, Paris 1969.

[22] J.-M. Rakotoson, *T-sets and relaxed solutions for parabolic equations,* journal of differential equations Belgium vol.111 n2, july 15, 1994.

[23] M. Ruzicka; *Electrorheological fluids: modeling and mathematical theory.* lecture Notes in Mathematics 1748, Springer-verlaag,Berlin, (2000).

[24] J. Simon. *Compact sets in the space* $L^p(0,T,B)$, Ann.Mat.Pura.Appl., 146(1987), pp. 65-96.

[25] E. Zeidler. *Nonlinear Functional Analysis and its Applications, Springer-Verlag (New York-Heidlberg, (1990)).*

[26] V. Zhikov; *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR Izvestiya **29**, (1987), no. 1, 33-66.

[27] C.Zhang – S.Zhou, *Renormalized and entropy solutions for nonlinear parabolic equations with variable exponents and* L^1*-data,* J. Differential Equations 248 (2010) 1376-1400.

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