Gulf Journal of Mathematics
Vol 6, Issue 4 (2018) 79-100

NONLINEAR ELLIPTIC EQUATIONS WITH MEASURE DATA IN MUSIELAK-ORLICZ SPACES

A. ABERQI[1*], J. BENNOUNA[2] AND M. ELMASSOUDI[3]

ABSTRACT. In this paper we will prove the existence of solutions of the unilateral problem

$$Au - div\Phi(x,u) + H(x,u,\nabla u) = \mu \tag{0.1}$$

in Musielak spaces, where A is a Leray-Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega)$, $\mu \in L(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$, where M and $\overline{M}$ are two complementary Musielak-Orlicz functions and both the first and the second lower terms Φ and H satisfies only the growth condition and $u \geq \zeta$ where ζ is a measurable function.

1. INTRODUCTION

Let Ω be a bounded open domain in $I\!R^N$, $(N \geq 2)$ and consider the following strongly nonlinear Dirichlet problem

$$\begin{cases} u \geq \zeta, \\ -\text{div}(a(x,u,\nabla u)) - \text{div}(\Phi(x,u)) + H(x,u,\nabla u) = \mu \quad \text{on} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

Where ζ is a measurable function, $A(u) = -\text{div}(a(x,u,\nabla u))$ is a Leray- Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega)$, with M a Musielak-Orlicz function and both the first and second lower order terms Φ and H satisfies only a growth conditions described by M and its complementary $\overline{M}$. The right hand-side μ is belong $L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$.

In the setting of Orlicz space, Gossez J. P. and Mustonen V. in [10] solved (1.1) in the variational case (i.e. $\mu \in W^{-1}E_{\overline{M}}(\Omega)$) Benkirane A. and Bennouna J. in [6] has been proved the existence and uniqueness of solutions of unilateral problem where $\Phi = H = 0$ and $\mu \in L^1(\Omega)$, Ahrouch L. and Rhoudaf M. in [1] with $\Phi = 0$ and H satisfied the sign condition and Ahrouch et al. in [2] have proved the existence results where $H = 0$, $\Phi \in \mathcal{C}^0(I\!R^N, I\!R^N)$ and $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$. In the Sobolev spaces with variable exponent in [5] Redwane H. et al. has been proved the existence of solutions for some nonlinear elliptic unilateral problems with measure data where H satisfies the sign condition, $\Phi \in \mathcal{C}^0(I\!R^N, I\!R^N)$, and

Date: Received: Dec 2, 2018
* Corresponding author.
2010 *Mathematics Subject Classification.* 47A15, 46A32, 47D20.
Key words and phrases. Musielak-Orlicz spaces, Nonlinear elliptic problems, Measure data.

$\mu \in L^1(\Omega) + W^{-1,p'(x)}(\Omega)$.
As far as the Musielak space is concerned, there are also interesting works. Indeed, in [8] Benkirane A. et al. generalized the work of Gossez J.P. and V. Mustonen V.[10] where they has proved the existence of solutions for the obstacle problem, in [12] Kellou A. M. and Benkirane A. has proved the existence of solution for a non-linear elliptic unilateral problems in Musielak-Orlicz spaces with L^1 data, $\Phi = 0$, H satisfies the sign condition.
It is our purpose in this paper to prove the existence of entropy solution for unilateral problem associated to (1.1) where Φ depends on x, t, u and satisfy only the growth condition and H is a nonlinear lower-order term having natural growth with respect to $|\nabla u|$. The second member of (1.1) as $\mu = f - div(F)$ with $f \in L^1(\Omega)$ and $F \in W^{-1}E_{\overline{M}}(\Omega)$.
The main difficulties of this problem are in the first the lack of coercivity lower order term Φ that makes the operator that governs the equation, non coercive. The second lower order term H is controlled by a non-polynomial growth (see (3.5)) and no sign condition is assumed. The function M defining Musielak-Orlicz space $W^1L_M(\Omega)$ does not satisfy the Δ_2-condition which makes us lose the reflexivity of Musielak space.
As an example of equations to which the present result can be applied, we give

$$\begin{cases} u \geq \zeta & \text{in} \quad \Omega, \\ -\Delta_M(u) + usin(|\nabla u|) = f + \text{div}(F) + c(x)\overline{M}_x^{-1}M(x, \alpha_0|u|) & \text{in} \quad \Omega, \end{cases}$$

where $-\Delta_M(u) = -\text{div}(\frac{m(x,|\nabla u|)}{|\nabla u|}.\nabla u)$, m is the derivative of M with respect to t, ζ is an admissible obstacle function and $c(.) \in (L(\Omega))^N$.
The structure of the paper is organized as follows: The section 2 contains some preliminaries in the Musielak-Sobolev space. In section 3, we give the essential assumptions to prove that the solution of the problem 1.1 belong to the space $W_0^1L_M(\Omega)$. In section 4, we establish the proof of main theorem (3.1).

2. Musielak-Orlicz spaces - notation and properties

2.1. **Musielak-Orlicz function.**

Let M be a real-valued function defined in $\Omega \times I\!R_+$ and satisfying conditions:

- $M(x, .)$ is a N-function for all $x \in \Omega$, (i.e. convex, non-decreasing, continuous, $M(x, 0) = 0$, $M(x, 0) > 0$ for $t > 0$, $\lim_{t\to 0} \sup_{x\in\Omega} \frac{M(x,t)}{t} = 0$ and $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{M(x,t)}{t} = \infty$).
- $M(., t)$ is a measurable function for all $t \geq 0$.

A function M which satisfies the above conditions is called a Musielak-Orlicz function.
Let $M_x(t) = M(x, t)$, we associate its non-negative reciprocal function M_x^{-1}, with respect to t, that is $M_x^{-1}(M(x, t)) = M(x, M_x^{-1}(t)) = t$.
Let M and P be two Musielak-Orlicz functions, we say that P grows essentially less rapidly than M at 0 (resp. near infinity), and we write $P \prec\prec M$, for every positive constant c, we have $\lim_{t\to 0} \big(\sup_{x\in\Omega} \frac{P(x, ct)}{M(x, t)}\big) = 0$ (resp. $\lim_{t\to\infty} \big(\sup_{x\in\Omega} \frac{P(x, ct)}{M(x, t)}\big) = 0\big)$.

Proposition 2.1. *([9]) Let $P \prec\prec M$ near infinity and $\forall t > 0$, $\sup_{x\in\Omega} P(x,t) < \infty$, then $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ such that*

$$P(x,t) \leq M(x,\epsilon t) + C_\epsilon, \forall t > 0. \tag{2.1}$$

2.2. **Musielak-Orlicz space.**

The Musielak-Orlicz space $L_M(\Omega)$ is define as

$$L_M(\Omega) = \{u : \Omega \to I\!R \quad \text{mesurable} : \quad \varrho_{M,\Omega}(\frac{u}{\lambda}) < \infty, \quad \text{for some} \quad \lambda > 0\}.$$

where $\varrho_{M,\Omega}(u) = \displaystyle\int_\Omega M(x,|u(x)|)dx$, equipped with the Luxemburg norm

$$\|u\|_M = \inf\big\{\lambda > 0 : \varrho_{M,\Omega}(\frac{u}{\lambda}) \leq 1\big\}.$$

Denote $\overline{M}(x,s) = \sup_{t\geq 0}(st - M(x,s))$ the conjugate Musielak-Orlicz function of M.

We define $E_M(\Omega)$ as the subset of $L_M(\Omega)$ of all measurable functions $u : \Omega \mapsto I\!R$ such that $\varrho_{M,\Omega}(\frac{u}{\lambda}) < \infty$ for all $\lambda > 0$. It is a separable space and $(E_M(\Omega))^* = L_{\overline{M}}(\Omega)$.

We define the Musielak-Orlicz-Sobolev space as

$$W^1L_M(\Omega) = \{u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), \quad \forall |\alpha| \leq 1\},$$

endowed with the norm

$$\|u\|^1_{M,\Omega} = \inf\{\lambda > 0 : \sum_{|\alpha|\leq 1} \varrho_{M,\Omega}(\frac{D^\alpha u}{\lambda}) \leq 1\}.$$

Lemma 2.1. *([11])(Approximation theorem) Let Ω be a bounded Lipschitz domain in $I\!R^N$ and let M and $\overline{M}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*

1. *There exists a constant $c > 0$ such that $\inf_{x\in\Omega} M(x,1) > c$,*
2. *There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$, we have*
$$\frac{M(x,t)}{M(y,t)} \leq |t|^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \textit{for all} \quad t \geq 1,$$
3. $\displaystyle\int_K M(y,\lambda)dx < \infty, \quad \forall\lambda > 0$ *and for every compact* $K \subset \Omega$,
4. *There exists a constant $C > 0$ such that $\overline{M}(y,t) \leq C$ a.e. in Ω.*

Under this assumptions $\mathcal{D}(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W^1_0L_M(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1_0L_M(\Omega)$ for the modular convergence.

