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RENORMALIZED SOLUTION OF NONLINEAR PARABOLIC EQUATIONS WITHOUT SIGN CONDITION AND GENERAL MEASURE DATA

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Abstract. We give an existence result of a renormalized solution for a class of nonlinear parabolic equations $\dfrac{\partial b(u)}{\partial t} - \operatorname{div}\Big(a(x,t,\nabla u)\Big) + h(u)|\nabla u|^p = \mu$, where the right side is a general measure and $b(u)$ is a strictly increasing C^1-function with $b(0)=0$, $-\operatorname{div}(a(x,t,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u and $h:\mathbb{R}\to\mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $(N\geq 1)$, T is a positive real number, and let $Q:=\Omega\times(0,T)$, $p>1$. We will consider the following nonlinear parabolic problem

$$\frac{\partial b(u)}{\partial t} - \operatorname{div}\Big(a(x,t,\nabla u)\Big) + h(u)|\nabla u|^p = \mu \ \text{ in } Q, \tag{1.1}$$

$$b(u)(t=0) = b(u_0) \ \text{ in } \Omega, \tag{1.2}$$

$$u = 0 \ \text{ on } \partial\Omega\times(0,T). \tag{1.3}$$

In Problem (1.1)-(1.3) the framework is the following: the data μ is a bounded Radon measure on Q, $b(s)$ is a strictly increasing C^1-function for every $s\in\mathbb{R}$ with u_0 belongs to $L^1(\Omega)$. The operator $-\operatorname{div}(a(x,t,\nabla u))$ is a Leray–Lions operator which is coercive and which grows like $|\nabla u|^{p-1}$ with respect to ∇u, (see assumptions (3.3), (3.4) and (3.5) of Section 3).

In the case where $b(u)=u$, and the right hand side is a bounded measure, the existence of a distributional solution was proved in [10], but due the lack of regularity of solution, the distributional formulation is not strong enough to provide uniqueness (see [40] for a counter example in the elliptic case). To overcome this difficulty the notion of renormalized solutions firstly introduced by DiPerna and Lions [23] for the study of Boltzmann equation was adapted to parabolic equations and (elliptic equations) with L^1 data (see [5, 11, 13, 32, 30, 31]). The

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equivalent notion of entropy solution has been developed independently by [6] for the study of nonlinear elliptic problems and by [36] in the parabolic case. Both renormalized or entropy solutions provide a convenient framework to deal with elliptic or parabolic equations with L^1 data. A large number of papers was then devoted to the study the existence of renormalized (or entropy) solution of parabolic problems with rough data under various assumptions and different contexts: in addition to the references already mentioned see, among others, [2, 3, 15, 16, 17, 19, 18, 20, 21, 33, 39].

Concerning the datum μ, the existence and uniqueness of renormalized solution of (1.1)-(1.3) has proved in [38] in the case where $b(u) = u$, $u_0 \in L^1(\Omega)$, $h = 0$, and for every measure μ which do not charge the sets of zero p-capacity , the so-called diffuse measures or soft measures, and we will use the symbol $\mu \in \mathcal{M}_0(Q)$ to denote them (see Section 2 for the definition). The importance of the measures not charging sets of null p-capacity was first observed in the stationary case in [8], and developed in the evolution case in [38].

For $\mu \in \mathcal{M}_0(Q)$, $u_0 \in L^1(\Omega)$ and $h = 0$ the existence and uniqueness of renormalized solution was proved in [18].

In the case where $\mu \in \mathcal{M}_0(Q)$, $h = 0$, and with the parabolic term on $b(x, u)$, the existence of renormalized solution of Problem (1.1)-(1.3) was proved in [37]. Concerning the lower order term, existence of renormalized solution of Problem (1.1)-(1.3) has proved in [19] in the case where $b(u) = b(x, u)$, $\mu \in L^1(Q)$, $u_0 \in L^1(Q)$ and h satisfies the sign condition. In the case $b(u) = b(x, u)$, $\mu \in L^1(Q)$, $u_0 \in L^1(Q)$ and without sign condition, the existence of renormalized solution of Problem (1.1)-(1.3) has proved in [1].

Notice that the definition of renormalized solution of problem (1.1)-(1.3) can be extended to case of general measure by adapting the idea of [25] for elliptic case and [34] for parabolic case.

For $\mu \in \mathcal{M}(Q)$ (the space of all bounded Radon measures on Q), $b(u) = u$ and $u_0 \in L^1(\Omega)$, and $h = 0$, the existence of renormalized solution was proved in [25] for elliptic case and [34] for parabolic case.

Our goal is to extend the approach in [37] to general, and possibly singular measure data and with lower order term which does not satisfies the sign condition.

The paper is organized as follows as follows. In Section 2 we give some preliminaries and, in particular, we provide the definition of parabolic capacity and some basic properties.
Section 3 is devoted to specify the assumptions on b, a, h, u_0 and μ and to give the definition of renormalized solution of (1.1)-(1.3) and see how the definition of renormalized solution does not depend on the decomposition (not uniquely determined) of the regular part of μ we mentioned above and to the statement of standard approximation argument we will use later. In Section 4 we establish (Theorem 4.1) the existence of such a solution.

2. Preliminaries on parabolic capacity

We recall the notion of p-capacity associated to our problem. Let $Q = \Omega\times(0,T)$ for any fixed $T > 0$, and let us recall that $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, endowed with its natural norm $\|.\|_{W_0^{1,p}(\Omega)} + \|.\|_{L^2(\Omega)}$ and

$$W = \Big\{u \in L^p(0,T;V), u_t \in L^{p'}(0,T;V')\Big\},$$

endowed with its natural norm $\|.\|_{L^p(0,T;V)} + \|.\|_{L^{p'}(0,T;V')}$, remark that W is continuously embedded in $C([0,T], L^2(\Omega))$, and if $1 < p < \infty$, then $C_c^\infty(\Omega \times [0,T])$ is dense in W.

