NONLINEAR ELLIPTIC EQUATIONS WITH MEASURE DATA IN MUSIELAK-ORLICZ SPACES

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ABSTRACT. In this paper we will prove the existence of solutions of the unilateral problem

\[ Au - \text{div}(\Phi(x,u)) + H(x,u,\nabla u) = \mu \]  \tag{0.1}

in Musielak spaces, where \( A \) is a Leray-Lions operator defined on \( D(A) \subset W^{1}_0 L_{\overline{M}}(\Omega) \), \( \mu \in L(\Omega) + W^{-1} E_{\overline{M}}(\Omega) \), where \( M \) and \( \overline{M} \) are two complementary Musielak-Orlicz functions and both the first and the second lower terms \( \Phi \) and \( H \) satisfy only the growth condition and \( u \geq \zeta \) where \( \zeta \) is a measurable function.

1. Introduction

Let \( \Omega \) be a bounded open domain in \( \mathbb{R}^N \), \( N \geq 2 \) and consider the following strongly nonlinear Dirichlet problem

\[
\begin{cases}
    u \geq \zeta, \\
    -\text{div}(a(x,u,\nabla u)) - \text{div}(\Phi(x,u)) + H(x,u,\nabla u) = \mu & \text{on } \Omega, \\
    u = 0 & \text{on } \partial \Omega.
\end{cases}
\]  \tag{1.1}

Where \( \zeta \) is a measurable function, \( A(u) = -\text{div}(a(x,u,\nabla u)) \) is a Leray- Lions operator defined on \( D(A) \subset W^{1}_0 L_{\overline{M}}(\Omega) \), with \( M \) a Musielak-Orlicz function and both the first and second lower order terms \( \Phi \) and \( H \) satisfies only a growth conditions described by \( M \) and its complementary \( \overline{M} \). The right hand-side \( \mu \) is belong \( L^1(\Omega) + W^{-1} E_{\overline{M}}(\Omega) \).

In the setting of Orlicz space, Gossez J. P. and Mustonen V. in [10] solved (1.1) in the variational case (i.e. \( \mu \in W^{-1} E_{\overline{M}}(\Omega) \)) Benkirane A. and Bennouna J. in [6] has been proved the existence and uniqueness of solutions of unilateral problem where \( \Phi = H = 0 \) and \( \mu \in L^1(\Omega) \), Ahrouch L. and Rhoudaf M. in [1] with \( \Phi = 0 \) and \( H \) satisfied the sign condition and Ahrouch et al. in [2] have proved the existence results where \( H = 0, \Phi \in C^0(\mathbb{R}^N, \mathbb{R}^N) \) and \( \mu \in L^1(\Omega) + W^{-1} E_{\overline{M}}(\Omega) \).

In the Sobolev spaces with variable exponent in [5] Redwane H. et al. has been proved the existence of solutions for some nonlinear elliptic unilateral problems with measure data where \( H \) satisfies the sign condition, \( \Phi \in C^0(\mathbb{R}^N, \mathbb{R}^N) \), and
\( \mu \in L^1(\Omega) + W^{-1,\Phi'(x)}(\Omega) \).

As far as the Musielak space is concerned, there are also interesting works. Indeed, in [8] Benkirane A. et al. generalized the work of Gossez J.P. and V. Mustonen V. [10] where they have proved the existence of solutions for the obstacle problem, in [12] Kellou A. M. and Benkirane A. has proved the existence of solution for a non-linear elliptic unilateral problems in Musielak-Orlicz spaces with \( L^1 \) data, \( \Phi = 0, H \) satisfies the sign condition.

It is our purpose in this paper to prove the existence of entropy solution for unilateral problem associated to (1.1) where \( \Phi \) depends on \( x, t, u \) and satisfy only the growth condition and \( H \) is a nonlinear lower-order term having natural growth with respect to \( |\nabla u| \). The second member of (1.1) as \( \mu = f - \text{div}(F) \) with \( f \in L^1(\Omega) \) and \( F \in W^{-1}E^N(\Omega) \).

The main difficulties of this problem are in the first the lack of coercivity lower order term \( \Phi \) that makes the operator that governs the equation, non coercive. The second lower order term \( H \) is controlled by a non-polynomial growth (see (3.5)) and no sign condition is assumed. The function \( M \) defining Musielak-Orlicz space \( W^1L^M(\Omega) \) does not satisfy the \( \Delta_2 \)-condition which makes us lose the reflexivity of Musielak space.

As an example of equations to which the present result can be applied, we give

\[
\begin{cases}
    u \geq \zeta & \text{in } \Omega, \\
    -\Delta_M(u) + u \sin(|\nabla u|) = f + \text{div}(F) + c(x)M^{-1}_x M(x, \alpha_0|u|) & \text{in } \Omega,
\end{cases}
\]

where \( -\Delta_M(u) = -\text{div}\left(\frac{m(x,|\nabla u|)}{|\nabla u|} \nabla u\right) \), \( m \) is the derivative of \( M \) with respect to \( t \), \( \zeta \) is an admissible obstacle function and \( c(.) \in (L(\Omega))^N \).

The structure of the paper is organized as follows: The section 2 contains some preliminaries in the Musielak-Sobolev space. In section 3, we give the essential assumptions to prove that the solution of the problem 1.1 belong to the space \( W^1_0L_M(\Omega) \). In section 4, we establish the proof of main theorem (3.1).

2. MUSIELAK-ORLICZ SPACES - NOTATION AND PROPERTIES

2.1. MUSIELAK-ORLICZ FUNCTION.

Let \( M \) be a real-valued function defined in \( \Omega \times IR_+ \) and satisfying conditions:

- \( M(x,.) \) is a N-function for all \( x \in \Omega \), (i.e. convex, non-decreasing, continuous, \( M(x,0) = 0, M(x,0) > 0 \) for \( t > 0 \), \( \lim_{t \to 0} \sup_{x \in \Omega} \frac{M(x,t)}{t} = 0 \) and \( \lim_{t \to \infty} \inf_{x \in \Omega} \frac{M(x,t)}{t} = \infty \).
- \( M(.,t) \) is a measurable function for all \( t \geq 0 \).

A function \( M \) which satisfies the above conditions is called a Musielak-Orlicz function.

Let \( M_x(t) = M(x,t) \), we associate its non-negative reciprocal function \( M_x^{-1} \), with respect to \( t \), that is \( M_x^{-1}(M(x,t)) = M(x,M_x^{-1}(t)) = t \).

Let \( M \) and \( P \) be two Musielak-Orlicz functions, we say that \( P \) grows essentially less rapidly than \( M \) at \( 0 \) (resp. near infinity), and we write \( P \ll M \), for every positive constant \( c \), we have \( \lim_{t \to 0} \left( \sup_{x \in \Omega} \frac{P(x,ct)}{M(x,t)} \right) = 0 \) (resp. \( \lim_{t \to \infty} \left( \sup_{x \in \Omega} \frac{P(x,ct)}{M(x,t)} \right) = 0 \)).
Proposition 2.1. ([9]) Let $P \ll M$ near infinity and $\forall t > 0$, sup$_{x \in \Omega} P(x, t) < \infty$, then $\forall \epsilon > 0$, $\exists C_\epsilon > 0$ such that

$$P(x, t) \leq M(x, \epsilon t) + C_\epsilon, \forall t > 0. \tag{2.1}$$

2.2. Musielak-Orlicz space.

The Musielak-Orlicz space $L_M(\Omega)$ is defined as

$$L_M(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable : } \varrho_{M, \Omega}(\frac{u}{\lambda}) < \infty, \text{ for some } \lambda > 0 \}.$$ 

where $\varrho_{M, \Omega}(u) = \int_\Omega M(x, |u(x)|)dx$, equipped with the Luxemburg norm

$$\|u\|_M = \inf \{ \lambda > 0 : \varrho_{M, \Omega}(\frac{u}{\lambda}) \leq 1 \}.$$ 

Denote $\overline{M}(x, s) = \sup_{t \geq 0}(st - M(x, s))$ the conjugate Musielak-Orlicz function of $M$.

We define $E_M(\Omega)$ as the subset of $L_M(\Omega)$ of all measurable functions $u : \Omega \mapsto \mathbb{R}$ such that $\varrho_{M, \Omega}(\frac{u}{\lambda}) < \infty$ for all $\lambda > 0$. It is a separable space and $(E_M(\Omega))^* = L_M(\Omega)$.