Exemple 2.1. *We give some example for a Musielak-Orlicz functions of approximation theorem*

- *$M_1(x,t) = |t|^{p(x)}$ with $p : \Omega \to [1,\infty)$ a measurable function with Log-Hölder continuite*
$$\frac{M_1(x,t)}{M_1(y,t)} = |t|^{p(x)-p(y)} \leq t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)} \quad \textit{for all} \quad t \geq 1.$$

- $M_2(x,t) = \alpha(x)(\exp(|t|) - 1 + |t|)$, $0 < \alpha(x) \in L^\infty(\Omega)$.

Remark that $M_1 \in \triangle_2$ *if* $p^+ := ess\sup_{x\in\Omega} p(x) < \infty$ *while* $M_2 \notin \triangle_2$.

Lemma 2.2. *([3], [4])(Modular Poincaré inequality) Under the assumptions of lemma 2.1, and by assuming that* $M(x,.)$ *decreases with respect to one of coordinate of* x*, there exists a constant* $\delta > 0$ *which depends only on* Ω *such that*

$$\int_\Omega M(x,|u|)dx \leq \int_\Omega M(x,\delta|\nabla u|)dx \quad \text{for all} \quad u \in W_0^1 L_M(\Omega). \tag{2.2}$$

Lemma 2.3. *([8]) Suppose that* Ω *satisfies the segment property and let* $u \in W_0^1 L_M(\Omega)$*. Then, there exists a sequence* $u_n \in \mathcal{D}(\Omega)$ *such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_M(\Omega).$$

Furthermore, if $u \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ *then* $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Lemma 2.4. *Let* $u_n, u \in L_M(\Omega)$*. If* $u_n \to u$ *with respect to the modular convergence, then* $u_n \rightharpoonup u$ *for* $\sigma(L_M, L_{\overline{M}})$.

Proof. Let $\lambda > 0$ such that $\int_\Omega M(x, \frac{u_n - u}{\lambda})dx \to 0$. Thus, for a subsequence, $u_n \to u$ a.e. in Ω. Take $v \in L_{\overline{M}}(\Omega)$. Multiplying v by a suitable constant, we can assume $\lambda v \in \mathcal{L}_{\overline{M}}(\Omega)$.
By Young's inequality,

$$|(u_n - u)v| \leq M(x, \frac{u_n - u}{\lambda}) + \overline{M}(x, \lambda v)$$

which implies, by Vitali's theorem, that $\int_\Omega |(u_n - u)v|dx \to 0$.

Truncation Operator: T_k, $k > 0$, denotes the truncation function at level k defined on $I\!R$ by $T_k(r) = \max(-k, \min(k, r))$.

3. Essential assumptions and Main results

Throughout this sequel we assume that Ω is an open bounded subset of $I\!R^N$ $(N \geq 2)$ and let M and P be two Musielak-Orlicz functions such that M and its complementary $\overline{M}$ satisfies conditions of Lemma 2.1, M is decreasing in x and $P \prec\prec M$.
$a : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in I\!R$, $\xi, \xi^* \in I\!R^N$, $\xi \neq \xi^*$

$$|a(x,s,\xi)| \leq \beta(a_0(x) + \overline{M}_x^{-1}(P(x,k_1|s|)) + \overline{M}_x^{-1}(M(x,k_2|\xi|))), \beta > 0, a_0(.) \in E_{\overline{M}}(\Omega), \tag{3.1}$$

$$(a(x,s,\xi) - a(x,s,\xi^*)(\xi - \xi^*) > 0, \tag{3.2}$$

$$a(x,s,\xi).\xi \geq \alpha M(x,|\xi|) + M(x,|s|). \tag{3.3}$$

$\Phi : \Omega \times I\!R \to I\!R^N$ is a Carathéodory function such that

$$|\Phi(x,s)| \leq c(x)\overline{M}_x^{-1} M(x, \alpha_0|s|), \tag{3.4}$$

where $c(.) \in L^\infty(\Omega)$ such that $\|c(.)\|_{L^\infty(\Omega)} < \frac{\alpha}{2}$ and $0 < \alpha_0 < \min(1, \frac{1}{\alpha})$.
$H : \Omega \times I\!R \times I\!R^N \to I\!R^N$ is a Carathéodory function such that

$$|H(x,s,\xi)| \leq h(x) + \rho(s)M(x,|\xi|), \tag{3.5}$$

$\rho : I\!R \to I\!R^+$ is continuous positive function which belong $L^\infty(I\!R)$ and h belong $L^1(\Omega)$.
$\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$ such that

$$\mu = f - \mathrm{div}(F), \tag{3.6}$$

with $f \in L^1(\Omega)$ and $F \in (E_{\overline{M}}(\Omega))^N$.

Given a negative measurable obstacle function $\zeta : \Omega \to I\!R$

$$K_\zeta = \{u \in W_0^1 L_M(\Omega) : u \geq \zeta \quad \text{a.e. in} \quad \Omega\}. \tag{3.7}$$

Definition 3.1. *A measurable function u defined on Ω is a entropy solution of problem (1.1), if it satisfies the following conditions:*

$$\begin{cases} u \in D(A) \cap W_0^1 L_M(\Omega), u \geq \zeta, \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega \Phi(x,u)\nabla T_k(u-v)dx \\ \displaystyle+\int_\Omega H(x,u,\nabla u)\nabla T_k(u-v)dx \leq \int_\Omega fT_k(u-v)dx + \int_\Omega F\nabla T_k(u-v)dx \\ \forall v \in K_\zeta \cap L^\infty(\Omega), \quad \forall k > 0. \end{cases} \tag{3.8}$$

The main result of the paper as follows,

Theorem 3.1. *(**Existence of entropy solutions**) Assume that* (3.1) – (3.7) *hold true. Then there exists at least one solution of the following unilateral problem (1.1) in the sense of the definition 3.1.*

4. Proof of theorem 3.1

Step 1: Approximate problem.
For each $n > 0$, we define the following approximations

$$a_n(x,s,\xi) = a(x,T_n(s),\xi) \quad \text{a.e. } x \in \Omega, \ \forall\, s \in I\!R, \ \forall\, \xi \in I\!R^N, \tag{4.1}$$

$$\Phi_n(x,s) = \Phi(x,T_n(s)) \quad \text{a.e. } x \in \Omega, \ \forall\, s \in I\!R, \tag{4.2}$$

$$H_n(x,s,\xi) = \frac{H(x,s,\xi)}{1 + \frac{1}{n}|H(x,s,\xi)|} \quad \text{a.e. } x \in \Omega, \ \forall\, s \in I\!R, \ \forall\, \xi \in I\!R^N, \tag{4.3}$$

$$f_n \text{ is a smooth function such that } f_n \to f \text{ strongly in } L^1(\Omega), \quad \text{and } \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{4.4}$$

And we define $sg_k(s) = \frac{T_k(s)}{k}$.
Let us now consider the approximate problem :

$$\begin{cases} u_n \in W_0^1 L_M(\Omega), \\ -div(a_n(x,u_n,\nabla u_n)) - div(\Phi_n(x,u_n)) \\ +H_n(x,u_n,\nabla u_n) - nT_n(u_n-\zeta)^- sg_{1/n}(u_n) = f_n - \mathrm{div}(F) \ in\ \Omega, \\ u_n(x) = 0 \quad on \quad \partial\Omega. \end{cases} \tag{4.5}$$

Since H_n is bounded for any fixed $n > 0$, there exists at last one solution $u_n \in W_0^1 L_M(\Omega)$ of (4.5)(see [10]).

Step 2: A priori estimates.