Let $U \subseteq Q$ be an open set, we define the parabolic p-capacity of U as

$$cap_p(U) = \inf\Big\{\|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q\Big\},$$

where as usual we set $inf\{\emptyset\} = +\infty$, then for any Borel set $B \subseteq Q$ we define

$$cap_p(B) = \inf\Big\{cap_p(U) : U \text{ open set of } Q, B \subseteq Q\Big\}.$$

We define the space $\mathcal{S}$ by

$$\mathcal{S} = \Big\{u \in L^p(0,T;W_0^{1,p}(\Omega)), u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\Big\},$$

endowed with its natural norm $\|.\|_{L^p(0,T;W_0^{1,p}(\Omega))} + \|.\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1(Q)}$.
We will denote by $\mathcal{M}(Q)$ the set of all Radon measures with bounded total variation on Q, while $\mathcal{M}_0(Q)$ is the set of all measures with bounded total variation over Q that do not charge the sets of zero p-capacity, that is if $\mu \in \mathcal{M}_0(Q)$, then $\mu(E) = 0$, for all $E \subseteq Q$ such that $cap_p(E) = 0$. We recall the following theorem

Theorem 2.1. *Let μ be a bounded measure in Q. If $\mu \in \mathcal{M}_0(Q)$ then there exists (f, g_1, g_2) such that $f \in L^1(Q)$, $g_1 \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, $g_2 \in L^p(0,T;V)$ and*

$$\int_Q \phi \, d\mu = \int_Q f\phi \, dx \, dt + \int_0^T \langle g_1, \phi\rangle \, dt - \int_0^T \langle \phi_t, g_2\rangle \, dt \quad \phi \in C_c^\infty(\Omega \times [0,T]).$$

Such a triplet (f, g_1, g_2) will called the decomposition of μ.

Proof. See [38]. □

So, if $\mu \in \mathcal{M}(Q)$, thanks to a well known decomposition result (see for instance [27]), we can split it into a sum (uniquely determined) of its absolutely continuous part μ_0 with respect to p-capacity and its singular part μ_s, that μ_s is concentrated on a set E of zero p-capacity; we will say that $\mu_s \perp cap_p$. Hence, if $\mu \in \mathcal{M}(Q)$, by Theorem 2.1, we have

$$\mu = f - \text{div}(G) + g_t + \mu_s^+ - \mu_s^-,$$

in the sense of distributions, for $f \in L^1(Q)$, $G \in (L^{p'}(Q))^N$, $g \in L^p(0,T;V)$, where μ_s^+ and μ_s^- are respectively the positive and the negative part of μ_s; note that the decomposition of the absolutely continuous part of μ in Theorem 2.1 is not uniquely determined. Let us state the following result that will be very useful in the squel; its proof relies on an easy application of Egorov and Dunford-Pettis theorems.

Proposition 2.2. *Let ρ_ε be a sequence of $L^1(Q)$ functions that converges to ρ weakly in $L^1(Q)$, and let σ_ε be sequence of functions in $L^\infty(Q)$ that is bounded in $L^\infty(Q)$ and converges to σ almost everywhere on Q. Then*

$$\lim_{\varepsilon\to 0}\int_Q \rho_\varepsilon\ \sigma_\varepsilon\ dxdt = \int_Q \rho\ \sigma\ dxdt.$$

Here are some notations we will use throughout this paper. For any non negative real number k we denote by $T_k(r) = min(k, max(r, -k))$ the truncation function at level k.
By $\langle .,. \rangle$ we mean the duality between suitable spaces in which functions are involved, in particular we will consider both the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$ and the duality between $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $W^{-1,p'}(\Omega) + L^1(\Omega)$.

3. Assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true: Ω is a bounded open set on $\mathbb{R}^N$ $(N \geq 1)$, $T > 0$ is given and we set $Q = \Omega \times (0,T)$.

$$b : \mathbb{R} \to \mathbb{R} \tag{3.1}$$

is a strictly increasing C^1-function with $b(0) = 0$, and there exist γ and $\Lambda > 0$ such that

$$\gamma \leq b'(s) \leq \Lambda, \tag{3.2}$$

for every $s \in \mathbb{R}$.

$$a : Q \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Carathéodory function} \tag{3.3}$$

$$a(x,t,\xi).\xi \geq \alpha|\xi|^p, \tag{3.4}$$

for almost every $(x,t) \in Q$, for every $\xi \in \mathbb{R}^N$, where $\alpha > 0$ given real number.

$$|a(x,t,\xi)| \leq \beta(L(x,t) + |\xi|^{p-1}), \tag{3.5}$$

for almost every $(x,t) \in Q$, for every $\xi \in \mathbb{R}^N$, where $\beta > 0$ given real number, L is a non negative function in $L^{p'}(Q)$.

$$[a(x,t,\xi) - a(x,t,\xi')][\xi - \xi'] > 0, \tag{3.6}$$

for any $(\xi,\xi') \in \mathbb{R}^{2N}$ and for almost every $(x,t) \in Q$.

$$h : \mathbb{R} \to \mathbb{R}^+ \tag{3.7}$$

is a continuous positive function that belongs to $L^1(\mathbb{R})$.

$$\mu \in \mathcal{M}(Q), \tag{3.8}$$

$$u_0 \text{ is an element of } L^1(\Omega) \text{ such that } b(u_0) \in L^1(\Omega). \tag{3.9}$$

To simplify notation, let us also define $v = b(u) - g$, the definition of a renormalized solution for Problem (1.1)-(1.3) is given below.

Definition 3.1. A measurable function u defined on Q is a renormalized solution of Problem (1.1)-(1.3) if

$$T_k(v) \in L^p(0,T;W_0^{1,p}(\Omega)) \ \forall k \geq 0\ ,\ v \in L^\infty(0,T;L^1(\Omega)) \ \text{ and } h(u)|\nabla u|^p \in L^1(Q), \tag{3.10}$$

and, for every function S in $W^{2,\infty}(\mathbb{R})$, which is piecewise C^1 and such that S' has a compact support and $S(0) = 0$, we have

$$S(v)_t - \operatorname{div}\Big(S'(v)a(x,t,\nabla u)\Big) + \ S''(v)a(x,t,\nabla u)\nabla v + S'(v)h(u)|\nabla u|^p \tag{3.11}$$

$$= fS'(v) - \operatorname{div}\Big(GS'(v)\Big) + GS''(v)\nabla v \ \text{ in } \mathcal{D}'(Q),$$

$$S(v)(t=0) = S(b(u_0)) \ \text{ in } L^1(\Omega). \tag{3.12}$$

For every $\psi \in C(\overline{Q})$, we have

$$\lim_{n\to\infty} \frac{1}{n}\int_{\{(x,t)\in Q\ ;\ n\leq v<2n\}} a(x,t,\nabla u)\nabla v\, \psi\, dx\, dt = \int_Q \psi\, d\mu_s^+, \tag{3.13}$$

$$\lim_{n\to\infty} \frac{1}{n}\int_{\{(x,t)\in Q\ ;\ -2n<v\leq -n\}} a(x,t,\nabla u)\nabla v\, \psi\, dx\, dt = \int_Q \psi\, d\mu_s^-. \tag{3.14}$$