We define the Musielak-Orlicz-Sobolev space as

$$W^1 L_M(\Omega) = \{ u \in L_M(\Omega) : D^\alpha u \in L_M(\Omega), \forall |\alpha| \leq 1 \},$$

endowed with the norm

$$\|u\|_{1, M, \Omega} = \inf \{ \lambda > 0 : \sum_{|\alpha| \leq 1} \varrho_{M, \Omega}(\frac{D^\alpha u}{\lambda}) \leq 1 \}.$$ 

Lemma 2.1. ([11]) (Approximation theorem) Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $M$ and $\overline{M}$ be two complementary Musielak-Orlicz functions which satisfy the following conditions:

1. There exists a constant $c > 0$ such that $\inf_{x \in \Omega} M(x, 1) > c$.
2. There exists a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$, we have
   $$\frac{M(x, t)}{\overline{M}(y, t)} \leq |t| \left( \frac{A}{\log(1/|x-y|)} \right) \text{ for all } t \geq 1,$$
3. $\int_K M(y, \lambda)dx < \infty, \forall \lambda > 0$ and for every compact $K \subset \Omega$,
4. There exists a constant $C > 0$ such that $\overline{M}(y, t) \leq C$ a.e. in $\Omega$.

Under these assumptions $D(\Omega)$ is dense in $L_M(\Omega)$ with respect to the modular topology, $D(\Omega)$ is dense in $W^1_0 L_M(\Omega)$ for the modular convergence and $D(\Omega)$ is dense in $W^1_0 L_M(\Omega)$ for the modular convergence.

Example 2.1. We give some example for a Musielak-Orlicz functions of approximation theorem

- $M_1(x, t) = |t|^{p(x)}$ with $p : \Omega \to [1, \infty)$ a measurable function with Log-Hölder continuous
  $$\frac{M_1(x, t)}{M_1(y, t)} = |t|^{p(x) - p(y)} \leq t \left( \frac{A}{\log(1/|x-y|)} \right) \text{ for all } t \geq 1.$$
\( M_2(x, t) = \alpha(x)(\exp(|t|) - 1 + |t|), \) \( 0 < \alpha(x) \in L^\infty(\Omega). \)

Remark that \( M_1 \in \Delta_2 \) if \( p^+ := \text{ess sup} p(x) < \infty \) while \( M_2 \notin \Delta_2. \)

Lemma 2.2. ([3], [4]) (Modular Poincaré inequality) Under the assumptions of lemma 2.1, and by assuming that \( M(x, \cdot) \) decreases with respect to one of coordinate of \( x, \) there exists a constant \( \delta > 0 \) which depends only on \( \Omega \) such that

\[
\int_\Omega M(x, |u|)dx \leq \int_\Omega M(x, \delta|\nabla u|)dx \quad \text{for all} \quad u \in \text{W}_0^1 \text{L}_M(\Omega). \tag{2.2}
\]

Lemma 2.3. ([8]) Suppose that \( \Omega \) satisfies the segment property and let \( u \in \text{W}_0^1 \text{L}_M(\Omega). \) Then, there exists a sequence \( u_n \in \mathcal{D}(\Omega) \) such that

\[
u_n \rightarrow u \quad \text{for modular convergence in} \quad \text{W}_0^1 \text{L}_M(\Omega).
\]

Furthermore, if \( u \in \text{W}_0^1 \text{L}_M(\Omega) \cap L^\infty(\Omega) \) then \( \|u_n\|_\infty \leq (N + 1)\|u\|_\infty. \)

Lemma 2.4. Let \( u_n, u \in \text{L}_M(\Omega) \). If \( u_n \rightarrow u \) with respect to the modular convergence, then \( u_n \rightarrow u \) for \( \sigma(\text{L}_M, \text{L}_M^\infty). \)

Proof. Let \( \lambda > 0 \) such that \( \int_\Omega M(x, \frac{u_n - u}{\lambda})dx \rightarrow 0. \) Thus, for a subsequence, \( u_n \rightarrow u \) a.e. in \( \Omega. \) Take \( v \in \text{L}_M^\infty(\Omega). \) Multiplying \( v \) by a suitable constant, we can assume \( \lambda v \in \text{L}_M^\infty(\Omega). \)

By Young’s inequality,

\[
|(u_n - u)v| \leq M(x, \frac{u_n - u}{\lambda}) + M(x, \lambda v)
\]

which implies, by Vitali’s theorem, that \( \int_\Omega |(u_n - u)v|dx \rightarrow 0. \)

Truncation Operator: \( T_k, k > 0, \) denotes the truncation function at level \( k \) defined on \( \mathbb{R} \) by \( T_k(r) = \max(-k, \min(k, r)). \)

3. Essential assumptions and Main results

Throughout this sequel we assume that \( \Omega \) is an open bounded subset of \( \mathbb{R}^N \) \((N \geq 2)\) and let \( M \) and \( P \) be two Musielak-Orlicz functions such that \( M \) and its complementary \( M^\ast \) satisfies conditions of Lemma 2.1, \( M \) is decreasing in \( x \) and \( P \prec \prec M. \)

\( a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is Carathéodory function such that for a.e. \( x \in \Omega \) and for all \( s \in \mathbb{R}, \) \( \xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^* \)

\[
|a(x, s, \xi)| \leq \beta(a_0(x) + M^{-1}_x(P(x, k_1|s|)) + M^{-1}_x(M(x, k_2|\xi|))), \beta > 0, a_0(.) \in E_{\text{L}_M^\infty}(\Omega), \tag{3.1}
\]

\[
(a(x, s, \xi) - a(x, s, \xi^*)(\xi - \xi^*)) > 0, \tag{3.2}
\]

\[
a(x, s, \xi), \xi \geq \alpha M(x, |\xi|) + M(x, |s|). \tag{3.3}
\]

\( \Phi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^N \) is a Carathéodory function such that

\[
|\Phi(x, s)| \leq c(x)M^{-1}_xM(x, \alpha_0|s|), \tag{3.4}
\]
where $c(\cdot) \in L^\infty(\Omega)$ such that $\|c(\cdot)\|_{L^\infty(\Omega)} < \frac{\alpha}{2}$ and $0 < \alpha_0 < \min(1, \frac{1}{\alpha})$.

$H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that

$$|H(x, s, \xi)| \leq h(x) + \rho(s)M(x, |\xi|), \quad (3.5)$$

$\rho : \mathbb{R} \to \mathbb{R}^+$ is continuous positive function which belong $L^\infty(\mathbb{R})$ and $h$ belong $L^1(\Omega)$.

$\mu \in L^1(\Omega) + W^{-1}E_{\mathcal{M}}(\Omega)$ such that

$$\mu = f - \text{div}(F), \quad (3.6)$$

with $f \in L^1(\Omega)$ and $F \in (E_{\mathcal{M}}(\Omega))^N$.

Given a negative measurable obstacle function $\zeta : \Omega \to \mathbb{R}$

$$K_\zeta = \{u \in W^1_0L_M(\Omega) : u \geq \zeta \text{ a.e. in } \Omega \}. \quad (3.7)$$

**Definition 3.1.** A measurable function $u$ defined on $\Omega$ is an entropy solution of problem $(1.1)$, if it satisfies the following conditions:

$$\begin{align*}
&\left\{ u \in D(A) \cap W^1_0L_M(\Omega), u \geq \zeta, \\
&\int_\Omega a(x, u, \nabla u)\nabla T_k(u - v)dx + \int_\Omega \Phi(x, u)\nabla T_k(u - v)dx \\
&\quad + \int_\Omega H(x, u, \nabla u)\nabla T_k(u - v)dx \leq \int_\Omega fT_k(u - v)dx + \int_\Omega F\nabla T_k(u - v)dx \\
&\forall v \in K_\zeta \cap L^\infty(\Omega), \forall k > 0.
\end{align*} \quad (3.8)$$

The main result of the paper as follows,

**Theorem 3.1. (Existence of entropy solutions)** Assume that $(3.1)$ – $(3.7)$ hold true. Then there exists at least one solution of the following unilateral problem $(1.1)$ in the sense of the definition 3.1.

4. Proof of Theorem 3.1

**Step 1: Approximate problem.**

For each $n > 0$, we define the following approximations

$$a_n(x, s, \xi) = a(x, T_n(s), \xi) \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \quad (4.1)$$

$$\Phi_n(x, s) = \Phi(x, T_n(s)) \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}, \quad (4.2)$$

$$H_n(x, s, \xi) = \frac{H(x, s, \xi)}{1 + \frac{1}{n}|H(x, s, \xi)|} \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \quad (4.3)$$

$f_n$ is a smooth function such that $f_n \to f$ strongly in $L^1(\Omega)$, and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$. \quad (4.4)

And we define $sg_k(s) = \frac{T_k(s)}{k}$.

Let us now consider the approximate problem:

$$\begin{align*}
&u_n \in W^1_0L_M(\Omega), \\
&-\text{div}(a_n(x, u_n, \nabla u_n)) - \text{div}(\Phi_n(x, u_n)) \\
&+ H_n(x, u_n, \nabla u_n) - nT_n(u_n - \zeta)^{-}sg_{1/n}(u_n) = f_n - \text{div}(F) \text{ in } \Omega, \\
&u_n(x) = 0 \quad \text{on } \partial\Omega.
\end{align*} \quad (4.5)$$
Since $H_n$ is bounded for any fixed $n > 0$, there exists at least one solution $u_n \in W_0^1 L_M(\Omega)$ of (4.5) (see [10]).