Lemma 4.1.
Let u_n be a solution of the approximate problem (4.5), then for all $k > 0$, there exists a constants C_1, C_2, C_3 and C_4 such that

$$\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))\nabla T_k(u_n)dx \leq kC_1 + C_2, \tag{4.6}$$

and

$$\int_\Omega M(x, |\nabla T_k(u_n)|)dx \leq kC_3 + C_4. \tag{4.7}$$

Proof. Fix $k > 0$,
Let a $\exp(G(u_n))T_k(u_n)^+$ as a test function in problem (4.5),where $G(s) = \int_0^s \frac{\rho(r)}{\alpha'}dr$, and $\alpha' > 0$ is a parameter to be specified later, we get

$$\int_\Omega a_n(x, u_n, \nabla u_n)\nabla\big(\exp(G(u_n))T_k(u_n)^+\big)dx \tag{4.8}$$

$$+ \int_\Omega \Phi_n(x, u_n))\nabla\big(\exp(G(u_n))T_k(u_n)^+\big)dx \tag{4.9}$$

$$+ \int_\Omega H(x, u_n, \nabla u_n)\exp(G(u_n))T_k(u_n)^+dx \tag{4.10}$$

$$+ \int_\Omega nT_n(u_n - \zeta)^- sg_{1/n}(u_n)\exp(G(u_n))T_k(u_n)^+dx \tag{4.11}$$

$$\leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|f_n\|_{L^1(\Omega)} + \int_\Omega F\nabla\big(\exp(G(u_n))T_k(u_n)^+\big)dx. \tag{4.12}$$

* For (4.9), we use (3.4), Lemma 2.2 and Young inequality, we get

$$\int_\Omega \Phi_n(x, u_n))\nabla(\exp(G(u_n))T_k(u_n)^+)dx$$

$$\begin{aligned}\leq \frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'}\big[\alpha_0 &\int_\Omega M(x, u_n)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dx \\ &+ \int_\Omega M(x, |\nabla u_n|)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dx\big] \\ &+\|c(.)\|_{L^\infty(\Omega)}\alpha_0 \int_{Q_\tau} M(x, |u_n|)\exp(G(u_n))dx \\ &+\|c(.)\|_{L^\infty(\Omega)} \int_\Omega M(x, |\nabla T_k(u_n)^+|)\exp(G(u_n))dx.\end{aligned}$$

* For (4.10), we have

$$\int_\Omega H_n(x, u_n, \nabla u_n)\exp(G(u_n))T_k(u_n)^+dx \leq k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|f\|_{L^1(\Omega)}$$
$$+ \int_\Omega \rho(u_n)\exp(G(u_n))M(x, |\nabla u_n|)T_k(u_n)^+dx.$$

* For (4.12), we have

$$\int_\Omega F\nabla\big(\exp(G(u_n))T_k(u_n)^+\big)dx \le \frac{k}{\alpha'}\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\|\rho\|_{L^\infty}\int_\Omega \overline{M}(x,\frac{|F|}{\epsilon_1})dx$$
$$+\frac{\epsilon_1}{\alpha'}\int_\Omega \rho(u_n)\exp(G(u_n))M(x,|\nabla u_n|)T_k(u_n)^+dx$$
$$+\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\int_\Omega \overline{M}(x,\frac{|F|}{\epsilon_1})dx+\epsilon_1\int_\Omega \exp(G(u_n))M(x,|\nabla T_k(u_n)^+|)dx.$$

Finally using the previous inequalities and (3.3), we obtain

$$\begin{cases} \frac{1}{\alpha'}\int_\Omega M(x,u_n)\rho(u_n)\exp(G(u_n)T_k(u_n)^+dx+\frac{\alpha}{\alpha'}\int_\Omega \rho(u_n)\exp(G(u_n))M(x,|\nabla T_k(u_n|))T_k(u_n)^+dx\\ +\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla(T_k(u_n)^+)dx\\ \frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'}\big[\alpha_0\int_\Omega M(x,|u_n|)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dx\\ +\int_\Omega M(x,|\nabla u_n|)\rho(u_n)\exp(G(u_n))T_k(u_n)^+dx\big]\\ +\|c(.)\|_{L^\infty(\Omega)}\alpha_0\int_\Omega M(x,|u_n|)\exp(G(u_n))\chi_{\{T_k(u_n)^+\}}dx\\ +\|c(.)\|_{L^\infty(\Omega)}\int_\Omega M(x,|\nabla T_k(u_n)^+|)\exp(G(u_n))dx\\ +\int_\Omega \rho(u_n)\exp(G(u_n))M(x,|\nabla u_n|)T_k(u_n)^+dx+\frac{\epsilon_1}{\alpha'}\int_\Omega \rho(u_n)\exp(G(u_n))M(x,|\nabla u_n|)T_k(u_n)^+dx\\ +\epsilon_1\int_\Omega M(x,|\nabla T_k(u_n)^+|)\exp(G(u_n))dx\\ +\int_\Omega nT_n(u_n-\zeta)^-sg_{1/n}(u_n)\exp(G(u_n))T_k(u_n)^+dx\\ +k\exp(\frac{\|\rho\|_{L^1}}{\alpha'})[\|f\|_{L^1(\Omega)}+\|h\|_{L^1(\Omega)}+\|\rho\|_{L^\infty}\int_\Omega \overline{M}(x,\frac{|F|}{\epsilon_1})dx]+\exp(\frac{\|\rho\|_{L^1}}{\alpha'})\int_\Omega \overline{M}(x,\frac{|F|}{\epsilon_1})dx \end{cases} \tag{4.13}$$

Using again (3.3) in (4.13) we get

$$(\frac{1-\alpha_0\|c(.)\|_{L^\infty(\Omega)}}{\alpha'})\int_\Omega M(x,|u_n|)\rho(u_n)\exp(G(u_n)T_k(u_n)^+dx$$
$$+\big[\frac{\alpha}{\alpha'}-\frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'}-1-\frac{\epsilon_1}{\alpha'}\big]\int_\Omega \rho(u_n)\exp(G(u_n)M(x,|\nabla u_n|)T_k(u_n)^+dx$$
$$+\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla(T_k(u_n)^+)dx$$
$$+\int_\Omega nT_n(u_n-\zeta)^-sg_{1/n}(u_n)\exp(G(u_n))T_k(u_n)^+dx\le$$
$$+\|c(.)\|_{L^\infty(\Omega)}\alpha_0\int_\Omega M(x,|u_n|)\exp(G(u_n))\chi_{T_k(u_n)^+}dx$$
$$+(\|c(.)\|_{L^\infty(\Omega)}+\epsilon_1)\int_\Omega M(x,|\nabla T_k(u_n)^+|)\exp(G(u_n))dx$$
$$+kc_1+c_2.$$

We choose α' and ϵ_1 such that $\alpha' = \frac{\alpha}{2}$ and $\epsilon_1 < \frac{\alpha}{2} - \|c(.)\|_{L^\infty(\Omega)}$, we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla(T_k(u_n)^+)dx$$
$$+\int_\Omega nT_n(u_n-\zeta)^- sg_{1/n}(u_n)\exp(G(u_n))T_k(u_n)^+dx \le$$
$$+(\frac{\|c(.)\|_{L^\infty(\Omega)}+\epsilon_1}{\alpha})[\alpha_0\alpha\int_\Omega M(x,|u_n|)\exp(G(u_n))\chi_{T_k(u_n)^+}dx$$
$$+\alpha\int_\Omega M(x,|\nabla T_k(u_n)^+|)\exp(G(u_n))dx] + kc_1 + c_2.$$

since $\alpha_0\alpha < 1$ and using (3.3) we get

$$[1-(\frac{\|c(.)\|_{L^\infty(\Omega)}+\epsilon_1)}{\alpha}]\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla(T_k(u_n)^+)dx$$
$$+\int_\Omega nT_n(u_n-\zeta)^- sg_{1/n}(u_n)\exp(G(u_n))T_k(u_n)^+dx$$
$$\le kc_1 + c_2,$$

we deduce

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla(T_k(u_n)^+)dx$$
$$+c'\int_\Omega nT_n(u_n-\zeta)^- sg_{1/n}(u_n)\exp(G(u_n))T_k(u_n)^+dx \le kc'c_1 + c'c_2, \tag{4.14}$$

where $\frac{1}{c'} = 1 - (\frac{\|c(.)\|_{L^\infty(\Omega)}+\epsilon_1}{\alpha})$.
It follow that

$$0 \le \int_\Omega nT_n(u-\zeta)^- sg_{1/n}(u_n)\exp(G(u_n))\frac{T_k(u_n)^+}{k}dx \le c_1,$$

we deduce by Fatou's lemma as $k \to 0$ that

$$0 \le \int_{\{u_n\ge 0\}} nT_n(u_n-\zeta)^- sg_{1/n}(u_n)\exp(G(u_n))dx \le c_1.$$

Return to (4.14), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla(T_k(u_n)^+)dx \le kc'c_1 + c'c_2,$$

as one has $\exp(G(u_n)) \ge 1$ a.e. in $\{x\in\Omega : 0 \le u_n \le k\}$ then

$$0 \le \int_{\{u_n\ge 0\}} nT_n(u_n-\zeta)^- sg_{1/n}(u_n)dx \le c_1, \tag{4.15}$$

and

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(T_k(u_n)^+)dx \le kc'c_1 + c'c_2. \tag{4.16}$$

By (3.3) and (4.16) we get

$$\int_\Omega M(x,|\nabla T_k(u_n)^+|)dx \le \frac{kc'c_1 + c'c_2}{\alpha}. \tag{4.17}$$