Remark 3.2. Note that thanks to our assumptions and the choice of S all terms in (3.11) are well defined (for more details see [18], Remark 3.2). Let also observe that $S(v)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ and $S(v) \in L^p(0,T;W_0^{1,p}(\Omega))$, which implies that $S(v) \in C([0,T],L^1(\Omega))$ (see [32]) and (3.11) makes a weak sense, indeed, since $\mu_0 \in \mathcal{M}_0(Q)$ and it is defined on the σ-algebra of the borelians of the open set Q, then μ_0 does not charge set at $t = 0$, which implies, in the weak sense, that $g(x,0) = 0$ for any g such that $(f, div(G), g)$ is a decomposition of μ_0, this explains (3.12), (see [38]).
A remark on the assumptions (3.2) is also necessary. As one could check later, due essentially to the presence of the term g_t in the decomposition of the measure μ,we are forced to assume $b'(s) \geq \gamma > 0$. We conjecture that this assumption is only technical and could be removed in order to deal with more general elliptic-parabolic problems (see for instance [4], [21]).

Now we give the following property of renormalized solutions; throughout the paper C will indicate any positive constant whose value may change from line to line.

Proposition 3.3. *Let $v = b(u) - g$ be a renormalized solution of problem (1.1)-(1.3). Then, for every $k > 0$, we have*

$$\int_Q |\nabla T_k(v)|^p\, dxdt \leq C(k+1), \tag{3.15}$$

where C is a positive constant not depending on k.

Proof. Using assumptions (3.2), following the same arguments as in [34], yields (3.15). □

Here, we give two results which show that the renormalized solution does not depend on the decomposition of the regular part of μ.

Lemma 3.4. *Let $\mu_0 \in \mathcal{M}_0(Q)$, and let (f, g_1, g_2) and $(\overline{f}, \overline{g_1}, \overline{g_2})$ to be two different decomposition of μ according to Theorem 2.1. Then we have $(g_2 - \overline{g_2})_t = \overline{f} - f + \overline{g_1} - g_1$ in distribution sense, $g_2 - \overline{g_2} \in C([0,T], L^1(\Omega))$ and $(g_2 - \overline{g_2})(0) = 0$.*

Proof. See [38], Lemma 2.29. □

The following result shows that the definition of renormalized solution does not depend on the decomposition of the absolutely continuous part of μ under the condition of bounded perturbations of time derivative part of μ_0, and due the estimate (3.15).

Proposition 3.5. *Let u be a renormalized solution of problem (1.1)-(1.3). Then, u satifies definition 3.1 for every decomposition (f, g_1, g_2) such that $g_2 - \overline{g_2} \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$.*

Proof. See [34], Proposition 3 and Remark 6. □

4. Existence result

This section is devoted to establish the following existence theorem.

Theorem 4.1. *Under assumptions (3.1)-(3.9), there exists at least a renormalized solution u of Problem (1.1)-(1.3).*

Proof. We will obtain the existence result by an approximation process, we approximate the measure $\mu \in \mathcal{M}(Q)$ by a sequence defined by

$$\mu^\varepsilon = f^\varepsilon - \operatorname{div}(G^\varepsilon) + \frac{\partial g^\varepsilon}{\partial t} + \lambda_+^\varepsilon - \lambda_-^\varepsilon \tag{4.1}$$

where $f^\varepsilon \in C_c^\infty(Q)$ is a sequence of functions which converges to f weakly in $L^1(Q)$, $G^\varepsilon \in (C_c^\infty(Q))^N$ is a sequence of functions which converges to G strongly in $(L^{p'}(Q))^N$, $g^\varepsilon \in C_c^\infty(Q)$ is a sequence of functions which converges to g strongly in $L^p(0,T; W_0^{1,p}(\Omega))$, and $\lambda_+^\varepsilon \in C_c^\infty(Q)$ (respectively λ_-^ε) is a sequence of non negatives functions that converges to μ_s^+ (respectively μ_s^-) in the narrow topology of measures. Moreover let $u_0^\varepsilon \in C_c^\infty(\Omega)$ such that

$$u_0^\varepsilon \in C_c^\infty(\Omega) : u_0^\varepsilon \to u_0 \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0. \tag{4.2}$$

We also assume

$$\|\mu^\varepsilon\|_{L^1(Q)} \le C|\mu|_{\mathcal{M}(Q)} \quad \text{and} \quad \|b(u_0^\varepsilon)\|_{L^1(Q)} \le C\|b(u_0)\|_{L^1(Q)}.$$

Let us now consider the following regularized problem:

$$u^\varepsilon \in L^p(0,T; W_0^{1,p}(\Omega)), \tag{4.3}$$

$$\int_0^T \langle \frac{\partial v^\varepsilon}{\partial t}, \varphi\rangle\, dt + \int_Q a(x,t,\nabla u^\varepsilon)\nabla\varphi\, dxdt + \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p \varphi\, dxdt = \int_Q f^\varepsilon \varphi\, dxdt + \int_Q G^\varepsilon \nabla\varphi\, dxdt + \int_Q \varphi\, d\lambda_+^\varepsilon - \int_Q \varphi\, d\lambda_-^\varepsilon \tag{4.4}$$

$$\forall \varphi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q),$$

$$b(u^\varepsilon)(t=0) = b(u_0^\varepsilon) \text{ in } \Omega, \tag{4.5}$$

where $v^\varepsilon = b(u^\varepsilon) - g^\varepsilon$. As a consequence, proving existence of a weak solution $u^\varepsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ of (4.3)-(4.5) is an easy task (see [9], [28]).
Now we prove the following proposition which gives some compactness results.

Proposition 4.2. *Let u^ε and v^ε be defined as before. Then*

$$\|u^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{4.6}$$

$$\int_Q |\nabla T_k(u^\varepsilon)|^p \, dxdt \le Ck, \tag{4.7}$$

$$u^\varepsilon \text{ is bounded in } L^q(0,T;W_0^{1,q}(\Omega)) \quad \forall\ 1 < q < p - \frac{N}{N+1}, \tag{4.8}$$

$$\|v^\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{4.9}$$

$$\int_Q |\nabla T_k(v^\varepsilon)|^p \, dxdt \le C(k+1), \tag{4.10}$$

$$h(u^\varepsilon)|\nabla u^\varepsilon|^p \text{ is bounded in } L^1(Q), \tag{4.11}$$

and, up to a subsequence, for any $k > 0$ we have

$$u^\varepsilon \to\ u \text{ a.e. on } Q \text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \text{ and strongly in } L^1(Q), \tag{4.12}$$

$$v^\varepsilon\ \to\ v \text{ a.e. on } Q \text{ weakly in } L^q(0,T;W_0^{1,q}(\Omega)) \text{ and strongly in } L^1(Q), \tag{4.13}$$

$$T_k(u^\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. on } Q, \tag{4.14}$$

$$T_k(v^\varepsilon) \rightharpoonup T_k(v) \text{ weakly in } L^p(0,T;W_0^{1,p}(\Omega)) \text{ and a.e. on } Q, \tag{4.15}$$