**Step 2: A priori estimates.**

**Lemma 4.1.**

Let $u_n$ be a solution of the approximate problem (4.5), then for all $k > 0$, there exists a constants $C_1, C_2, C_3$ and $C_4$ such that

$$
\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) dx \leq kC_1 + C_2,
$$

(4.6)

and

$$
\int_\Omega M(x, |\nabla T_k(u_n)|) dx \leq kC_3 + C_4.
$$

(4.7)

**Proof.** Fix $k > 0$,

Let a $\exp(G(u_n))T_k(u_n)^+$ as a test function in problem (4.5), where $G(s) = \int_0^s \frac{\rho(r)}{\alpha'} dr$, and $\alpha' > 0$ is a parameter to be specified later, we get

$$
\int_\Omega a_n(x, u_n, \nabla u_n) \nabla \left( \exp(G(u_n))T_k(u_n)^+ \right) dx
$$

(4.8)

$$
+ \int_\Omega \Phi_n(x, u_n) \nabla \left( \exp(G(u_n))T_k(u_n)^+ \right) dx
$$

(4.9)

$$
+ \int_\Omega H(x, u_n, \nabla u_n) \exp(G(u_n))T_k(u_n)^+ dx
$$

(4.10)

$$
+ \int_\Omega nT_n(u_n - \zeta)^{-sg} L_1(u_n) \exp(G(u_n))T_k(u_n)^+ dx
$$

(4.11)

* For (4.9), we use (3.4), Lemma 2.2 and Young inequality, we get

$$
\int_\Omega \Phi_n(x, u_n) \nabla \left( \exp(G(u_n))T_k(u_n)^+ \right) dx
$$

$$
\leq k \exp\left( \frac{\|\rho\|_{L^1(\Omega)}}{\alpha'} \right) ||f_n||_{L^1(\Omega)} + \int_\Omega F \nabla \left( \exp(G(u_n))T_k(u_n)^+ \right) dx.
$$

(4.12)

* For (4.10), we have

$$
\int_\Omega H_n(x, u_n, \nabla u_n) \exp(G(u_n))T_k(u_n)^+ dx \leq k \exp\left( \frac{\|\rho\|_{L^1(\Omega)}}{\alpha'} \right) ||f||_{L^1(\Omega)}
$$

$$
+ \int_\Omega \rho(u_n) \exp(G(u_n)) M(x, |\nabla u_n|) T_k(u_n)^+ dx.
$$
* For (4.12), we have
\[
\int_{\Omega} F \nabla (\exp(G(u_n))T_k(u_n)^+) dx \leq \frac{k}{\alpha'} \exp\left( \frac{\|\rho\|_{L^1}}{\alpha'} \right) \int_{\Omega} M(x, |F|) dx \\
+ \frac{\epsilon_1}{\alpha} \int_{\Omega} \rho(u_n) \exp(G(u_n)) M(x, |\nabla u_n|) T_k(u_n)^+ dx \\
+ \exp\left( \frac{\|\rho\|_{L^1}}{\alpha'} \right) \int_{\Omega} M(x, |F|) dx + \epsilon_1 \int_{\Omega} \exp(G(u_n)) M(x, |\nabla T_k(u_n)^+|) dx.
\]
Finally using the previous inequalities and (3.3), we obtain
\[
\left\{ \begin{array}{l}
\frac{1}{\alpha'} \int_{\Omega} M(x, u_n) \rho(u_n) \exp(G(u_n)) T_k(u_n)^+ dx + \frac{\alpha}{\alpha'} \int_{\Omega} \rho(u_n) \exp(G(u_n)) M(x, |\nabla T_k(u_n)|) T_k(u_n)^+ dx \\
+ \int_{\Omega} a(x, u_n, \nabla u_n) \exp(G(u_n)) \nabla (T_k(u_n)^+) dx \\
+ \frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'} \left[ \alpha_0 \int_{\Omega} M(x, |u_n|) \rho(u_n) \exp(G(u_n)) T_k(u_n)^+ dx \\
+ \int_{\Omega} M(x, |\nabla u_n|) \rho(u_n) \exp(G(u_n)) T_k(u_n)^+ dx \right] \\
+ \frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'} \int_{\Omega} M(x, |\nabla T_k(u_n)^+|) \exp(G(u_n)) dx \\
+ \int_{\Omega} \rho(u_n) \exp(G(u_n)) M(x, |\nabla u_n|) T_k(u_n)^+ dx + \frac{\epsilon_1}{\alpha'} \int_{\Omega} \rho(u_n) \exp(G(u_n)) M(x, |\nabla u_n|) T_k(u_n)^+ dx \\
+ \epsilon_1 \int_{\Omega} M(x, |\nabla T_k(u_n)^+|) \exp(G(u_n)) dx \\
+ \int_{\Omega} nT_n(u_n - \zeta)^- \text{sg}_1(u_n) \exp(G(u_n)) T_k(u_n)^+ dx \\
+ k \exp\left( \frac{\|\rho\|_{L^1}}{\alpha'} \right) \|f\|_{L^1(\Omega)} + \|h\|_{L^1(\Omega)} + \|\rho\|_{L^\infty} \int_{\Omega} M(x, |F|) dx + \exp\left( \frac{\|\rho\|_{L^1}}{\alpha'} \right) \int_{\Omega} M(x, |F|) dx
\end{array} \right.
\]
Using again (3.3) in (4.13) we get
\[
\left( 1 - \alpha_0 \frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'} \right) \int_{\Omega} M(x, |u_n|) \rho(u_n) \exp(G(u_n)) T_k(u_n)^+ dx \\
+ \left[ \frac{\alpha}{\alpha'} - \frac{\|c(.)\|_{L^\infty(\Omega)}}{\alpha'} \right] \int_{\Omega} \rho(u_n) \exp(G(u_n)) M(x, |\nabla u_n|) T_k(u_n)^+ dx \\
+ \int_{\Omega} a(x, u_n, \nabla u_n) \exp(G(u_n)) \nabla (T_k(u_n)^+) dx \\
+ \int_{\Omega} nT_n(u_n - \zeta)^- \text{sg}_1(u_n) \exp(G(u_n)) T_k(u_n)^+ dx \leq \\
+ \|c(.)\|_{L^\infty(\Omega)} \alpha_0 \int_{\Omega} M(x, |u_n|) \exp(G(u_n)) \chi T_k(u_n)^+ dx \\
+ (\|c(.)\|_{L^\infty(\Omega)} + \epsilon_1) \int_{\Omega} M(x, |\nabla T_k(u_n)^+|) \exp(G(u_n)) dx \\
+ kc_1 + c_2.
\]
We choose \( \alpha' \) and \( \epsilon_1 \) such that \( \alpha' = \frac{\alpha}{2} \) and \( \epsilon_1 < \frac{\alpha}{2} - \|c(.)\|_{L^\infty(\Omega)} \), we obtain
\[
\int_\Omega a(x, u_n, \nabla u_n) \exp(G(u_n)) \nabla (T_k(u_n)^+) dx \\
+ \int_\Omega nT_n(u_n - \zeta) - sg_{1/n}(u_n) \exp(G(u_n))T_k(u_n)^+ dx \leq \\
\left( \frac{\|c(.)\|_{L^\infty(\Omega)} + \epsilon_1}{\alpha} \right) \left[ \alpha_0 \alpha \int_\Omega M(x, |u_n|) \exp(G(u_n)) \chi_{T_k(u_n)} dx \\
+ \alpha \int_\Omega M(x, |\nabla T_k(u_n)|) \exp(G(u_n)) dx \right] \leq kc_1 + c_2.
\]

since \( \alpha_0 \alpha < 1 \) and using (3.3) we get
\[
\int_\Omega a(x, u_n, \nabla u_n) \exp(G(u_n)) \nabla (T_k(u_n)^+) dx \\
+ c' \int_\Omega nT_n(u_n - \zeta) - sg_{1/n}(u_n) \exp(G(u_n))T_k(u_n)^+ dx \leq kc_1 + c_2,
\]

(4.14)

where \( \frac{1}{c'} = 1 - \left( \frac{\|c(.)\|_{L^\infty(\Omega)} + \epsilon_1}{\alpha} \right) \).