Similarly, taking $\exp(-G(u_n)T_k(u_n)^-$ as a test function in problem (4.5), we get

$$+\int_\Omega a_n(x,u_n,\nabla u_n)\nabla(\exp(-G(T_k(u_n))\nabla T_k(u_n)^-)dx \tag{4.18}$$

$$+\int_\Omega \Phi_n(x,u_n))\nabla(\exp(-G(T_k(u_n))\nabla T_k(u_n)^-)dx \tag{4.19}$$

$$+\int_\Omega H(x,u_n,\nabla u_n)\exp(-G(T_k(u_n))T_k(u_n)^- dx \tag{4.20}$$

$$+\int_\Omega nT_n(u_n-\zeta)^- sg_{1/n}(u_n)\exp(-G(u_n))T_k(u_n)^- dx \tag{4.21}$$

$$\geq \int_\Omega f_n \exp(-G(T_k(u_n))T_k(u_n)^- dx - \int_\Omega F\nabla(\exp(-G(T_k(u_n))\nabla T_k(u_n)^-)dx, \tag{4.22}$$

and using same techniques, we obtain also

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla(T_k(u_n)^-)dx$$

$$-c'\int_\Omega nT_n(u-\zeta)^- sg_{1/n}(u_n)\exp(-G(u_n))T_k(u_n)^- dx \leq kc'c_1 + c'c_2. \tag{4.23}$$

One has also $\exp(-G(u_n)) \geq 1)$ a.e. in $\{x\in\Omega : -k \leq u_n \leq 0\}$, then as above, it follow that

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)dx \leq kc'c_1 + c'c_2, \tag{4.24}$$

$$0 \leq -\int_{\{u_n\leq 0\}} nT_n(u_n-\zeta)^- sg_{1/n}(u_n)dx \leq c_1, \tag{4.25}$$

and

$$\int_\Omega M(x,|\nabla T_k(u_n)^-|)dx \leq \frac{kc'c_1 + c'c_2}{\alpha}. \tag{4.26}$$

Combining now (4.16) and (4.24), we get

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)dx \leq kC_1 + C_2, \tag{4.27}$$

Of the same with (4.17) and (4.26), we get

$$\int_\Omega M(x,|\nabla T_k(u_n)|)dx \leq kC_3 + C_4. \tag{4.28}$$

We conclude that $T_k(u_n)$ is bounded in $W_0^1L_M(\Omega)$ independently of n and for any $k>0$, so there exists a subsequence still denoted by u_n such that

$$T_k(u_n) \rightharpoonup \xi_k \quad \text{weakly in} \quad W_0^1L_M(\Omega). \tag{4.29}$$

On the other hand, using (4.28), we have

$$\inf_{x\in\Omega} M(x,\frac{k}{\delta})\text{meas}\{|u_n|>k\} \leq \int_{\{|u_n|>k\}} M(x,\frac{|T_k(u_n)|}{\delta})dx$$

$$\leq \int_\Omega M(x,|\nabla T_k(u_n)|)dx \leq kC_3 + C_4.$$

Then

$$\text{meas}\{|u_n| > k\} \leq \frac{kC_3 + C_4}{\inf_{x\in\Omega} M(x, \frac{k}{\delta})},$$

for all n and for all k.

Assuming that there exists a positive function ψ such that $\lim_{t\to\infty} \frac{\psi(t)}{t} = +\infty$ and
$\psi(t) \leq ess\inf_{x\in\Omega} M(x,t)$, $\forall t \geq 0$. Thus, we get

$$\lim_{k\to\infty} \text{meas}\{|u_n| > k\} = 0. \tag{4.30}$$

□

Step 3: Convergence of u_n and boundedness of $a_n(x, T_k(u_n), \nabla T_k(u_n))$
Now we turn to prove the almost every convergence of u_n and convergence of $a_n(x, T_k(u_n), \nabla T_k(u_n))$.

Proposition 4.1. *Let u_n be a solution of the approximate problem, then*

$$u_n \to u \quad a.e \ in \quad \Omega, \tag{4.31}$$

$$a_n(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k \quad in \quad (L_{\overline{M}}(\Omega))^N, \quad for \quad \sigma(\Pi L_{\overline{M}}, \Pi E_M), \tag{4.32}$$

for some $\varpi_k \in (L_{\overline{M}}(\Omega))^N$.
Proof.
Proof of (4.31) **:** Let $\eta > 0$ and $\epsilon > 0$ then

$$\begin{aligned}\text{meas}\{|u_n - u_m| > \eta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ +\text{meas}\{|T_k(u_n) - T_k(u_m)| > \eta\}.\end{aligned}$$

By (4.29), we assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω. thus, there exists $k(\epsilon) > 0$ such that $meas\{|T_k(u_n) - T_k(u_m)| > \eta\} < \epsilon$ for all $n, m > n_0$.
Finally using (4.30) allows as to prove that u_n is a Cauchy sequence in measure in Ω and then converges almost everywhere to some measurable function u.
Proof of (4.32) **:**
We shall prove that $\{a(x, T_k(u_n), \nabla T_k(u_n))\}_n$ is bounded in $\left(L_{\overline{M}}(\Omega)\right)^N$ for all $k > 0$.

Let $w \in (E_M(\Omega))^N$ be arbitrary. By condition (3.2) we have,

$$(a(x, u_n, \nabla u_n) - a(x, u_n, w))(\nabla u_n - w) > 0.$$

Then

$$\int_{\{|u_n|\leq k\}} a(x, u_n, \nabla u_n) w dx \leq \int_{\{|u_n|\leq k\}} a(x, u_n, \nabla u_n)\nabla u_n dx + \int_{\{|u_n|\leq k\}} a(x, u_n, w)(w - \nabla u_n)dx,$$

by (3.1) we have for $\nu > \beta$

$$\begin{aligned}\int_{\{|u_n|\leq k\}} \overline{M}(x, \frac{|a(x, u_n, \frac{w}{k_2})|}{3\beta})dx &\leq \frac{\beta}{3\nu}[\int_\Omega \overline{M}(x, |a_0(x)|)dx + \int_\Omega P(x, k_1|T_k(u_n)|)dx + M(x, |w|)dx] \\ &\leq \frac{\beta}{3\nu}[\int_\Omega \overline{M}(x, |a_0(x)|)dx + \int_\Omega P(x, k_1 k)dx + M(x, |w|)dx].\end{aligned} \tag{4.33}$$

Thus $\{a(x, T_k(u_n), \frac{w}{k_2})\}$ is bounded in $(L_{\overline{M}}(\Omega))^N$
by (4.33),(4.6) and by the theorem of Banach-Steinhaus, the sequence $\{a(x, T_k(u_n), \nabla T_k(u_n))\}$ remains bounded in $(L_{\overline{M}}(\Omega))^N$ and we conclude (4.32).
□

Step 4: Almost everywhere convergence of the gradients.
To have that the gradient converges almost everywhere, we need to prove this proposition.

Proposition 4.2. *Let u_n be a solution of the approximate problem (4.5), then*

$$(1)\qquad \lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx = 0, \tag{4.34}$$

(2) for a subsequence as $n \to \infty$

$$\nabla u_n \to \nabla u \quad a.e. \ in \quad \Omega. \tag{4.35}$$

Proof.
(1) Taking the function $Z_m(u_n) = T_1(u_n - T_m(u_n))^-$ and multiplying the approximating equation (4.5) by the test function $\exp(-G(u_n))Z_m(u_n)$ we get

$$\begin{cases} \displaystyle\int_\Omega a_n(x,u_n,\nabla u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dx + \int_\Omega \Phi_n(x,u_n)\nabla(\exp(-G(u_n))Z_m(u_n))dx \\ \displaystyle + \int_\Omega H_n(x,u_n,\nabla u_n)\exp(-G(u_n))Z_m(u_n)dx \\ \displaystyle + \int_\Omega nT_n(u_n-\zeta)^- sg_{1/n}(u_n)\exp(-G(u_n))Z_m(u_n)\, dx \\ \displaystyle = \int_\Omega f_n \exp(-G(u_n))Z_m(u_n)\, dx + \int_\Omega F\nabla(\exp(-G(u_n))Z_m(u_n))dx. \end{cases} \tag{4.36}$$

Using the same argument in step 2, we obtain

$$\begin{aligned} \int_\Omega M(x,|\nabla Z_m(u_n)|)dx \le \exp(\frac{\|\rho\|_{L^1}}{\alpha'})C[\int_\Omega f_n Z_m(u_n)\, dx + \int_\Omega h(x)Z_m(u_n)dx \\ +\|\rho\|_{L^\infty}\int_\Omega \overline{M}(x,\frac{|F|}{\epsilon_1})Z_m(u_n)dx] \\ +\exp(\frac{\|\rho\|_{L^1}}{\alpha'})C\int_{\{-(m+1)\le u_n\le -m\}}\overline{M}(x,\frac{|F|}{\epsilon_1})dx, \end{aligned} \tag{4.37}$$

where $\dfrac{1}{C} = [1-(\dfrac{\|c(.)\|_{L^\infty(\Omega)}+\epsilon_1}{\alpha})]$.
Passing to the limit as $n \to +\infty$, since the pointwise convergence of u_n and strongly convergence in $L^1(\Omega)$ of f_n, we get

$$\begin{aligned} \lim_{n\to+\infty}\int_\Omega M(x,|\nabla Z_m(u_n)|)dx \le \exp(\frac{\|\rho\|_{L^1}}{\alpha'})C[\int_\Omega fZ_m(u)\, dx + \int_\Omega h(x)Z_m(u)dx \\ +\|\rho\|_{L^\infty}\int_\Omega \overline{M}(x,\frac{|F|}{\epsilon_1})Z_m(u)dx] \\ +\exp(\frac{\|\rho\|_{L^1}}{\alpha'})C\int_{\{-(m+1)\le u\le -m\}}\overline{M}(x,\frac{|F|}{\epsilon_1})dx. \end{aligned}$$