Proof. We prove (4.6) and (4.7), using $T_k(u^\varepsilon)^+$ as a test function in (4.4) and we integrate in $]0,t[$ we get

$$\begin{aligned}\int_\Omega B_k(u^\varepsilon)(t)\,dx + \int_0^t\int_\Omega a(x,t,\nabla u^\varepsilon)\nabla T_k(u^\varepsilon)^+\,dx\,ds + \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p T_k(u^\varepsilon)^+\,dxds \\ = \int_0^t\int_\Omega \mu^\varepsilon T_k(u^\varepsilon)^+\,dx\,ds + \int_\Omega B_k(u_0^\varepsilon)\,dx,\end{aligned} \tag{4.16}$$

for almost every $t \in (0,T)$, and where $B_k(s) = \int_0^s T_k(r)^+ b'(r)\,dr$. Using assumption (3.4) and since $B_k(u^\varepsilon) \ge 0$, from (4.16) we obtain that $T_k(u^\varepsilon)^+$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, and similarly by taking $T_k(u^\varepsilon)^-$ as a test function in (4.4) yields (4.7). By (3.2) we have

$$B_1(s) \ge \gamma \int_0^s T_1(r)^+\,dr \quad \forall s \in \mathbb{R},$$

and since $\int_0^s T_1(r)^+\,dr \ge s - 1\ \forall s \in \mathbb{R}^+$, we obtain

$$\int_\Omega u^\varepsilon(t)\,dx \le \frac{1}{\gamma}(\|b(u_0^\varepsilon)\|_{L^1(\Omega)} + |\mu|_{\mathcal{M}(Q)}) + \ meas(\Omega).$$

Similarly by taking $T_k(u^\varepsilon)^-$ as a test function in (4.4) we conclude that u^ε is bounded in $L^\infty(0,T;L^1(\Omega))$, which yields (4.6). Now, using (4.6), (4.7), and arguing as in [10] we obtain (4.8). Taking $T_k(v^\varepsilon)^+$ as test function in (4.4) and we integrate in $]0,t[$, by assumptions (3.2), (3.4), (3.5), (3.7) and by means of Young's inequality one obtains

$$\int_\Omega \overline{T_k}(v^\varepsilon)(t)dx + \frac{\alpha}{2}\int_{\{|v^\varepsilon|\le k\}} b'(u^\varepsilon)|\nabla u^\varepsilon|^p dxdt \tag{4.17}$$
$$\le C(\|G^\varepsilon\|^{p'}_{L^{p'}(Q)} + \|L\|^{p'}_{L^{p'}(Q)} + \|\nabla g^\varepsilon\|^p_{L^p(Q)})$$
$$+k(\|f^\varepsilon\|_{L^1(Q)} + \|b(u_0^\varepsilon)\|_{L^1(\Omega)} + \|\lambda^\varepsilon_-\|_{L^1(Q)} + \|\lambda^\varepsilon_+\|_{L^1(Q)}),$$

where $\overline{T_k}(s) = \displaystyle\int_0^s T_k(r)^+\,dr \ \forall s \in \mathbb{R}^+$. Similarly by taking $T_k(v^\varepsilon)^-$ as a test function in (4.4) we deduce that (4.9) and (4.10) hold true.
Now we prove that (4.11) holds, we use $\rho_k(u^\varepsilon) = \int_0^{u^\varepsilon} h(s)\chi_{\{s>k\}}\,ds$ as test function in (4.4) we obtain

$$\int_\Omega B_k(u^\varepsilon)(T)\,dx + \int_Q a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon h(u^\varepsilon)\chi_{\{u^\varepsilon>k\}}\,dxdt \tag{4.18}$$
$$\le (\int_k^\infty h(s)\,ds)\|\mu^\varepsilon\|_{L^1(Q)} + \int_\Omega B_k(u_0^\varepsilon)\,dx$$
$$\le C\int_k^\infty h(s)\,ds,$$

where $B_k(s) = \displaystyle\int_0^s \rho_k(r)b'(r)\,dr$.
By assumptions (3.4) and (3.7) we obtain that $\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\chi_{\{u^\varepsilon>k\}}\,dxdt$ is bounded, similarly taking $\rho_k(u^\varepsilon) = \int_{u^\varepsilon}^0 h(s)\chi_{\{s<-k\}}\,ds$ as test function in (4.4) we obtain the same result.
We have

$$\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\,dxdt \le \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\chi_{\{|u^\varepsilon|<k\}}\,dxdt + \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\chi_{\{|u^\varepsilon|>k\}}\,dxdt,$$

due to (3.7), (4.7) and (4.18) we get (4.11). By (3.2), (4.8), (4.11) and since μ^ε is bounded in $L^1(Q)$, one obtain that $\frac{\partial b(u^\varepsilon)}{\partial t}$ is bounded in $L^1(0,T;W^{-1,1}(\Omega))$, using compactness arguments (see [41]) yield (4.12) and (4.13). □

Let us introduce for $k\ge 0$ fixed, the time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been first introduced by R. Landes. More recently, it has been exploited to solve a few nonlinear evolution problems with L^1 or measure data. This specific time regularization of $T_k(u)$ (for fixed $k\ge 0$) is defined as follows. Let $(v_0^\nu)_\nu$ in $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ such that $\|v_0^\nu\|_{L^\infty(\Omega)}\le k$, for all $\nu>0$, and $v_0^\nu \to T_k(u_0)$ a.e. in Ω with $\frac{1}{\nu}\|v_0^\nu\|_{L^p(\Omega)}\to 0$ as $\nu\to+\infty$.
For fixed $k\ge 0$ and $\nu>0$, let us consider the unique solution $T_k(u)_\nu \in L^\infty(Q)\cap L^p(0,T,W_0^{1,p}(\Omega))$ of the monotone problem:

$$\frac{\partial T_k(u)_\nu}{\partial t} + \nu(T_k(u)_\nu - T_k(u)) = 0 \text{ in } \mathcal{D}'(Q),$$

$$T_k(u)_\nu(t=0) = v_0^\nu \text{ in } \Omega.$$

The behavior of $T_k(u)_\nu$ as $\nu \to +\infty$ is investigated in [29] (see also [24]) and we just recall here that:

$$T_k(u)_\nu \to T_k(u) \text{ strongly in } L^p(0,T,W_0^{1,p}(\Omega)) \text{ a.e. in } Q \; as \; \nu \to +\infty$$

with $\|T_k(u)_\nu\|_{L^\infty(\Omega)} \leq k$ for any $\nu > 0$, and $\frac{\partial T_k(u)_\nu}{\partial t} \in L^p(0,T,W_0^{1,p}(\Omega))$.
Here and in the rest of paper $\omega(\varepsilon, n, \delta, \mu)$ will indicate any quantity that vanishes as the parameters go to their limit point with in the same order in which they appear, that is, for example

$$\overline{\lim_{\nu\to\infty}}\;\overline{\lim_{\delta\to 0}}\;\overline{\lim_{n\to\infty}}\;\overline{\lim_{\varepsilon\to 0}}\,|\omega(\varepsilon,n,\delta,\nu)| = 0.$$

Now we prove the following proposition

Proposition 4.3. *The sequence (∇u^ε) converges to ∇u a.e. in Q,*

In order to prove the Proposition 4.3 we give the following result .

Lemma 4.4. *Let z^ε be a sequence in $L^p(0,T;W_0^{1,p}(\Omega)) \cap C^0([0,T];L^2(\Omega))$ such that $z^\varepsilon(.,0) = 0$, and $(z^\varepsilon)_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, suppose that z^ε converges almost everywhere in Q to a function z such that $T_k(z) \in L^p(0,T;W_0^{1,p}(\Omega))$ for every $k > 0$. then we have*

$$\int_0^T \langle \frac{\partial z^\varepsilon}{\partial t}, T_\sigma(z^\varepsilon - T_k(z)_\nu)) \rangle \, ds\, dt \geq \omega(n,\nu,k,\sigma) \tag{4.19}$$

Proof. See [7] □

Remark 4.5. The proof of Lemma 4.4 extends the case of sequences (u_0^ε) converging in L^1, the only difference being to include the initial condition u_0^ε in Landes approximation. This technical point can be found e.g. in [32].

Now we are ready to prove the Proposition 4.3

Proof. We follow the method used in [7], by the monotonicity of $a(x,t,\xi)$, the result will be proved if, up to subsequences still denoted by u^ε (for simplicity of notation, we will omit the dependence of a on x and t),

$$\Big[(a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u)\Big]^\theta \to 0. \tag{4.20}$$

Note that (4.20) will be true if we show that

$$\int_Q \Big[(a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u)\Big]^\theta dxdt = \omega(\varepsilon), \tag{4.21}$$

for some $\theta > 0$, thanks to Proposition 4.2, the following estimate holds

$$meas(\{|b(u)| \geq k\}) = \omega(k),$$

and for simplicity we set $w = b(u)$.
We can write

$$\int_Q \Big[(a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u)\Big]^\theta dxdt$$

$$= \int_{\{|w|\geq k\}} \Big[(a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u)\Big]^\theta dxdt$$
$$+ \int_{\{|w|<k\}} \Big[(a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u)\Big]^\theta dxdt$$
$$= I_{\varepsilon,k} + J_{\varepsilon,k}.$$

Since u^ε is bounded in $L^q(0,T;W_0^{1,q}(\Omega))$ for some $q<p$, we can choose $\theta < \frac{q}{p} < 1$, so that using Hölder inequality, we obtain

$$|I_{\varepsilon,k}| \leq c\Big(\int_Q (|L|^{p'/p} + |\nabla u| + |\nabla u^\varepsilon|)^q\Big)^{\theta p/q} meas(\{|w|\geq k\})^{1-\theta p/q}$$
$$\leq c\; meas(\{|w|\geq k\})^{1-\theta p/q},$$

and so $I_{\varepsilon,k} = \omega(\varepsilon,k)$. On the other hand,

$$J_{\varepsilon,k} = \int_{\{|w|<k\}} \Big[\Big(a(\nabla u^\varepsilon) - a(\nabla u\chi_{\{|w|<k\}})\Big)\Big(\nabla u^\varepsilon - \nabla u\Big)\Big]^\theta dxdt$$
$$\leq \int_Q \Big[\Big(a(\nabla u^\varepsilon) - a(\nabla u\chi_{\{|w|<k\}})\Big)\Big(\nabla u^\varepsilon - \nabla u\chi_{\{|w|<k\}}\Big)\Big]^\theta dxdt$$

We define

$$\Psi_{\varepsilon,k} = \Big(a(\nabla u^\varepsilon) - a(\nabla u\chi_{\{|w|<k\}})\Big)\Big(\nabla u^\varepsilon - \nabla u\chi_{\{|w|<k\}}\Big),$$

and we have

$$\int_Q \Big[(a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u)\Big]^\theta dxdt \tag{4.22}$$
$$\leq \int_Q \Psi^\theta_{\varepsilon,k}\chi_{\{|w^\varepsilon - T_k(w)_\nu|\leq\sigma\}} + \int_Q \Psi^\theta_{\varepsilon,k}\chi_{\{|w^\varepsilon - T_k(w)_\nu|>\sigma\}} + \omega(\varepsilon,k),$$

since $\Psi^\theta_{\varepsilon,k}$ is bounded in $L^{q/\theta p}(Q)$ and since $\chi_{\{|w-T_k(w)|>\sigma\}}$ converges to zero almost everywhere in Q as k tends to infinity, we obtain

$$\int_Q \Psi^\theta_{\varepsilon,k}\chi_{\{|w^\varepsilon - T_k(w)_\nu|>\sigma\}} = \omega(\varepsilon,\nu,k),$$

Thus, (4.21) becomes

$$\int_Q \Big[(a(\nabla u^\varepsilon) - a(\nabla u)).(\nabla u^\varepsilon - \nabla u)\Big]^\theta dxdt$$
$$\leq \int_Q \Psi^\theta_{\varepsilon,k}\chi_{\{|w^\varepsilon - T_k(w)_\nu|\leq\sigma\}} + \omega(\varepsilon,\nu,k).$$

Using Hölder inequality (with exponents $1/\theta$ and $1/1-\theta$) the last integral is smaller than

$$meas(Q)^{1-\theta}\Big(\int_Q \Psi_{\varepsilon,k}\chi_{\{|w^\varepsilon - T_k(w)_\nu|\leq\sigma\}}\Big)^\theta,$$

so that (4.21) will be proved if we can show that

$$\int_Q \Psi_{\varepsilon,k}\chi_{\{|w^\varepsilon - T_k(w)_\nu|\leq\sigma\}} = \omega(\varepsilon,\nu,k,\sigma). \tag{4.23}$$