It follows that
\[
0 \leq \int_\Omega nT_n(u - \zeta) - sg_{1/n}(u_n) \exp(G(u_n)) \frac{T_k(u_n)^+}{k} dx \leq c_1.
\]

we deduce by Fatou's lemma as \( k \to 0 \) that
\[
0 \leq \int_{\{u_n \geq 0\}} nT_n(u_n - \zeta) - sg_{1/n}(u_n) \exp(G(u_n)) dx \leq c_1.
\]

Return to (4.14), we have
\[
\int_\Omega a(x, u_n, \nabla u_n) \exp(G(u_n)) \nabla (T_k(u_n)^+) dx \leq kc'c_1 + c'c_2,
\]
as one has \( \exp(G(u_n)) \geq 1 \) a.e. in \( \{ x \in \Omega : 0 \leq u_n \leq k \} \) then
\[
0 \leq \int_{\{u_n \geq 0\}} nT_n(u_n - \zeta) - sg_{1/n}(u_n) dx \leq c_1,
\]

(4.15)

and
\[
\int_\Omega a(x, u_n, \nabla u_n) \nabla (T_k(u_n)^+) dx \leq kc'c_1 + c'c_2.
\]

(4.16)

By (3.3) and (4.16) we get
\[
\int_\Omega M(x, |\nabla T_k(u_n)^+|) dx \leq \frac{kc'c_1 + c'c_2}{\alpha}.
\]
Similarly, taking $\exp(-G(u_n)T_k(u_n)^-)\Phi_n(x,u_n)$ as a test function in problem (4.5), we get
\begin{align}
+ \int_{\Omega} a_n(x,u_n,\nabla u_n)\nabla \left(\exp(-G(T_k(u_n)))\nabla T_k(u_n)^-\right)dx \\
+ \int_{\Omega} \Phi_n(x,u_n)\nabla \left(\exp(-G(T_k(u_n)))\nabla T_k(u_n)^-\right)dx \\
+ \int_{\Omega} H(x,u_n,\nabla u_n)\exp(-G(T_k(u_n)))\nabla T_k(u_n)^-dx \\
+ \int_{\Omega} nT_n(u_n - \zeta)^-sg_1/n(u_n)\exp(-G(u_n))T_k(u_n)^-dx \\
\geq \int_{\Omega} f_n\exp(-G(T_k(u_n)))T_k(u_n)^-dx - \int_{\Omega} F\nabla \left(\exp(-G(T_k(u_n)))\nabla T_k(u_n)^-\right)dx,
\end{align}
and using same techniques, we obtain also
\begin{align}
\int_{\Omega} a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla (T_k(u_n)^-)dx \\
- c' \int_{\Omega} nT_n(u - \zeta)^-sg_1/n(u_n)\exp(-G(u_n))T_k(u_n)^-dx \leq kc_1' + c_2'.
\end{align}
One has also $\exp(-G(u_n)) \geq 1$ a.e. in $\{x \in \Omega : -k \leq u_n \leq 0\}$, then as above, it follow that
\begin{align}
\int_{\Omega} a(x,u_n,\nabla u_n)\nabla T_k(u_n)dx \leq kc_1' + c_2', \\
0 \leq - \int_{\{u_n \leq 0\}} nT_n(u_n - \zeta)^-sg_1/n(u_n)dx \leq c_1,
\end{align}
and
\begin{align}
\int_{\Omega} M(x,|\nabla T_k(u_n)^-|)dx \leq \frac{kc_1' + c_2'}{\alpha}.
\end{align}
Combining now (4.16) and (4.24), we get
\begin{align}
\int_{\Omega} a(x,u_n,\nabla u_n)\nabla T_k(u_n)dx \leq kC_1 + C_2,
\end{align}
Of the same with (4.17) and (4.26), we get
\begin{align}
\int_{\Omega} M(x,|\nabla T_k(u_n)|)dx \leq kC_3 + C_4.
\end{align}
We conclude that $T_k(u_n)$ is bounded in $W^1_0L_M(\Omega)$ independently of $n$ and for any $k > 0$, so there exists a subsequence still denoted by $u_n$ such that
\begin{align}
T_k(u_n) \rightharpoonup \xi_k \quad \text{weakly in} \quad W^1_0L_M(\Omega).
\end{align}
On the other hand, using (4.28), we have
\begin{align}
\inf_{x \in \Omega} M(x,\frac{k}{\delta}\text{meas}\{|u_n| > k\}) \leq \int_{\{|u_n| > k\}} M(x,\frac{|T_k(u_n)|}{\delta})dx \\
\leq \int_{\Omega} M(x,|\nabla T_k(u_n)|)dx \leq kC_3 + C_4.
\end{align}
Then
\[ \text{meas}\{|u_n| > k \} \leq \frac{kC_3 + C_4}{\inf_{x \in \Omega} M(x, \frac{\delta}{3})}, \]
for all \( n \) and for all \( k \).

Assuming that there exists a positive function \( \psi \) such that \( \lim_{t \to \infty} \frac{\psi(t)}{t} = +\infty \)
and \( \psi(t) \leq \text{ess inf}_{x \in \Omega} M(x, t), \forall t \geq 0 \). Thus, we get
\[ \lim_{k \to \infty} \text{meas}\{|u_n| > k \} = 0. \quad (4.30) \]

\[ \square \]

**Step 3: Convergence of \( u_n \) and boundedness of \( a_n(x, T_k(u_n), \nabla T_k(u_n)) \)**

Now we turn to prove the almost every convergence of \( u_n \) and convergence of \( a_n(x, T_k(u_n), \nabla T_k(u_n)) \).

**Proposition 4.1.** Let \( u_n \) be a solution of the approximate problem, then
\[ u_n \to u \quad \text{a.e in} \quad \Omega, \quad (4.31) \]
\[ a_n(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varpi_k \quad \text{in} \quad (L_{M^*}^\infty(\Omega))^N, \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_M), \quad (4.32) \]
for some \( \varpi_k \in (L_{M^*}^\infty(\Omega))^N \).

**Proof.**

**Proof of (4.31)**: Let \( \eta > 0 \) and \( \epsilon > 0 \) then
\[ \text{meas}\{|u_n - u_m| > \eta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \eta\}. \]

By (4.29), we assume that \( (T_k(u_n))_n \) is a Cauchy sequence in measure in \( \Omega \).
thus, there exists \( k(\epsilon) > 0 \) such that \( \text{meas}\{|T_k(u_n) - T_k(u_m)| > \eta\} < \epsilon \) for all \( n, m > n_0 \).

Finally using (4.30) allows us to prove that \( u_n \) is a Cauchy sequence in measure in \( \Omega \) and then converges almost everywhere to some measurable function \( u \).

**Proof of (4.32)**:

We shall prove that \( \{a_n(x, T_k(u_n), \nabla T_k(u_n))\}_n \) is bounded in \( (L_{M^*}^\infty(\Omega))^N \) for all \( k > 0 \).

Let \( w \in (E^*_M(\Omega))^N \) be arbitrary. By condition (3.2) we have,
\[ (a(x, u_n, \nabla u_n) - a(x, u_n, w))(\nabla u_n - w) > 0. \]

Then
\[ \int_{\{|u_n| \leq k\}} a(x, u_n, \nabla u_n)w dx \leq \int_{\{|u_n| \leq k\}} a(x, u_n, \nabla u_n)\nabla u_n dx + \int_{\{|u_n| \leq k\}} a(x, u_n, w)(w - \nabla u_n) dx, \]
by (3.1) we have for \( \nu > \beta \)
\[ \int_{\{|u_n| \leq k\}} \frac{M(x, |a(x, u_n, w)| \frac{w}{k_2})}{3\beta} dx \leq \frac{\beta}{3\nu} \int_{\Omega} M(x, |a_0(x)|) dx + \int_{\Omega} P(x, k_1|T_k(u_n)|) dx + M(x, |w|) dx \]
\[ \leq \frac{\beta}{3\nu} \int_{\Omega} M(x, |a_0(x)|) dx + \int_{\Omega} P(x, k_1 k) dx + M(x, |w|) dx. \quad (4.33) \]
Thus \( \{a(x, T_k(u_n), \frac{w}{K})\} \) is bounded in \((L^N_{\text{loc}}(\Omega))^N\) by (4.33), (4.6) and by the theorem of Banach-Steinhaus, the sequence \( \{a(x, T_k(u_n), \nabla T_k(u_n))\} \) remains bounded in \((L^N_{\text{loc}}(\Omega))^N\) and we conclude (4.32).

\[ \square \]

**Step 4: Almost everywhere convergence of the gradients.**

To have that the gradient converges almost everywhere, we need to prove this proposition.