Using Lebesgue's theorem and passing to the limit as $m \to +\infty$, in the all term of the right-hand side, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_\Omega M(x,|\nabla Z_m(u_n)|)dx = 0. \tag{4.38}$$

On the other hand, by (3.4) and Young inequality, for $n > m+1$ we obtain

$$\int_\Omega |\Phi_n(x,u_n)\exp(-G(u_n))\nabla Z_m(u_n)|dx$$

$$\leq \exp(\frac{\|\rho\|_{L^1}}{\alpha'})[\int_{\{-(m+1)\leq u_n\leq -m\}} M(x,\alpha_0|T_{m+1}(u_n)|)dx + \int_\Omega M(x,|\nabla Z_m(u_n)|)dx]$$

Using the pointwise convergence of u_n and by Lebesgue's theorem, it follows

$$\lim_{n\to+\infty}\int_\Omega |\Phi_n(x,u_n)\exp(-G(u_n))\nabla Z_m(u_n)|dx$$

$$\leq \exp(\frac{\|\rho\|_{L^1}}{\alpha'})[\int_{\{-(m+1)\leq u\leq -m\}} M(x,\alpha_0|T_{m+1}(u)|)dx + \lim_{n\to+\infty}\int_\Omega M(x,|\nabla Z_m(u_n)|)dx],$$

passing to the limit in as $m \to +\infty$, we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_\Omega \Phi_n(x,u_n)\exp(-G(u_n))\nabla Z_m(u_n)dx = 0.$$

Finally passing to the limit in (4.37), we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{-(m+1)\leq u_n\leq -m\}} a_n(x,u_n,\nabla u_n)\nabla u_n dx = 0.$$

In the same way we take $Z_m(u_n) = T_1(u_n - T_m(u_n))^+$ and multiplying the approximating equation (4.5) by the test function $\exp(G(u_n))Z_m(u_n)$ and we also obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\leq u_n\leq m+1\}} a_n(x,u_n,\nabla u_n)\nabla u_n dx = 0.$$

On the above we get (4.34).

(2) For the almost everywhere convergence of the gradients, we use the following lemma.

Lemma 4.2. *Under the assumptions (3.1)–(3.6), let z_n be a sequence in $W_0^1L_M(\Omega)$ such that:*

$$z_n \rightharpoonup z \quad \textit{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{4.39}$$

$$(a(x,z_n,\nabla z_n)) \quad \textit{is bounded in } (L_{\overline{M}}(\Omega))^N, \tag{4.40}$$

$$\int_\Omega [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]dx \to 0, \tag{4.41}$$

as n and s tend to $+\infty$, and where χ_s is the characteristic function of $\Omega_s = \{x\in\Omega; |\nabla z|\leq s\}$ then,

$$\nabla z_n \to \nabla z \quad \textit{a.e. in } \Omega, \tag{4.42}$$

$$\lim_{n\to+\infty}\int_\Omega a(x,z_n,\nabla z_n)\nabla z_n dx = \int_\Omega a(x,z,\nabla z)\nabla z dx, \tag{4.43}$$

$$M(x, |\nabla z_n|) \to M(x, |\nabla z|) \quad in\ L^1(\Omega). \tag{4.44}$$

Proof. see [7]. □

Let $v_j \in \mathcal{D}(\Omega)$ be a sequence such that $v_j \to u$ in $W_0^1 L_M(\Omega)$ for the modular convergence.

We introduce a sequence of increasing $\mathbf{C}^1(I\!R)$-functions S_m such that

$S_m(r) = 1$ for $|r| \le m$, $S_m(r) = m+1-|r|$, for $m \le |r| \le m+1$, $S_m(r) = 0$ for $|r| \ge m+1$

for any $m \ge 1$ and we denote by $\epsilon(n, \eta, j, m)$ all quantities (possibly different) such that

$$\lim_{m\to+\infty} \lim_{j\to+\infty} \lim_{\eta\to+\infty} \lim_{\mu\to+\infty} \lim_{n\to+\infty} \epsilon(n, \eta, j, m) = 0,$$

the main estimate is

For fixed $k \ge 0$, let $W_\eta^{n,j} = T_\eta(T_k(u_n) - T_k(v_j))^+$ and $W_\eta^j = T_\eta(T_k(u) - T_k(v_j))^+$. Multiplying the approximating equation by $\exp(G(u_n))W_\eta^{n,j} S_m(u_n)$ and using the same technique in step 2 we obtain:

$$\begin{cases} + \displaystyle\int_\Omega a_n(x, u_n, \nabla u_n) \exp(G(u_n))\nabla(W_\eta^{n,j}) S_m(u_n) dx \\ + \displaystyle\int_\Omega a_n(x, u_n, \nabla u_n)\nabla u_n \exp(G(u_n)) W_\eta^{n,j} S'_m(u_n)\, dx \\ - \displaystyle\int_\Omega \Phi_n(x, u_n) \exp(G(u_n))\nabla(W_\eta^{n,j}) S_m(u_n)\, dx - \int_\Omega \Phi_n(x, u_n)\nabla u_n \exp(G(u_n)) W_\eta^{n,j} S'_m(u_n) dx \\ \le \displaystyle\int_\Omega f_n \exp(G(u_n)) W_\eta^{n,j} S_m(u_n)\, dx + \int_\Omega h(x) \exp(G(u_n)) W_\eta^{n,j} S_m(u_n) dx \\ + \displaystyle\int_\Omega F \exp(G(u_n))\nabla(W_\eta^{n,j}) S_m(u_n)\, dx + \int_\Omega F\nabla u_n \exp(G(u_n)) W_\eta^{n,j} S'_m(u_n) dx. \end{cases} \tag{4.45}$$

Now we pass to the limit in (4.45) for k real number fixed.
In order to perform this task we prove below the following results for any fixed $k \ge 0$:

$$\int_\Omega \Phi_n(x, u_n) S_m(u_n) \exp(G(u_n))\nabla(W_\eta^{n,j})\, dx = \epsilon(n, j) \quad \text{for any } m \ge 1, \tag{4.46}$$

$$\int_\Omega \Phi_n(x, u_n)\nabla u_n S'_m(u_n) \exp(G(u_n)) W_\eta^{n,j}\, dx = \epsilon(n, j) \qquad \text{for any } m \ge 1, \tag{4.47}$$

$$\int_\Omega a_n(x, u_n, \nabla u_n)\nabla u_n S'_m(u_n) \exp(G(u_n)) W_\eta^{n,j}\, dx \le \epsilon(n, m), \tag{4.48}$$

$$\int_\Omega a_n(x, u_n, \nabla u_n) S_m(u_n) \exp(G(u_n))\nabla(W_\eta^{n,j})\, dx \le C\eta + \epsilon(n, j, m), \tag{4.49}$$

$$\int_\Omega f_n S_m(u_n) \exp(G(u_n)) W_\eta^{n,j}\, dx + \int_\Omega h(x) \exp(G(u_n)) W_\eta^{n,j} S_m(u_n)\, dx \le C\eta + \epsilon(n, \eta), \tag{4.50}$$

$$+\int_\Omega F\exp(G(u_n))\nabla(W_\eta^{n,j})S_m(u_n)\,dx \int_\Omega F\nabla u_n \exp(G(u_n))W_\eta^{n,j}S'_m(u_n)dx \le \epsilon(n,m,j,\eta), \tag{4.51}$$