Now recalling the definition of $\Psi_{\varepsilon,k}$ we can write by (3.2)

$$\int_Q \Psi_{\varepsilon,k}\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}} \tag{4.24}$$
$$\le \frac{1}{\gamma}\Big(\int_Q b'(u^\varepsilon)a(\nabla u^\varepsilon)\Big(\nabla u^\varepsilon - \nabla u\chi_{\{|w|\le k\}}\Big)\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}}$$
$$-\int_Q b'(u^\varepsilon)a(\nabla u\chi_{\{|w|\le k\}})\Big(\nabla u^\varepsilon - \nabla u\chi_{\{|w|\le k\}}\Big)\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}}\Big)$$

By Proposition 4.2 and since $|T_k(w)_\nu| \le k$ we obtain

$$\int_Q b'(u^\varepsilon)a(\nabla u\chi_{\{|w|\le k\}})\Big(\nabla u^\varepsilon - \nabla u\chi_{\{|w|\le k\}}\Big)\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}} \tag{4.25}$$
$$= \int_Q b'(u)a(\nabla u\chi_{\{|w|\le k\}})\Big(\nabla u\chi_{\{|w|\le k+\sigma\}} - \nabla u\chi_{\{|w|\le k\}}\Big)\Big)\chi_{\{|w - T_k(w)_\nu|\le\sigma\}} + \omega(\varepsilon)$$
$$= \omega(\varepsilon).$$

On the other hand we have

$$\int_Q b'(u^\varepsilon)a(\nabla u^\varepsilon)\Big(\nabla u^\varepsilon - \nabla u\chi_{\{|w|\le k\}}\Big)\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}} \tag{4.26}$$
$$= \int_Q a(\nabla u^\varepsilon)\nabla(w^\varepsilon - T_k(w)_\nu)\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}}$$
$$+\int_Q b'(u^\varepsilon)a(\nabla u^\varepsilon)(b'(u^\varepsilon)^{-1}\nabla T_k(w)_\nu - b'(u)^{-1}\nabla T_k(w))\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}}.$$

We have

$$\int_Q b'(u^\varepsilon)a(\nabla u^\varepsilon)(b'(u^\varepsilon)^{-1}\nabla T_k(w)_\nu - b'(u)^{-1}\nabla T_k(w))\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}}$$
$$= \int_Q a(b'(u^\varepsilon)^{-1}\nabla T_{k+\sigma}(w^\varepsilon)\nabla T_k(w)_\nu\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}}$$
$$-\int_Q b'(u^\varepsilon)b'(u)^{-1}a(b'(u^\varepsilon)^{-1}\nabla T_{k+\sigma}(w^\varepsilon)\nabla T_k(w)\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}},$$

since $a(b'(u^\varepsilon)^{-1}\nabla T_{k+\sigma}(w^\varepsilon))$ converge weakly to $\Gamma_{k,\sigma}$ in $L^{p'}(Q)$, $b'(u^\varepsilon)$ converges to $b'(u)$ $\star$-weakly in $L^\infty(Q)$, and almost everywhere in Q by Propostion 2.2 we obtain

$$\int_Q b'(u^\varepsilon)a(\nabla u^\varepsilon)(b'(u^\varepsilon)^{-1}\nabla T_k(w)_\nu - b'(u)^{-1}\nabla T_k(w))\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}}$$
$$= \int_Q \Gamma_{k,\sigma}(\nabla T_k(w)_\nu - \nabla T_k(w))\chi_{\{|w - T_k(w)_\nu|\le\sigma\}} + \omega(\varepsilon)$$
$$= \omega(\varepsilon,\nu).$$

Hence (4.24), (4.25) and (4.26) imply that

$$\int_Q \Psi_{\varepsilon,k}\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}} \le \int_Q a(\nabla u^\varepsilon)\nabla(w^\varepsilon - T_k(w)_\nu)\chi_{\{|w^\varepsilon - T_k(w)_\nu|\le\sigma\}} + \omega(\varepsilon,\nu).$$

Now we use the equation solved by u^ε. Taking $T_\sigma(w^\varepsilon - T_k(w)_\nu)^+$ in (4.4) we obtain

$$\int_0^T \langle \frac{\partial w^\varepsilon}{\partial t}, T_\sigma(w^\varepsilon - T_k(w)_\nu)^+\rangle \, dt + \int_Q a(\nabla u^\varepsilon)\nabla T_\sigma(w^\varepsilon - T_k(w)_\nu)^+ \, dxdt$$
$$+ \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p T_\sigma(w^\varepsilon - T_k(w)_\nu)^+ \, dxdt = \int_Q \mu^\varepsilon T_\sigma(w^\varepsilon - T_k(w)_\nu)^+ \, dxdt.$$

By property of μ^ε we have

$$\Big| \int_Q \mu^\varepsilon T_\sigma(w^\varepsilon - T_k(w)_\nu) \, dxdt \Big| \leq \sigma \|\mu^\varepsilon\|_{L^1(Q)} \leq \sigma |\mu|_{\mathcal{M}(Q)}$$

By Lemma 4.4 and assumption (3.7) we obtain

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_\sigma(w^\varepsilon - T_k(w)_\nu)^+ \, dxdt \leq \omega(\varepsilon,\nu,\sigma).$$

Similarly we take $T_\sigma(w^\varepsilon - T_k(w)_\nu)^-$ as test function in (4.4) we deduce

$$\int_Q a(\nabla u^\varepsilon)\nabla T_\sigma(w^\varepsilon - T_k(w)_\nu) \, dxdt \leq \omega(\varepsilon,\nu,\sigma). \tag{4.27}$$

Then we obtain (4.27) and therefore (4.21). □

Now we give the basic result about approximate capacitary potential.