**Proposition 4.2.** Let \( u_n \) be a solution of the approximate problem (4.5), then

\[
\begin{align*}
(1) \quad & \lim_{m \to \infty} \limsup_{n \to \infty} \int_{\{m \leq |u_n| \leq m+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0, \\
(2) \quad & \text{for a subsequence as } n \to \infty \quad \nabla u_n \to \nabla u \quad \text{a.e. in } \Omega.
\end{align*}
\]

**Proof.**

(1) Taking the function \( Z_m(u_n) = T_1(u_n - T_m(u_n))^- \) and multiplying the approximating equation (4.5) by the test function \( \exp(-G(u_n)) \) we get

\[
\begin{cases}
\int a_n(x, u_n, \nabla u_n) \nabla (\exp(-G(u_n)) Z_m(u_n)) dx + \int \Phi_n(x, u_n) \nabla (\exp(-G(u_n)) Z_m(u_n)) dx \\
+ \int H_n(x, u_n, \nabla u_n) \exp(-G(u_n)) Z_m(u_n) dx \\
+ \int nT_n(u_n - \zeta)^-sg_{1/n}(u_n) \exp(-G(u_n)) Z_m(u_n) dx \\
= \int f_n \exp(-G(u_n)) Z_m(u_n) dx + \int F \nabla (\exp(-G(u_n)) Z_m(u_n)) dx.
\end{cases}
\]

Using the same argument as in step 2, we obtain

\[
\begin{align*}
\int \Omega \max(x, |\nabla Z_m(u_n)|) dx & \leq \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) C \left[ \int \Omega f_n Z_m(u_n) dx + \int \Omega h(x) Z_m(u_n) dx \\
& + \|\rho\|_{L^\infty} \int \Omega \max(x, \frac{|F|}{\epsilon_1}) Z_m(u_n) dx \\
& + \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) C \int_{\{-(m+1) \leq u_n \leq m\}} \max(x, \frac{|F|}{\epsilon_1}) dx,
\end{align*}
\]

where \( \frac{1}{C} = [1 - (\frac{\|c(\cdot)\|_{L^\infty(\Omega)} + \epsilon_1)] \).

Passing to the limit as \( n \to +\infty \), since the pointwise convergence of \( u_n \) and strongly convergence in \( L^1(\Omega) \) of \( f_n \), we get

\[
\begin{align*}
\lim_{n \to +\infty} \int \Omega \max(x, |\nabla Z_m(u_n)|) dx & \leq \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) C \left[ \int \Omega f Z_m(u) dx + \int \Omega h(x) Z_m(u) dx \\
& + \|\rho\|_{L^\infty} \int \Omega \max(x, \frac{|F|}{\epsilon_1}) Z_m(u) dx \\
& + \exp\left(\frac{\|\rho\|_{L^1}}{\alpha'}\right) C \int_{\{-(m+1) \leq u \leq m\}} \max(x, \frac{|F|}{\epsilon_1}) dx.
\end{align*}
\]
Using Lebesgue’s theorem and passing to the limit as \( m \to +\infty \), in the all term of the right-hand side, we get
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \int_{\Omega} M(x, |\nabla Z_m(u_n)|) dx = 0. \tag{4.38}
\]
On the other hand, by (3.4) and Young inequality, for \( n > m + 1 \) we obtain
\[
\int_{\Omega} |\Phi_n(x, u_n) \exp(-G(u_n)) \nabla Z_m(u_n)| dx
\]
\[
\leq \exp(\frac{\|\rho\|_{L^1}}{\alpha'}) \left[ \int_{\{-m+1\} \leq u_n \leq -m\}} M(x, \alpha_0[T_{m+1}(u_n)]) dx + \int_{\Omega} M(x, |\nabla Z_m(u_n)|) dx \right]
\]
Using the pointwise convergence of \( u_n \) and by Lebesgue’s theorem, it follows
\[
\lim_{n \to +\infty} \int_{\Omega} |\Phi_n(x, u_n) \exp(-G(u_n)) \nabla Z_m(u_n)| dx
\]
\[
\leq \exp(\frac{\|\rho\|_{L^1}}{\alpha'}) \left[ \int_{\{-m+1\} \leq u_n \leq -m\}} M(x, \alpha_0[T_{m+1}(u)]) dx + \lim_{n \to +\infty} \int_{\Omega} M(x, |\nabla Z_m(u_n)|) dx \right],
\]
passing to the limit in (4.37), we get
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \int_{\Omega} \Phi_n(x, u_n) \exp(-G(u_n)) \nabla Z_m(u_n) dx = 0.
\]
Finally passing to the limit in (4.37), we get
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \int_{\{-(m+1) \leq u_n \leq -(m+1)\}} a_n(x, u_n, \nabla u_n) \nabla u_n dx = 0.
\]
In the same way we take \( Z_m(u_n) = T_1(u_n - T_m(u_n))^+ \) and multiplying the approximating equation (4.5) by the test function \( \exp(G(u_n))Z_m(u_n) \) and we also obtain
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \int_{\{m \leq u_n \leq m+1\}} a_n(x, u_n, \nabla u_n) \nabla u_n dx = 0.
\]
On the above we get (4.34).

(2) For the almost everywhere convergence of the gradients, we use the following lemma.

**Lemma 4.2.** Under the assumptions (3.1)–(3.6), let \( z_n \) be a sequence in \( W^1_L M(\Omega) \) such that:
\[
z_n \to z \quad \text{for} \quad \sigma(\Pi M(\Pi E_{\overline{M}}),
\]
\[
(a(x, z_n, \nabla z_n)) \quad \text{is bounded in} \quad (L^1_M(\Omega))^N,
\]
\[
\int_{\Omega} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z \chi_s)] [\nabla z_n - \nabla z \chi_s] dx \to 0,
\]
\[
as n \text{ and } s \text{ tend to } +\infty, \text{ and where } \chi_s \text{ is the characteristic function of } \Omega_s = \{ x \in \Omega; |\nabla z| \leq s \} \text{ then},
\]
\[
\nabla z_n \to \nabla z \quad a.e. \text{ in } \Omega,
\]
\[
\lim_{n \to +\infty} \int_{\Omega} a(x, z_n, \nabla z_n) \nabla z_n dx = \int_{\Omega} a(x, z, \nabla z) \nabla z dx,
\]
\[ M(x, |\nabla z_n|) \rightarrow M(x, |\nabla z|) \quad \text{in } L^1(\Omega). \] (4.44)

**Proof.** see [7].

Let \( v_j \in D(\Omega) \) be a sequence such that \( v_j \rightarrow u \) in \( W^1_0 L_M(\Omega) \) for the modular convergence.

We introduce a sequence of increasing \( C^1(\mathbb{R}) \)-functions \( S_m \) such that 
\[ S_m(r) = 1 \quad \text{for } |r| \leq m, \quad S_m(r) = m+1-|r|, \quad \text{for } m \leq |r| \leq m+1, \quad S_m(r) = 0 \quad \text{for } |r| \geq m+1 \]
for any \( m \geq 1 \) and we denote by \( \epsilon(n, \eta, j, m) \) all quantities (possibly different) such that
\[ \lim_{m \rightarrow +\infty} \lim_{j \rightarrow +\infty} \lim_{\eta \rightarrow +\infty} \lim_{n \rightarrow +\infty} \epsilon(n, \eta, j, m) = 0, \]
the main estimate is

For fixed \( k \geq 0 \), let \( W^{n,j}_\eta = T_\eta(T_k(u_n) - T_k(v_j))^+ \) and \( W^{n,j}_\eta = T_\eta(T_k(u) - T_k(v_j))^+ \). Multiplying the approximating equation by \( \exp(G(u_n))W^{n,j}_\eta S_m(u_n) \) and using the same technique in step 2 we obtain:

\[
\begin{align*}
\int_\Omega (a_n(x, u_n, \nabla u_n) \exp(G(u_n)) \nabla W^{n,j}_\eta) S_m(u_n) dx + \\
\int_\Omega a_n(x, u_n, \nabla u_n) \nabla u_n \exp(G(u_n)) W^{n,j}_\eta S'_m(u_n) dx - \\
\int_\Omega \Phi_n(x, u_n) \exp(G(u_n)) \nabla W^{n,j}_\eta S_m(u_n) dx - \\
\int_\Omega \Phi_n(x, u_n) \nabla u_n \exp(G(u_n)) W^{n,j}_\eta S'_m(u_n) dx \\
\leq \int_\Omega f_n \exp(G(u_n)) W^{n,j}_\eta S_m(u_n) dx + \int_\Omega h(x) \exp(G(u_n)) W^{n,j}_\eta S_m(u_n) dx \\
+ \int_\Omega F \exp(G(u_n)) \nabla W^{n,j}_\eta S_m(u_n) dx + \int_\Omega F \nabla u_n \exp(G(u_n)) W^{n,j}_\eta S'_m(u_n) dx.
\end{align*}
\] (4.45)

Now we pass to the limit in (4.45) for \( k \) real number fixed.