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]dx \to 0. \tag{4.52}$$

Proof of (4.46): If we take $n > m+1,$ we get

$$\Phi_n(x,u_n)\exp(G(u_n))S_m(u_n) = \Phi(x,T_{m+1}(u_n))\exp(G(T_{m+1}(u_n)))S_m(T_{m+1}(u_n)),$$

then $\Phi_n(x,u_n)\exp(G(u_n))S_m(u_n)$ is bounded in $L_{\overline{M}}(Q)$, thus, by using the pointwise convergence of u_n and Lebesgue's theorem we obtain $\Phi_n(x,u_n)\exp(G(u_n))S_m(u_n) \to \Phi(x,u)\exp(G(u))S_m(u)$ with the modular convergence as $n \to +\infty$, then $\Phi_n(x,u_n)\exp(G(u_n))S_m(u_n) \to \Phi(x,u)\exp(G(u))S_m(u)$ for $\sigma(\Pi L_{\overline{M}}, \Pi L_M)$.
In the other hand $\nabla W_\eta^{n,j} = \nabla T_k(u_n) - \nabla(T_k(v_j))$ for $0 \le T_k(u_n) - (T_k(v_j)) \le \eta$ converge to $\nabla T_k(u) - \nabla(T_k(v_j))$ weakly in $(L_M(\Omega))^N$, then

$$\int_\Omega \Phi_n(x,u_n)\exp(G(u_n))S_m(u_n)\nabla W_\eta^{n,j}\,dx \to \int_\Omega \Phi(x,u)S_m(u)\exp(G(u))\nabla W_\eta^j\,dx$$

as $n \to +\infty$.
By using the modular convergence of W_η^j as $j \to +\infty$ and letting μ tends to infinity, we get (4.46).
Proof of (4.47):
For $n > m+1 > k$, we have $\nabla u_n S'_m(u_n) = \nabla T_{m+1}(u_n)$ a.e. in Ω. By the almost every where convergence of u_n we have $\exp(G(u_n))W_\eta^{n,j} \to \exp(G(u))W_\eta^j$ in $L^\infty(\Omega)$ weak-* and since the sequence $(\Phi_n(x,T_{m+1}(u_n)))_n$ converge strongly in $E_{\overline{M}}(\Omega)$ then

$$\Phi_n(x,T_{m+1}(u_n))\exp(G(u_n))\ W_\eta^{n,j} \to \Phi(x,T_{m+1}(u))\ \exp(G(u))W_\eta^j$$

converge strongly in $E_{\overline{M}}(\Omega)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_M(\Omega))^N$ as $n \to +\infty$ we have

$$\int_{\{m\le|u_n|\le m+1\}} \Phi_n(x,T_{m+1}(u_n))\nabla u_n S'_m(u_n)\exp(G(u_n))W_\eta^{n,j}\,dx$$
$$\to \int_{\{m\le|u|\le m+1\}} \Phi(x,u))\nabla u\exp(G(u))W_\eta^j\,dx$$

as $n \to +\infty$.
with the modular convergence of W_η^j as $j \to +\infty$ and letting $\mu \to +\infty$ we get (4.47).
Proof of (4.48):
For (4.48), we have

$$\int_\Omega a_n(x,u_n,\nabla u_n)S'_m(u_n)\nabla u_n\exp(G(u_n))W_\eta^{n,j}\,dx$$
$$= \int_{\{m\le|u_n|\le m+1\}} a_n(x,u_n,\nabla u_n)S'_m(u_n)\nabla u_n\exp(G(u_n))W_\eta^{n,j}\,dx$$

$$\leq \eta C \int_{\{m\leq |u_n|\leq m+1\}} a_n(x,u_n,\nabla u_n)\nabla u_n\,dx.$$

Using (4.34), we get

$$\int_\Omega a_n(x,u_n,\nabla u_n)S'_m(u_n)\nabla u_n \exp(G(u_n))W_\eta^{n,j}\,dx \leq \epsilon(n,m).$$

Proof of (4.50):
Since $S_m(r)\leq 1$ and $W_\eta^{n,j}\leq \eta$ we get
$\int_\Omega f_n S_m(u_n)\exp(G(u_n))W_\eta^{n,j}dx \leq \epsilon(n,\eta)$, and $\int_\Omega h(x)\exp(G(u_n))W_\eta^{n,j}S_m(u_n)dx \leq \epsilon(\eta)$.

Proof of (4.51):
Denoting by $I_{F,1} = \int_\Omega F\exp(G(u_n))\nabla(W_\eta^{n,j})S_m(u_n)\,dx$
and by $I_{F,2} = \int_\Omega F\nabla u_n \exp(G(u_n))W_\eta^{n,j}S'_m(u_n)dx$.
For the first integral we have

$$I_{F,1} \leq \exp(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'})\int_\Omega F\,\nabla W_\eta^{n,j}dx \leq \epsilon(\eta).$$

Since $T_k(u_n)$ and $T_k(v_j)$ converge weakly in $W^{0,1}L_M(\Omega)$, we deduce

$$I_{F,1} \leq \epsilon(n,j,\eta).$$

For the first integral we know that $\nabla u_n S'_m(u_n) = T_{m+1}(u_n)$ and using (3.3) we get

$$I_{F,2} \leq \exp(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'})[\epsilon_1\int_\Omega \overline{M}(x,\frac{F}{\epsilon_1})W_\eta^{n,j}dx+\epsilon_1\eta\int_{m\leq|u_n|\leq m+1}a_n(x,u_n,\nabla u_n)\nabla u_n dx]$$
$$\leq \epsilon(n,m,j,\eta).$$

Proof of (4.49):

$$\begin{aligned}&\int_\Omega a_n(x,u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla W_\eta^{n,j}\,dx\\ &= \int_{\{|u_n|\leq k\}\cap\{0\leq T_k(u_n)-T_k(v_j))\leq \eta\}} a_n(x,T_k(u_n),\nabla T_k(u_n))S_m(u_n)\exp(G(u_n))(\nabla T_k(u_n)-\nabla T_k(v_j))\,dx\\ &\quad -\int_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(v_j))\leq \eta\}} a_n(x,u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla T_k(v_j)\,dx.\end{aligned} \tag{4.53}$$

Since $a_n(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{M}}(\Omega))^N$, there exist some $\varpi_{k+\eta}\in (L_{\overline{M}}(\Omega))^N$ such that $a_n(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)) \to \varpi_{k+\eta}$ weakly in $(L_{\overline{M}}(\Omega))^N$. Consequently:

$$\begin{aligned}&\int_{\{|u_n|>k\}\cap\{0\leq T_k(u_n)-T_k(v_j))\leq \eta\}} a_n(x,u_n,\nabla u_n)S_m(u_n)\exp(G(u_n))\nabla T_k(v_j)\,dx\\ &= \int_{\{|u|>k\}\cap\{0\leq T_k(u)-T_k(v_j))\leq \eta\}} S_m(u)\exp(G(u))\nabla T_k(v_j)\varpi_{k+\eta}\,dx + \epsilon(n),\end{aligned} \tag{4.54}$$

where we have used the fact that

$$S_m(u_n)\exp(G(u_n))\nabla T_k(\upsilon_j))\chi_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j))\le\eta\}}$$
$$\to S_m(u)\exp(G(u))\nabla T_k(\upsilon_j))\chi_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j))\le\eta\}}$$

strongly in $(E_M(\Omega))^N$.
Letting $j\to+\infty$, we obtain

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(\upsilon_j))\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(\upsilon_j)\varpi_{k+\eta}\,dx$$
$$=\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u))\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(u)\varpi_{k+\eta}\,dx+\epsilon(n,j).$$

One has,

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(u))\le\eta\}} S_m(u)\exp(G(u))\nabla T_k(u)\varpi_{k+\eta}\,dx=\epsilon(n,j).$$

By (4.45)-(4.50), (4.53) and (4.54) we obtain

$$\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j))|\le\eta\}} a_n(x,T_k(u_n),\nabla T_k(u_n))S_m(u_n)\exp(G(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j))\,dx$$
$$\le C\eta+\epsilon(n,j,m),$$

we know that $\exp(G(u_n))\ge 1$ and $S_m(u_n)=1$ for $|u_n|\le k$ then

$$\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(\upsilon_j))|\le\eta\}} a_n(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n)-\nabla T_k(\upsilon_j))\,dx$$
$$\le C\eta+\epsilon(n,j,m). \tag{4.55}$$

Proof of (4.52):
Setting for $s>0$, $\Omega^s=\{x\in\Omega:|\nabla T_k(u)|\le s\}$ and $\Omega^s_j=\{x\in\Omega:|\nabla T_k(\upsilon_j)|\le s\}$ and denoting by χ^s and χ^s_j the characteristic functions of Ω^s and Ω^s_j respectively, we deduce that letting $0<\delta<1$, define

$$\Theta_{n,k}=(a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n)-\nabla T_k(u)).$$

For $s>0$, we have

$$0\le\int_{\Omega^s}\Theta^\delta_{n,k}\,dx=\int_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}}\,dx+\int_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{T_k(u_n)-T_k(\upsilon_j)>\eta\}}\,dx.$$

The first term of the right-side hand, with the Hölder inequality

$$\int_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}}\,dx\le(\int_{\Omega^s}\Theta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}}\,dx)^\delta(\int_{\Omega^s}dx)^{1-\delta}$$
$$\le C_1(\int_{\Omega^s}\Theta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(\upsilon_j)\le\eta\}}\,dx)^\delta.$$

Also using the Hölder inequality, the second term of the right-side hand is

$$\int_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{T_k(u_n)-T_k(\upsilon_j)>\eta\}}\,dx\le(\int_{\Omega^s}\Theta_{n,k}\,dx)^\delta(\int_{T_k(u_n)-T_k(\upsilon_j)>\eta)}dx)^{1-\delta},$$

since $a(x, T_k(u_n), \nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(\Omega))^N$, while $\nabla T_k(u_n)$ is bounded in $(L_M(\Omega))^N$ then

$$\int_{\Omega^s} \Theta_{n,k}^{\delta} \chi_{\{T_k(u_n)-T_k(\upsilon_j)>\eta\}} \, dx \leq C_2 meas\{x \in \Omega : T_k(u_n) - T_k(\upsilon_j) > \eta\}^{1-\delta}.$$