Lemma 4.6. *Let $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}(Q)$ where μ_s^+ and μ_s^- are concentrated respectively, on two disjoint E^+ and E^- of zero p-capacity. Then, for every $\delta > 0$, there exist two compact sets $K_\delta^+ \subseteq E^+$ and $K_\delta^- \subseteq E^-$ such that*

$$\mu_s^+(E^+ \backslash K_\delta^+) \leq \delta, \quad \mu_s^-(E^+ \backslash K_\delta^-) \leq \delta, \tag{4.28}$$

and there exist ψ_δ^+, $\psi_\delta^- \in C_0^1(Q)$, such that

$$\psi_\delta^+ \equiv 1 \text{ and } \psi_\delta^- \equiv 1 \text{ respectively on } K_\delta^+ \text{ and } K_\delta^-, \tag{4.29}$$

$$0 \leq \psi_\delta^+, \quad \psi_\delta^- \leq 1, \tag{4.30}$$

$$supp(\psi_\delta^+) \cap supp(\psi_\delta^-) \equiv \emptyset. \tag{4.31}$$

Moreover

$$\|\psi_\delta^+\|_{\mathcal{S}} \leq \delta, \quad \|\psi_\delta^-\|_{\mathcal{S}} \leq \delta, \tag{4.32}$$

and in particular, there exists a decomposition of $(\psi_\delta^+)_t$ and a decomposition of $(\psi_\delta^-)_t$ such that

$$\|(\psi_\delta^+)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^+)_t^2\|_{L^1(Q)} \leq \frac{\delta}{3}, \tag{4.33}$$

$$\|(\psi_\delta^-)_t^1\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))} \leq \frac{\delta}{3}, \quad \|(\psi_\delta^-)_t^2\|_{L^1(Q)} \leq \frac{\delta}{3}. \tag{4.34}$$

Both ψ_δ^+ and ψ_δ^- converges to zero $*-$weakly in $L^\infty(Q)$, in $L^1(Q)$, and up to subsequences, almost everywhere as δ vanishes. Moreover, if λ_+^ε and λ_-^ε are as in (4.1) we have

$$\int_Q \psi_\delta^- \, d\lambda_+^\varepsilon = \omega(\varepsilon,\delta), \quad \int_Q \psi_\delta^- \, d\mu_s^+ \leq \delta, \tag{4.35}$$

$$\int_Q \psi_\delta^+ \, d\lambda_-^\varepsilon = \omega(\varepsilon, \delta), \quad \int_Q \psi_\delta^+ \, d\mu_s^- \leq \delta, \tag{4.36}$$

$$\int_Q (1 - \psi_\delta^+ \psi_\eta^+) \, d\lambda_+^\varepsilon = \omega(\varepsilon, \delta, \eta), \quad \int_Q (1 - \psi_\delta^+ \psi_\eta^+) \, d\mu_s^+ \leq \delta + \eta, \tag{4.37}$$

$$\int_Q (1 - \psi_\delta^- \psi_\eta^-) \, d\lambda_-^\varepsilon = \omega(\varepsilon, \delta, \eta), \quad \int_Q (1 - \psi_\delta^- \psi_\eta^-) \, d\mu_s^- \leq \delta + \eta. \tag{4.38}$$

Proof. See [34], Lemma 5. □

In what follows we will always refer to subsequences of both ψ_δ^+ and ψ_δ^- that satisfy all the convergence results stated in Lemma 4.6.
Now we will prove the following theorem

Theorem 4.7. *Let v^ε and v be as before. Then, for every $k > 0$*

$$T_k(v^\varepsilon) \to T_k(v) \text{ strongly in } L^p(0, T; W_0^{1,p}(\Omega)).$$

Proof. Let us first introduce the following function that we will use in the proof of Theorem 4.7.

$$H_n(s) = \begin{cases} 1 & \text{if } |s| \leq n, \\ \dfrac{2n - s}{n} & \text{if } \ n < s \leq 2n, \\ \dfrac{2n + s}{n} & \text{if } \ -2n < s \leq -n, \\ 0 & \text{if } \ |s| > 2n. \end{cases}$$

Let also introduce another auxiliary function in terms of H_n by $B_n(s) = 1 - H_n(s)$.

Our aim is to prove the following asymptotic estimate:

$$\overline{\lim_{\varepsilon \to 0}} \int_Q a(x, t, \nabla u^\varepsilon) \nabla T_k(v^\varepsilon) \, dxdt \leq \int_Q a(x, t, \nabla u) \nabla T_k(v) \, dx \, dt. \tag{4.39}$$

In order to prove (4.39), we shall follow several steps.
⋆ **Step 1**.
For every $\delta, \eta > 0$, let $\psi_\delta^+, \psi_\eta^+$, ψ_δ^- and ψ_η^- as in Lemma 4.6 and let E^+ and E^- be the sets where, respectively, μ_s^+ and μ_s^- are concentrated. Setting $\Phi_{\delta,\eta} = \psi_\delta^+ \psi_\eta^+ + \psi_\delta^- \psi_\eta^-$, we can write

$$\begin{aligned} &\int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon) \, dx \, dt \\ &= \int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon) \Phi_{\delta,\eta} \, dxdt \\ &+ \int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon) (1 - \Phi_{\delta,\eta}) \, dxdt. \end{aligned} \tag{4.40}$$

Now, if $n > k$, since $a(x, t, \nabla u^\varepsilon \chi_{\{|v^\varepsilon| \leq 2n\}}) \nabla T_k(v)_\nu$ is weakly compact in $L^1(Q)$ as ε goes to zero, $H_n(v^\varepsilon)$ converges to $H_n(v)$ ⋆-weakly in $L^\infty(Q)$, and almost everywhere in Q, by Proposition 2.2 we have

$$\overline{\lim_{\varepsilon \to 0}} \int_Q a(x, t, \nabla u^\varepsilon) \nabla (T_k(v^\varepsilon) - T_k(v)_\nu) H_n(v^\varepsilon) \Phi_{\delta,\eta} \, dxdt \tag{4.41}$$

$$= \overline{\lim_{\varepsilon\to 0}}\Big[\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt\Big] - \int_Q a(x,t,\nabla u)\nabla T_k(v)_\nu H_n(v)\Phi_{\delta,\eta}\,dxdt$$

$$= \overline{\lim_{\varepsilon\to 0}}\Big[\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt\Big] - \int_Q a(x,t,\nabla u)\nabla T_k(v)\Phi_{\delta,\eta}\,dxdt + \omega(\nu).$$

Since $\Phi_{\delta,\eta}$ converges to zero $*$- weakly in $L^\infty(Q)$ as δ goes to zero,

$$\int_Q a(x,t,\nabla u)\nabla T_k(v)\Phi_{\delta,\eta}\,dxdt = \omega(\delta).$$

Therefore, if we prove that

$$\overline{\lim_{\eta\to 0}}\ \overline{\lim_{\delta\to 0}}\ \overline{\lim_{\varepsilon\to 0}}\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt \le 0, \tag{4.42}$$

then we can conclude

$$\overline{\lim_{\eta\to 0}}\ \overline{\lim_{\delta\to 0}}\ \overline{\lim_{\varepsilon\to 0}}\int_Q a(x,t,\nabla u^\varepsilon)\nabla (T_k(v^\varepsilon) - T_k(v)_\nu)H_n(v^\varepsilon)\Phi_{\delta,\eta}\,dxdt \le 0. \tag{4.43}$$