In order to perform this task we prove below the following results for any fixed \( k \geq 0 \):  
\[
\int_\Omega \Phi_n(x, u_n) S_m(u_n) \exp(G(u_n)) \nabla W^{n,j}_\eta dx = \epsilon(n, j) \quad \text{for any } m \geq 1, \] (4.46)
\[
\int_\Omega \Phi_n(x, u_n) \nabla u_n S'_m(u_n) \exp(G(u_n)) W^{n,j}_\eta dx = \epsilon(n, j) \quad \text{for any } m \geq 1, \] (4.47)
\[
\int_\Omega a_n(x, u_n, \nabla u_n) \nabla u_n S'_m(u_n) \exp(G(u_n)) W^{n,j}_\eta dx \leq \epsilon(n, m), \] (4.48)
\[
\int_\Omega a_n(x, u_n, \nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla W^{n,j}_\eta dx \leq C \eta + \epsilon(n, j, m), \] (4.49)
\[
\int_\Omega f_n S_m(u_n) \exp(G(u_n)) W^{n,j}_\eta dx + \int_\Omega h(x) \exp(G(u_n)) W^{n,j}_\eta S_m(u_n) dx \leq C \eta + \epsilon(n, \eta), \] (4.50)
In the other hand
\[ + \int_\Omega F \exp(G(u_n)) \nabla (W^{n,j}_{\eta}) S_m(u_n) \, dx \int_\Omega F \nabla u_n \exp(G(u_n)) W^{n,j}_{\eta} S'_m(u_n) \, dx \leq \epsilon(n,m,j,\eta), \] (4.51)
\[
\int_\Omega \left[ a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] \, dx \to 0.
\] (4.52)

**Proof of (4.46):** If we take \( n > m + 1 \), we get
\[
\Phi_n(x, u_n) \exp(G(u_n)) S_m(u_n) = \Phi(x, T_{m+1}(u_n)) \exp(G(T_{m+1}(u_n))) S_m(T_{m+1}(u_n)),
\]
then \( \Phi_n(x, u_n) \exp(G(u_n)) S_m(u_n) \) is bounded in \( L^\infty(\Omega) \), thus, by using the pointwise convergence of \( u_n \) and Lebesgue’s theorem we obtain
\[
\Phi_n(x, u_n) \exp(G(u_n)) S_m(u_n) \to \Phi(x, u) \exp(G(u)) S_m(u)
\]
with the modular convergence of \( \sigma(\Pi_L M, \Pi_L M) \).
In the other hand \( \nabla W^{n,j}_{\eta} = \nabla T_k(u_n) - \nabla (T_k(v_j)) \) for \( 0 \leq T_k(u_n) - (T_k(v_j)) \leq \eta \) converge to \( \nabla T_k(u) - \nabla (T_k(v_j)) \) weakly in \( (L_M(\Omega))^N \), then
\[
\int_\Omega \Phi_n(x, u_n) \exp(G(u_n)) S_m(u_n) \nabla W^{n,j}_{\eta} \, dx \to \int_\Omega \Phi(x, u) S_m(u) \exp(G(u)) \nabla W^{j}_{\eta} \, dx
\]
as \( n \to +\infty \).
By using the modular convergence of \( W^{j}_{\eta} \) as \( j \to +\infty \) and letting \( \mu \) tends to infinity, we get (4.46).

**Proof of (4.47):**
For \( n > m + 1 > k \), we have \( \nabla u_n S'_m(u_n) = \nabla T_{m+1}(u_n) \) a.e. in \( \Omega \). By the almost everywhere convergence of \( u_n \) we have \( \exp(G(u_n)) W^{n,j}_{\eta} \to \exp(G(u)) W^{j}_{\eta} \) in \( L^\infty(\Omega) \) weak-* and since the sequence \( (\Phi_n(x, T_{m+1}(u_n)))_n \) converge strongly in \( E_M(\Omega) \) then
\[
\Phi_n(x, T_{m+1}(u_n)) \exp(G(u_n)) W^{n,j}_{\eta} \to \Phi(x, T_{m+1}(u)) \exp(G(u)) W^{j}_{\eta}
\]
converge strongly in \( E_M(\Omega) \) as \( n \to +\infty \). By virtue of \( \nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u) \) weakly in \( (L_M(\Omega))^N \) as \( n \to +\infty \) we have
\[
\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(x, T_{m+1}(u_n)) \nabla u_n S'_m(u_n) \exp(G(u_n)) W^{n,j}_{\eta} \, dx
\]
\[
\to \int_{\{m \leq |u| \leq m+1\}} \Phi(x, u) \nabla u \exp(G(u)) W^{j}_{\eta} \, dx
\]
as \( n \to +\infty \).
with the modular convergence of \( W^{j}_{\eta} \) as \( j \to +\infty \) and letting \( \mu \to +\infty \) we get (4.47).

**Proof of (4.48):**
For (4.48), we have
\[
\int_\Omega a_n(x, u_n, \nabla u_n) S'_m(u_n) \nabla u_n \exp(G(u_n)) W^{n,j}_{\eta} \, dx
\]
\[
= \int_{\{m \leq |u_n| \leq m+1\}} a_n(x, u_n, \nabla u_n) S'_m(u_n) \nabla u_n \exp(G(u_n)) W^{n,j}_{\eta} \, dx
\]
\[ \leq \eta C \int_{|a_n| \leq m+1} a_n(x, u_n, \nabla u_n) \nabla u_n \, dx. \]

Using (4.34), we get
\[ \int_{\Omega} a_n(x, u_n, \nabla u_n) S_m(u_n) \nabla u_n \exp(G(u_n)) W_{n,j} \, dx \leq \epsilon(n, m). \]

**Proof of (4.50):**

Since \( S_m(r) \leq 1 \) and \( W_{n,j} \leq \eta \) we get
\[ \int_{\Omega} f_n S_m(u_n) \exp(G(u_n)) W_{n,j} \, dx \leq \epsilon(n, \eta), \]
and
\[ \int_{\Omega} h(x) \exp(G(u_n)) W_{n,j} S_m(u_n) \, dx \leq \epsilon(\eta). \]

**Proof of (4.51):**

Denoting by \( I_{F,1} = \int_{\Omega} F \exp(G(u_n)) \nabla(W_{n,j} S_m(u_n)) \, dx \)
and by \( I_{F,2} = \int_{\Omega} F \nabla u_n \exp(G(u_n)) W_{n,j} S_m(u_n) \, dx \).
For the first integral we have
\[ I_{F,1} \leq \exp\left( \frac{\|\rho\| L^1(\mathbb{R})}{\alpha'} \right) \int_{\Omega} F \nabla W_{n,j} \, dx \leq \epsilon(\eta). \]

Since \( T_k(u_n) \) and \( T_k(v_j) \) converge weakly in \( W^{0,1} L_M(\Omega) \), we deduce
\[ I_{F,1} \leq \epsilon(n, j, \eta). \]

For the first integral we know that \( \nabla u_n S_m(u_n) = T_{m+1}(u_n) \) and using (3.3) we get
\[ I_{F,2} \leq \exp\left( \frac{\|\rho\| L^1(\mathbb{R})}{\alpha'} \right) \left[ \epsilon_1 \int_{\Omega} M(x, F, \eta) W_{n,j} \, dx + \epsilon_1 \eta \int_{m \leq |u_n| \leq m+1} a_n(x, u_n, \nabla u_n) \nabla u_n \, dx \right] \]
\[ \leq \epsilon(n, m, j, \eta). \]

**Proof of (4.49):**

\[ \int_{\Omega} a_n(x, u_n, \nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla W_{n,j} \, dx \]
\[ = \int_{\{|u_n| \leq k\} \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a_n(x, T_k(u_n), \nabla T_k(u_n)) S_m(u_n) \exp(G(u_n)) (\nabla T_k(u_n) - \nabla T_k(v_j)) \, dx \]
\[ - \int_{\{|u_n| > k\} \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a_n(x, u_n, \nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla T_k(v_j) \, dx. \]

Since \( a_n(x, T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n)) \) is bounded in \((L_{M'}(\Omega))^N\), there exist some \( \varpi_{k+\eta} \in (L_{M'}(\Omega))^N \) such that \( a_n(x, T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n)) \to \varpi_{k+\eta} \) weakly in \((L_{M'}(\Omega))^N\). Consequently:
\[ \int_{\{|u| > k\} \cap \{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} a_n(x, u_n, \nabla u_n) S_m(u_n) \exp(G(u_n)) \nabla T_k(v_j) \, dx \]
\[ = \int_{\{|u| > k\} \cap \{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} S_m(u) \exp(G(u)) \nabla T_k(v_j) \varpi_{k+\eta} \, dx + \epsilon(n), \quad (4.54) \]
where we have used the fact that
\[ S_m(u_n) \exp(G(u_n)) \nabla T_k(v_j) \chi_{\{|u_n| > k\} \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \]
\[ \to S_m(u) \exp(G(u)) \nabla T_k(v_j) \chi_{\{|u| > k\} \cap \{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} \]
strongly in \((E_M(\Omega))^N\).