We obtain

$$\int_{\Omega^s} \Theta_{n,k}^{\delta} \, dx \leq C_1 (\int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta\}} \, dx)^{\delta}$$
$$+ C_2 meas\{x \in \Omega : T_k(u_n) - T_k(\upsilon_j) > \eta\}^{1-\delta}.$$

On the other hand

$$\int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta\}} \, dx$$
$$\leq \int_{\{0 \leq T_k(u)-T_k(\upsilon_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx$$

For each $s > r, r > 0,$ one has

$$0 \leq \int_{\Omega^r \cap \{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta)\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u)) \, dx$$

$$\leq \int_{\Omega^s \cap \{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta)\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u)) \, dx$$

$$= \int_{\Omega^s \cap \{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta)\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s))(\nabla T_k(u_n) - \nabla T_k(u)\chi_s) \, dx$$

$$\leq \int_{\Omega \cap \{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta)\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi^s))(\nabla T_k(u_n) - \nabla T_k(u)\chi^s) \, dx$$

$$= \int_{\{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)\chi_j^s) \, dx$$

$$+ \int_{\{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s) \, dx$$

$$+ \int_{\{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(\upsilon_j)\chi_j^s) - a(x, T_k(u_n), \nabla T_k(u)\chi^s))\nabla T_k(u_n) \, dx$$

$$- \int_{\{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(\upsilon_j)\chi_j^s)\nabla T_k(\upsilon_j)\chi_j^s \, dx$$

$$+ \int_{\{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u)\chi^s)\nabla T_k(u)\chi^s \, dx$$

$$= I_1(n, j, s) + I_2(n, j) + I_3(n, j) + I_4(n, j) + I_5(n).$$

passing to the limit as n, j, μ, and $s \to +\infty$

$$I_1 = \int_{\{0 \leq T_k(u_n)-T_k(\upsilon_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)) \, dx$$

$$- \int_{\{0\leq T_k(u_n)-T_k(\upsilon_j)\leq\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(\upsilon_j))\,dx$$

$$- \int_{\{0\leq T_k(u_n)-T_k(\upsilon_j)\leq\eta\}} a(x,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)\chi_j^s))\,dx.$$

Using (4.55), the first term of the right-hand side, we get

$$\int_{\{0\leq T_k(u_n)-T_k(\upsilon_j)\leq\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(\upsilon_j))\,dx$$

$$\leq C\eta + \epsilon(n,m,j,s) - \int_{\{|u|>k\cap 0\leq T_k(u)-T_k(\upsilon_j)\leq\eta\}} a(x,T_k(u),0)\nabla T_k(\upsilon_j)\,dx$$

$$\leq C\eta + \epsilon(n,m,j).$$

The second term of the right-hand side tends to

$$\int_{\{0\leq T_k(u)-T_k(\upsilon_j)\leq\eta\}} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(\upsilon_j))\,dx,$$

since $a(x,T_k(u_n),\nabla T_k(u_n))$ is bounded in $(L_{\overline{M}}(\Omega))^N$, there exist some $\varpi_k \in (L_{\overline{M}}(\Omega))^N$ such that (for a subsequence still denoted by u_n

$$a(x,T_k(u_n),\nabla T_k(u_n)) \to \varpi_k \quad \text{in} \quad (L_M(\Omega))^N \quad \text{for} \quad \sigma(\Pi L_{\overline{M}}, \Pi E_M).$$

In view of the fact that

$$(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(\upsilon_j))\chi_{\{0\leq T_k(u_n)-T_k(\upsilon_j)\leq\eta\}} \to (\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(\upsilon_j))\chi_{\{0\leq T_k(u)-T_k(\upsilon_j)\leq\eta\}}$$

Strongly in $(E_M(\Omega))^N$ as $n \to +\infty$.
The third term of the right-hand side tends to

$$\int_{\{0\leq T_k(u)-T_k(\upsilon_j)\leq\eta\}} a(x,T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))(\nabla T_k(u) - \nabla T_k(\upsilon_j)\chi_j^s))\,dx.$$

Since

$$a(x,T_k(u_n),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{\{0\leq T_k(u_n)-T_k(\upsilon_j)\leq\eta\}} \to a(x,T_k(u),\nabla T_k(\upsilon_j)\chi_j^s))\chi_{\{0\leq T_k(u)-T_k(\upsilon_j)\leq\eta\}}$$

in $(E_{\overline{M}}(\Omega))^N$ while

$$(\nabla T_k(u_n) - \nabla T_k(\upsilon_j)\chi_j^s)) \to (\nabla T_k(u) - \nabla T_k(\upsilon_j)\chi_j^s))$$

in $(L_M(\Omega))^N$ for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ Passing to limit as $j \to +\infty$ and $\mu \to +\infty$ and using Lebesgue's theorem, we have

$$I_1 \leq C\eta + \epsilon(n,j,s).$$

For what concerns I_2, by letting $n \to +\infty$, we have

$$I_2 \to \int_{\{0\leq T_k(u)-T_k(\upsilon_j)\leq\eta\}} \varpi_k(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\,dx.$$

Since $a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k$ in $(L_{\overline{M}}(\Omega))^N$, for $\sigma(\Pi L_{\overline{M}}, \Pi E_M)$ while

$$(\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}} \to (\nabla T_k(\upsilon_j)\chi_j^s - \nabla T_k(u)\chi^s)\chi_{\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}}$$

strongly in $(E_M(\Omega))^N$.
Passing to limit $j \to +\infty$, and using Lebesgue's theorem, we have

$$I_2 = \epsilon(n, j).$$

Similar ways as above give

$$I_3 = \epsilon(n, j),$$

$$I_4 = \int_{\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}} a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\, dx + \epsilon(n, j, s, m),$$

$$I_5 = \int_{\{0\le T_k(u)-T_k(\upsilon_j)\le\eta\}} a(x, T_k(u), \nabla T_k(u))\nabla T_k(u)\, dx + \epsilon(n, j, s, m).$$

Finally, we obtain

$$\int_{\Omega^s} \Theta_{n,k}\, dx \le C_1(C\eta + \epsilon(n, \eta, m))^\delta + C_2(\epsilon(n,))^{1-\delta}.$$

Which yields, by passing to the limit sup over n, j ,μ, s and η

$$\int_{\{T_\eta(T_k(u_n)-T_k(\upsilon_j))\ge 0\}\cap\Omega^r} [(a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))]^\delta\, dx = \epsilon(n). \tag{4.56}$$

Taking on the hand the function $W_\eta^{n,j} = T_\eta(T_k(u_n) - T_k(\upsilon_j))^-$ and $W_\eta^j = T_\eta(T_k(u) - T_k(\upsilon_j))^-$. Multiplying the approximating equation by $\exp(G(u_n)))W_\eta^{n,j}S_m(u_n)$, we obtain

$$\int_{\{T_\eta(T_k(u_n)-T_k(\upsilon_j))\le 0\}\cap\Omega^r} [(a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))]^\delta\, dx = \epsilon(n), \tag{4.57}$$

by (4.56) and (4.57) we get

$$\int_{\Omega^r} [(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))]^\delta\, dx = \epsilon(n).$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Ω^r, and since r is arbitrary,

$$\nabla u_n \to \nabla u \text{ a.e. in } \quad \Omega^r.$$

□

Step 5: Equi-integrability of the nonlinearity sequence:
We shall prove that $H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u)$ strongly in $L^1(\Omega)$.
Consider $g_0(u_n) = \displaystyle\int_0^{u_n} \rho(s)\chi_{\{s>h\}} ds$ and multiply (4.5) by $\exp(G(T_k(u_n)))g_0(u_n)$, we get after using the same technique in step 2,

$$\int_{\{u_n>h\}} \rho(u_n)M(x, \nabla u_n)dx \leq$$

$$(\int_h^{+\infty} \rho(s)dx) \exp\big(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'}\big)[\|f\|_{L^1(\Omega)} + \|h(x)\|_{L^1(\Omega)} + \frac{\|\rho\|_{L^\infty(I\!R)}}{\alpha'} \int_\Omega \overline{M}(x, \frac{F}{\epsilon_1})dx]$$
$$+ \exp\big(\frac{\|\rho\|_{L^1(I\!R)}}{\alpha'} \|\rho\|_{L^\infty(I\!R)} \int_{\{u_n>h\}} \overline{M}(x, \frac{F}{\epsilon_1})dx.$$

Since $\rho \in L^1(I\!R)$, we get

$$\lim_{h\to 0} \sup_{n\in I\!N} \int_{\{u_n>h\}} \rho(u_n)M(x, \nabla u_n)dx = 0.$$

Similarly, let $g_0(u_n) = \displaystyle\int_{u_n}^0 \rho(s)\chi_{\{s<-h\}}dx$ in (4.5), we have also

$$\lim_{h\to 0} \sup_{n\in I\!N} \int_{\{u_n<-h\}} \rho(u_n)M(x, \nabla u_n)dx = 0.$$

We conclude that

$$\lim_{h\to 0} \sup_{n\in I\!N} \int_{\{|u_n|>h\}} \rho(u_n)M(x, \nabla u_n)dx = 0. \tag{4.58}$$

Let $D \subset \Omega$ then

$$\int_D \rho(u_n)M(x, \nabla u_n)dx \leq \max_{\{|u_n|\leq h\}}(\rho(x)) \int_{D\cap\{|u_n|\leq h\}} M(x, \nabla u_n)dx + \int_{D\cap\{|u_n|>h\}} \rho(u_n)M(x, \nabla u_n)dx$$

Consequently $\rho(u_n)M(x, \nabla u_n)$ is equi-integrable. Then $\rho(u_n)M(x, \nabla u_n)$ converge to $\rho(u)M(x, \nabla u)$ strongly in $L^1(I\!R)$. By (3.5), we get our result.