Letting \(j \to +\infty\), we obtain
\[ \int_{\{|u| > k\} \cap \{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} S_m(u) \exp(G(u)) \nabla T_k(v_j) \varpi_{k+\eta} \, dx \]
\[ = \int_{\{|u| > k\} \cap \{0 \leq T_k(u) - T_k(v) \leq \eta\}} S_m(u) \exp(G(u)) \nabla T_k(u) \varpi_{k+\eta} \, dx + \epsilon(n, j). \]

One has,
\[ \int_{\{|u| > k\} \cap \{0 \leq T_k(u) - T_k(v) \leq \eta\}} S_m(u) \exp(G(u)) \nabla T_k(u) \varpi_{k+\eta} \, dx = \epsilon(n, j). \]

By (4.45)-(4.50), (4.53) and (4.54) we obtain
\[ \int_{\{|u_n| \leq k\} \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a_n(x, T_k(u_n), \nabla T_k(u_n)) S_m(u_n) \exp(G(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)) \, dx \]
\[ \leq C\eta + \epsilon(n, j, m), \]
we know that \(\exp(G(u_n)) \geq 1\) and \(S_m(u_n) = 1\) for \(|u_n| \leq k\) then
\[ \int_{\{|u_n| \leq k\} \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a_n(x, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(v_j)) \, dx \]
\[ \leq C\eta + \epsilon(n, j, m). \quad (4.55) \]

Proof of (4.52):
Setting for \(s > 0\), \(\Omega^s = \{x \in \Omega : |\nabla T_k(u)| \leq s\}\) and \(\Omega^s_j = \{x \in \Omega : |\nabla T_k(v_j)| \leq s\}\) and denoting by \(\chi^s\) and \(\chi^s_j\) the characteristic functions of \(\Omega^s\) and \(\Omega^s_j\) respectively, we deduce that letting \(0 < \delta < 1\), define
\[ \Theta_{n,k} = (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)). \]

For \(s > 0\), we have
\[ 0 \leq \int_{\Omega^s} \Theta_{n,k} \, dx = \int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \, dx + \int_{\Omega^s} \Theta_{n,k} \chi_{\{T_k(u_n) - T_k(v_j) > \eta\}} \, dx. \]

The first term of the right-side hand, with the Hölder inequality
\[ \int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \, dx \leq \left( \int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \, dx \right)^\delta (\int_{\Omega^s} \, dx)^{1-\delta} \]
\[ \leq C_1 \left( \int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \, dx \right)^\delta. \]

Also using the Hölder inequality, the second term of the right-side hand is
\[ \int_{\Omega^s} \Theta_{n,k} \chi_{\{T_k(u_n) - T_k(v_j) > \eta\}} \, dx \leq (\int_{\Omega^s} \Theta_{n,k} \, dx)^\delta \left( \int_{T_k(u_n) - T_k(v_j) > \eta} \, dx \right)^{1-\delta}, \]
since \(a(x, T_k(u_n), \nabla T_k(u_n))\) is bounded in \((L^\infty(\Omega))^N\), while \(\nabla T_k(u_n)\) is bounded in \((L_M(\Omega))^N\) then
\[
\int_{\Omega^s} \Theta^\delta_{n,k} \chi_{\{T_k(u_n) - T_k(v_j) > \eta\}} \, dx \leq C_2 \text{meas}\{x \in \Omega : T_k(u_n) - T_k(v_j) > \eta\}^{1-\delta}.
\]
We obtain
\[
\int_{\Omega^s} \Theta^\delta_{n,k} \, dx \leq C_1(\int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \, dx)^\delta + C_2 \text{meas}\{x \in \Omega : T_k(u_n) - T_k(v_j) > \eta\}^{1-\delta}.
\]
On the other hand
\[
\int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \, dx
\]
\[
\leq \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u)) \, dx
\]
For each \(s > r, r > 0\), one has
\[
0 \leq \int_{\Omega^s \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u)) \, dx
\]
\[
\leq \int_{\Omega^s \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u)) \, dx
\]
\[
= \int_{\Omega^s \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u) \chi_s)(\nabla T_k(u_n) - \nabla T_k(u) \chi_s) \, dx
\]
\[
\leq \int_{\Omega^s \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u) \chi_s)(\nabla T_k(u_n) - \nabla T_k(u) \chi_s) \, dx
\]
\[
= \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(v_j) \chi_s)(\nabla T_k(u_n) - \nabla T_k(v_j) \chi_s) \, dx
\]
\[
+ \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(v_j) \chi_s - \nabla T_k(u) \chi_s) \, dx
\]
\[
+ \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} (a(x, T_k(u_n), \nabla T_k(v_j) \chi_s) - a(x, T_k(u_n), \nabla T_k(u) \chi_s)) \nabla T_k(u_n) \, dx
\]
\[
- \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(v_j) \chi_s) \nabla T_k(v_j) \chi_s \, dx
\]
\[
+ \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u) \chi_s) \nabla T_k(u) \chi_s) \, dx
\]
\[
= I_1(n, j, s) + I_2(n, j) + I_3(n, j) + I_4(n, j) + I_5(n).
\]
passing to the limit as \(n, j, \mu, \) and \(s \to +\infty\)
\[
I_1 = \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j)) \, dx
\]
- \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)) \, dx

- \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) \, dx.

Using (4.55), the first term of the right-hand side, we get

\int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} a(x, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(v_j) - \nabla T_k(v_j)) \, dx

\leq C\eta + \epsilon(n, m, j, s) - \int_{\{|u| > k, 0 \leq T_k(u) - T_k(v_j) \leq \eta\}} a(x, T_k(u), 0)\nabla T_k(v_j) \, dx

\leq C\eta + \epsilon(n, m, j).

The second term of the right-hand side tends to

\int_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} \varpi_k(\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j)) \, dx,

since \(a(x, T_k(u_n), \nabla T_k(u_n))\) is bounded in \((L^\infty)\), there exist some \(\varpi_k \in (L^\infty)\) such that (for a subsequence still denoted by \(u_n\))

\(a(x, T_k(u_n), \nabla T_k(u_n)) \to \varpi_k \) in \((L^\infty)\) for \(\sigma(\Pi L^\infty, \Pi E_M)\).

In view of the fact that

\((\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j))\chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \to (\nabla T_k(v_j)\chi_j^s - \nabla T_k(v_j))\chi_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}}\)

Strongly in \((E_M)\) as \(n \to +\infty\).

The third term of the right-hand side tends to

\int_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} a(x, T_k(u), \nabla T_k(v_j)\chi_j^s)(\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s) \, dx.

Since

\(a(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s)\chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \to a(x, T_k(u), \nabla T_k(v_j)\chi_j^s)\chi_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}}\)

in \((E_M)\) while

\((\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s) \to (\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s)\)

in \((L^\infty)\) for \(\sigma(\Pi L^\infty, \Pi E_M)\) Passing to limit as \(j \to +\infty\) and \(\mu \to +\infty\) and using Lebesgue’s theorem, we have

\(I_1 \leq C\eta + \epsilon(n, j, s)\).

For what concerns \(I_2\), by letting \(n \to +\infty\), we have

\(I_2 \to \int_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} \varpi_k(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi_j^s) \, dx\).
Since \( a(x, T_k(u_n), \nabla T_k(u_n)) \to \varpi_k \) in \((L^\infty(\Omega))^N\), for \( \sigma(\Pi L^\infty, \Pi E_M) \) while

\[
(\nabla T_k(v_j) \chi_j^u - \nabla T_k(u) \chi^u) \chi_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} \to (\nabla T_k(v_j) \chi_j^u - \nabla T_k(u) \chi^u) \chi_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}}
\]

strongly in \((E_M(\Omega))^N\).

Passing to limit \( j \to +\infty \), and using Lebesgue’s theorem, we have

\[
I_2 = \epsilon(n, j).
\]

Similar ways as above give

\[
I_3 = \epsilon(n, j),
\]

\[
I_4 = \int_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx + \epsilon(n, j, s, m),
\]

\[
I_5 = \int_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} a(x, T_k(u), \nabla T_k(u)) \nabla T_k(u) \, dx + \epsilon(n, j, s, m).
\]

Finally, we obtain

\[
\int_{\Omega^r} \Theta_{n,k} \, dx \leq C_1(C \eta + \epsilon(n, \eta, m)) + C_2(\epsilon(n, \eta))^{1-\delta}.
\]

Which yields, by passing to the limit sup over \( n, j, \mu, s \) and \( \eta \)

\[
\int_{\{T_k(u_n) - T_k(v_j) \geq 0\} \cap \Omega^r} [(a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))] \, dx = \epsilon(n).
\] (4.56)

Taking on the hand the function \( W^{n,j}_\eta = T_k(u_n) - T_k(v_j) \)

and \( W^{j}_\eta = T_k(u) - T_k(v_j) \). Multiplying the approximating equation by \( \exp(G(u_n))W^{n,j}_\eta S_m(u_n) \), we obtain

\[
\int_{\{T_k(u_n) - T_k(v_j) \leq 0\} \cap \Omega^r} [(a(T_k(u_n), \nabla T_k(u_n)) - a(T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))] \, dx = \epsilon(n),
\] (4.57)

by (4.56) and (4.57) we get

\[
\int_{\Omega^r} [(a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))] \, dx = \epsilon(n).
\]

Thus, passing to a subsequence if necessary, \( \nabla u_n \to \nabla u \) a.e. in \( \Omega^r \), and since \( r \) is arbitrary,

\[
\nabla u_n \to \nabla u \text{ a.e. in } \Omega^r.
\]
Step 5: Equi-integrability of the nonlinearity sequence:

We shall prove that \( H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u) \) strongly in \( L^1(\Omega) \).

Consider \( g_0(u_n) = \int_0^{\|u_n\|} \rho(s) \chi_{\{|s|>h\}} ds \) and multiply (4.5) by \( \exp(G(T_k(u_n)))g_0(u_n) \), we get after using the same technique in step 2,

\[
\int_{\{u_n>h\}} \rho(u_n)M(x, \nabla u_n)dx \leq \\
\left( \int_h^{+\infty} \rho(s)ds \right) \exp \left( \frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'} \right) \left[ \|f\|_{L^1(\Omega)} + \|h(x)\|_{L^1(\Omega)} + \frac{\|\rho\|_{L^\infty(\mathbb{R})}}{\alpha'} \right] \int_\Omega M(x, \frac{F}{\epsilon_1})dx \\
+ \exp \left( \frac{\|\rho\|_{L^1(\mathbb{R})}}{\alpha'} \frac{\|\rho\|_{L^\infty(\mathbb{R})}}{\epsilon_1} \right) \int_{\{u_n>h\}} M(x, \frac{F}{\epsilon_1})dx.
\]

Since \( \rho \in L^1(\mathbb{R}) \), we get

\[
\lim_{h \to 0} \sup_{n \in \mathbb{N}} \int_{\{u_n>h\}} \rho(u_n)M(x, \nabla u_n)dx = 0.
\]

Similarly, let \( g_0(u_n) = \int_{u_n}^{0} \rho(s) \chi_{\{|s|<h\}} ds \) in (4.5), we have also

\[
\lim_{h \to 0} \sup_{n \in \mathbb{N}} \int_{\{u_n<-h\}} \rho(u_n)M(x, \nabla u_n)dx = 0.
\]

We conclude that

\[
\lim_{h \to 0} \sup_{n \in \mathbb{N}} \int_{\{|u_n|>h\}} \rho(u_n)M(x, \nabla u_n)dx = 0. \tag{4.58}
\]

Let \( D \subset \Omega \) then

\[
\int_D \rho(u_n)M(x, \nabla u_n)dx \leq \max_{\{|u_n|\leq h\}} (\rho(u)) \int_{D \cap \{|u_n| \leq h\}} M(x, \nabla u_n)dx + \int_{D \cap \{|u_n| > h\}} \rho(u_n)M(x, \nabla u_n)dx
\]

Consequently \( \rho(u_n)M(x, \nabla u_n) \) is equi-integrable. Then \( \rho(u_n)M(x, \nabla u_n) \) converge to \( \rho(u)M(x, \nabla u) \) strongly in \( L^1(\mathbb{R}) \). By (3.5), we get our result.

Step 6: We show that \( u \) satisfies (3.8)

Firstly show that \( u \geq \zeta \) a.e. in \( \Omega \).

In fact, from (4.15) and (4.25) we get

\[
0 \leq \int_\Omega T_n(u_n - \zeta^-)dx \leq \frac{c_1}{n}.
\]

Let \( n \) tends to \( +\infty \) we obtain

\[
\int_\Omega (u - \zeta^-)dx = 0,
\]

then \( (u - \zeta)^- = 0 \) a.e. in \( \Omega \); thus \( u \geq \zeta \) a.e. in \( \Omega \).

Secondly passing Now to the limit in (4.59) to show that \( u \) satisfies the equation (4.5).

Let \( v \in K_\zeta \cap L^\infty(\Omega) \), then by lemma (2.3) there exists \( v_j \in \mathcal{D}(\Omega) \) such that \( v_j \to v \) in \( W^1_0 L_M(\Omega) \) for the modular convergence in \( W^1_0 L_M(\Omega) \).
with \( \|v_j\|_{L^\infty(\Omega)} \leq (N + 1)\|v\|_{L^\infty(\Omega)} \), and \( v_j \in K_\zeta \).

Pointwise multiplication of the approximate equation (4.5) by \( T_k(u_n - v_j) \), we get
\[
\begin{align*}
\int a_n(x, u_n, \nabla u_n)) \nabla T_k(u_n - v_j)dx + \int \Phi_n(x, u_n) \nabla T_k(u_n - v_j)dx \\
+ \int H_n(x, u_n) \nabla T_k(u_n - v_j)dx + \int nT_n(u_n - \zeta) - s g_{1/n}(u_n) T_k(u_n - v_j)dx \\
= \int f_n T_k(u_n - v_j)dx - \int F \nabla T_k(u_n - v_j)dx.
\end{align*}
\]

We pass to the limit as in (4.59), \( n \) tend to \( +\infty \) and \( j \) tend to \( +\infty \):

- We follow same way in [6] to prove that
\[
\liminf_{j \to \infty} \liminf_{n \to \infty} \int a(x, u_n, \nabla u_n) \nabla T_k(u_n - v_j)dx \geq \int a(x, u, \nabla u) \nabla T_k(u - v)dx.
\]

- For \( n \geq k + (N + 1)\|v\|_{L^\infty(\Omega)} \)
\[
\Phi_n(x, u_n) \nabla T_k(u_n - v_j) = \Phi(x, T_k+(N+1)\|v\|_{L^\infty(\Omega)}(u_n)) \nabla T_k(u_n - v_j).
\]

The pointwise convergence of \( u_n \) to \( u \) as \( n \) tend to \( +\infty \) and (3.4), then
\[
\Phi(x, T_k+(N+1)\|v\|_{L^\infty(\Omega)}(u_n)) \nabla T_k(u_n - v_j) \rightarrow \Phi(x, T_k+(N+1)\|v\|_{L^\infty(\Omega)}(u)) \nabla T_k(u - v_j)
\]
weakly for \( \sigma(\Pi L_v, \Pi L_{\Omega}) \).

In a similar way, we obtain
\[
\lim_{j \to \infty} \int_\Omega \Phi(x, T_k+(N+1)\|v\|_{L^\infty(\Omega)}(u_n)) \nabla T_k(u_n - v_j)dx = \int_\Omega \Phi(x, T_k+(N+1)\|v\|_{L^\infty(\Omega)}(u)) \nabla T_k(u - v)dx
\]
\[
= \int_\Omega \Phi(x, u) \nabla T_k(u - v)dx.
\]

- Limit of \( H_n(x, u_n, \nabla u_n) T_k(u_n - v_j) \):
Since \( H_n(x, u_n, \nabla u_n) \) converge strongly to \( H(x, u, \nabla u) \) in \( L^1(\Omega) \) and the pointwise convergence of \( u_n \) to \( u \) as \( n \to +\infty \), it is possible to prove that \( H_n(x, u_n, \nabla u_n) T_k(u_n - v_j) \) converge to \( H(x, u, \nabla u) T_k(u - v_j) \) in \( L^1(\Omega) \) and
\[
\lim_{j \to \infty} \int_\Omega H(x, u, \nabla u) T_k(u_n - v_j)dx = \int_\Omega H(x, u, \nabla u) T_k(u - v)dx.
\]

- Since \( f_n \) converge strongly to \( f \) in \( L^1(\Omega) \), and \( T_k(u_n - v_j) \to T_k(u - v_j) \) weakly* in \( L^\infty(\Omega) \), we have
\[
\int \Omega f_n T_k(u_n - v_j)dx \to \int \Omega f T_k(u - v_j)dx \text{ as } n \to \infty,
\]
and
\[
\int \Omega f T_k(u_n - v_j)dx \to \int \Omega f T_k(u - v)dx \text{ as } j \to \infty.
\]

Also, it easy to get \( \lim \lim_{j \to \infty} \int_\Omega F \nabla T_k(u_n - v_j)dx = \int_\Omega F \nabla T_k(u - v)dx \).

Finally Since \( -\int_\Omega T_n(u_n - \zeta) T_k(u_n - v_j)dx \geq 0 \), we obtain (3.8).

As a conclusion, the proof of Theorem (3.1) is complete.
References


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