Step 6: We show that u satisfies (3.8)
Firstly show that $u \geq \zeta$ a.e. in Ω.
In fact, from (4.15) and (4.25) we get

$$0 \leq \int_\Omega T_n(u_n - \zeta)^- dx \leq \frac{c_1}{n}.$$

Let n tends to $+\infty$ we obtain

$$\int_\Omega (u - \zeta)^- dx = 0,$$

then $(u - \zeta)^- = 0$ a.e. in Ω; thus $u \geq \zeta$ a.e. in Ω.
Secondly passing Now to the limit in (4.59) to show that u satisfies the equation (4.5).
Let $v \in K_\zeta \cap L^\infty(\Omega)$, then by lemma (2.3) there exists $v_j \in \mathcal{D}(\Omega)$ such that $v_j \to v$ in $W_0^1L_M(\Omega)$ for the modular convergence in $W_0^1L_M(\Omega)$

with $\|v_j\|_{L^\infty(\Omega)} \leq (N+1)\|v\|_{L^\infty(\Omega)}$, and $v_j \in K_\zeta$.
Pointwise multiplication of the approximate equation (4.5) by $T_k(u_n - v_j)$, we get

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega a_n(x, u_n, \nabla u_n))\nabla T_k(u_n - v_j)dx + \int_\Omega \Phi_n(x, u_n)\nabla T_k(u_n - v_j)dx \\ \displaystyle + \int_\Omega H_n(x, u_n)\nabla T_k(u_n - v_j)dx + \int_\Omega nT_n(u_n - \zeta)^- sg_{1/n}(u_n)T_k(u_n - v_j)dx \\ \displaystyle = \int_\Omega f_n T_k(u_n - v_j)dx - \int_\Omega F\nabla T_k(u_n - v_j)dx. \end{array} \right. \tag{4.59}$$

We pass to the limit as in (4.59), n tend to $+\infty$ and j tend to $+\infty$:

- We follow same way in [6] to prove that

$$\liminf_{j\to\infty} \liminf_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v_j)dxds \geq \int_\Omega a(x, u, \nabla u)\nabla T_k(u - v)dxds.$$

- For $n \geq k + (N+1)\|v\|_{L^\infty(\Omega)}$

  $\Phi_n(x, u_n)\nabla T_k(u_n - v_j) = \Phi(x, T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u_n))\nabla T_k(u_n - v_j)$.

  The pointwise convergence of u_n to u as n tends to $+\infty$ and (3.4), then

$$\Phi(x, T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u_n))\nabla T_k(u_n - v_j) \rightharpoonup \Phi(x, T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u))\nabla T_k(u - v_j)$$

  weakly for $\sigma(\Pi L_v, \Pi L_{\overline{M}})$.
  In a similar way, we obtain

$$\lim_{j\to\infty} \int_\Omega \Phi(x, T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u))\nabla T_k(u - v_j)dx = \int_\Omega \Phi(x, T_{k+(N+1)\|v\|_{L^\infty(\Omega)}}(u))\nabla T_k(u - v)dx$$
$$= \int_\Omega \Phi(x, u)\nabla T_k(u - v)dx.$$

- Limit of $H_n(x, u_n, \nabla u_n)T_k(u_n - v_j)$:
  Since $H_n(x, u_n, \nabla u_n)$ converge strongly to $H(x, u, \nabla u)$ in $L^1(\Omega)$ and and the pointwise convergence of u_n to u as $n \to +\infty$, it is possible to prove that $H_n(x, u_n, \nabla u_n)T_k(u_n - v_j)$ converge to $H(x, u, \nabla u)T_k(u - v_j)$ in $L^1(\Omega)$ and

$$\lim_{j\to\infty} \int_\Omega H(x, u, \nabla u)T_k(u - v_j)dx = \int_\Omega H(x, u, \nabla u)T_k(u - v)dx.$$

- Since f_n converge strongly to f in $L^1(\Omega)$, and $T_k(u_n - v_j) \to T_k(u - v_j)$ weakly* in $L^\infty(\Omega)$, we have $\displaystyle\int_\Omega f_n T_k(u_n - v_j)dx \to \int_\Omega f T_k(u - v_j)dx$ as $n \to \infty$,
  and $\displaystyle\int_\Omega f T_k(u - v_j)dx \to \int_\Omega f T_k(u - v)dx$ as $j \to \infty$.
  Also, it easy to get $\displaystyle\lim_{j\to\infty} \lim_{n\to\infty} \int_\Omega F\nabla T_k(u_n - v_j)dx = \int_\Omega F\nabla T_k(u - v)dx$.
  Finally Since $\displaystyle -\int_\Omega T_n(u_n - \zeta)^- T_k(u_n - v_j)dx \geq 0$, we obtain (3.8).

As a conclusion, the proof of Theorem (3.1) is complete.

References

[1] L. Aharouch and M. Roudaf, *Strongly nonlinear elliptic unilateral problems in Orlicz space and L^1 data*, v6, issue 2, article 54, (2005).

[2] L. Aharouch, E. Azroul and M. Rhoudaf, *Nonlinear unilateral problems in Orlicz spaces*, Applicationes Mathematicae 33, 2 (2006), pp. 217-241

[3] A. Ahmed, A. Benkirane, M.S.B. Elemine Vall and A. Touzani, *Existence of entropy solutions for nonlinear elliptic equations in Musielak framework with L^1 data*, Bol. Soc. Parana. Mat. (3) 36, no. 1, (2018), pp. 125-150.

[4] Y. Ahmida, I. Chlebicka, P. Gwiazda and A. Youssfi, *Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces*, Journal of Functional Analysis, V. 275, I 9, (2018), 2538-2571.

[5] E. Azroul, H. Redwane and C. Yazough, *Existence of solutions for some nonlinear elliptic unilateral problems with measure data*, E.J.A.T.D.E., No. 43, (2013), 1-21.

[6] A. Benkirane, J. Bennouna, *Existence and uniqueness of solution of unilateral problems with L^1-data in Orlicz spaces*, Abstract and Applied Analysis 7:2, (2002), 85-102.

[7] A. Benkirane, M.B. Elemine Vall, A. Ahmed and A.Touzani, *Existence of entropy solutions for nonlinear elliptic equations in Musielak framework with L^1 data*, Bol. Soc. Paran. Mat. v. 36 1, (2018), 125-150.

[8] A. Benkirane and M. Sidi El Vally, *Variational inequalities in Musielak-Orlicz-Sobolev spaces*, Bull. Belg. Math. Soc. Simon Stevin, (2014), pp. 787-811.

[9] M. Elmassoudi, A. Aberqi and J. Bennouna, *Existence of Entropy Solutions in Musielak-Orlicz Spaces Via a Sequence of Penalized Equations*, Bol. Soc. Paran. Mat., In press. Doi:10.5269/bspm.v38i6.37269.

[10] J. P. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces*, Nonlinear Anal. 11, (1987), 379-492.

[11] M. A. Kellou and A. Benkirane, *Elliptic inequalities with L^1 data in Musielak-Orlicz spaces*, Monatsh Math. DOI 10.1007/s00605-016-1010-1, (2016).

[12] M. A. Kellou and A. Benkirane, *Non-linear Elliptic Unilateral Problems in Musielak-Orlicz spaces with L^1 data*, Bol. Soc. Paran. Mat.(3s.) v. 36 1 (2018), 51-78.

[1]National School of Applied Sciences Fez, University of Fez, Laboratory LISA, Morocco.

Email address: [1]aberqi_ahmed@yahoo.fr

[2,3]Laboratory LAMA, Department of Mathematics,, University of Fez, Faculty of Sciences Dhar El Mahraz,, B.P 1796 Atlas Fez, Morocco.

Email address: [2]jbennouna@hotmail.com, [3]elmassoudi09@gmail.com