$\star$ **Step 2**. Near to E.
Before proving (4.41), we first show the following result

Lemma 4.8. *Let u^ε be a solution of (4.3)-(4.5). Let η be a positive real number, and let φ^η_+ and φ^η_- be two non negative functions in $C_c^\infty(Q)$ such that*

$$0 \le \varphi^\eta_+ \le 1, \qquad 0 \le \varphi^\eta_- \le 1,$$

and

$$0 \le \int_Q \varphi^\eta_-\,d\mu_s^+ \le \eta, \qquad 0 \le \int_Q \varphi^\eta_+\,d\mu_s^- \le \eta, \tag{4.44}$$

we then have

$$\frac{1}{n}\int_{\{-2n<v^\varepsilon\le -n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \varphi^\eta_+\,dx\,dt = \omega(\varepsilon,n,\eta), \tag{4.45}$$

$$\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \varphi^\eta_-\,dx\,dt = \omega(\varepsilon,n,\eta). \tag{4.46}$$

Proof. Let us prove (4.46); let $\beta_n(s) = B_n(s^+)$, we can choose $\beta_n(v^\varepsilon)\varphi^\eta_-$ as test function in (4.4) and rearranging conveniently all terms we have

$$\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon \varphi^\eta_-\,dx\,dt + \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\beta_n(v^\varepsilon)\varphi^\eta_-\,dxdt + \int_Q \beta_n(v^\varepsilon)\varphi^\eta_-\,d\lambda^\varepsilon_-$$

$$= \frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(x,t,\nabla u^\varepsilon)\nabla g^\varepsilon \varphi^\eta_-\,dx\,dt$$

$$+ \int_Q \overline{\beta_n}(v^\varepsilon)\frac{d\varphi^\eta_-}{dt}\,dxdt - \int_Q a(x,t,\nabla u^\varepsilon)\nabla\varphi^\eta_-\beta_n(v^\varepsilon)\,dxdt + \int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi^\eta_-\,dxdt$$

$$- \int_0^T < \mathrm{div}(G^\varepsilon), \beta_n(v^\varepsilon)\varphi^\eta_- > \,dt + \int_Q \beta_n(v^\varepsilon)\varphi^\eta_-\,d\lambda^\varepsilon_+,$$

where $\overline{\beta_n}(s) = \displaystyle\int_0^s \beta_n(r)\,dr$. Using the fact that $\displaystyle\int_Q \beta_n(v^\varepsilon)\varphi_-^\eta\,d\lambda_-^\varepsilon \geq 0$ and by assumptions (3.2), (3.4), (3.5), (3.7) and Young's inequality we obtain

$$\frac{1}{n}\int_{\{n\leq v^\varepsilon<2n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\varphi_-^\eta\,dx\,dt \leq \frac{C}{n}\int_Q \left(|\nabla g^\varepsilon|^p + |L|^{p'}\right)dx\,dt$$

$$+\int_Q \overline{\beta_n}(v^\varepsilon)\frac{d\varphi_-^\eta}{dt}\,dx\,dt - \int_Q a(x,t,\nabla u^\varepsilon).\nabla\varphi_-^\eta\beta_n(v^\varepsilon)\,dx\,dt + \int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi_-^\eta\,dx\,dt$$

$$-\int_0^T < \mathrm{div}(G^\varepsilon), \beta_n(v^\varepsilon)\varphi_-^\eta > \,dt + \int_Q \beta_n(v^\varepsilon)\varphi_-^\eta\,d\lambda_+^\varepsilon.$$

By Proposition 4.2 and 4.3 we have $a(x,t,\nabla u^\varepsilon)$ converges weakly to $a(x,t,\nabla u)$ in $(L^{q'}(Q))^N$ as ε goes to 0 for every $q' < 1 + \frac{1}{(N+1)(p-1)}$, since φ_-^η belongs to $C_c^\infty(Q)$ and $\beta_n(v^\varepsilon)$ converges to $\beta_n(v)$ a.e. in Q and $\star$-weakly in $L^\infty(Q)$ as ε goes to zero and $\beta_n(v)$ converges to 0 a.e. in Q and $\star$-weakly in $L^\infty(Q)$ as n goes to $+\infty$, thanks to Proposition 2.2, we obtain

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla\varphi_-^\eta\beta_n(v^\varepsilon)\,dx\,dt = \omega(\varepsilon,n).$$

Since $\overline{\beta_n}(v^\varepsilon)$ converges to $\overline{\beta_n}(v)$ in $L^1(Q)$ as ε goes to 0, and $\overline{\beta_n}(v)$ converges to 0 in $L^1(Q)$ as n goes to $+\infty$, we obtain

$$\int_Q \overline{\beta_n}(v^\varepsilon)\frac{d\varphi_-^\eta}{dt}\,dx\,dt = \omega(\varepsilon,n).$$

Moreover, the weak $L^1(Q)$ convergence of f^ε to f and thanks to Proposition 2.2 we obtain

$$\int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi_-^\eta\,dxdt = \omega(\varepsilon,n).$$

Due the strong convergence of $\mathrm{div}(G^\varepsilon)$ to $\mathrm{div}(G)$ in $L^{p'}(0,T,W^{-1,p'}(\Omega))$ and the weak convergence in $L^p(0,T,W_0^{1,p}(\Omega))$ of $\beta_n(v^\varepsilon)$ to $\beta_n(v)$ and $\beta_n(v)$ to 0 strongly in $L^p(0,T,W_0^{1,p}(\Omega))$ (this facts is an easy consequence of the estimate on the truncates of u^ε in Proposition 4.2), we obtain

$$\int_0^T < \mathrm{div}(G^\varepsilon), \beta_n(v^\varepsilon)\varphi_-^\eta > \,dxdt = \omega(\varepsilon,n).$$

Finally, by (4.44) and since $\beta_n(v^\varepsilon)$ is non negative and bounded and φ_-^η is continuous, we have

$$\int_Q \beta_n(v^\varepsilon)\varphi_-^\eta\,d\lambda_+^\varepsilon \leq \int_Q \varphi_-^\eta\,d\mu_s^+ + \omega(\varepsilon) = \omega(\varepsilon,\eta).$$

Putting together all these facts lead to (4.46), while (4.45) can be obtained in the same way choosing $\beta_n(s) = B_n(s^-)$ and $\beta_n(v^\varepsilon)\varphi_+^\eta$ as test function in (4.4). $\square$

Now let us check (4.42). For fixed $k>0$, we choose $(k-T_k(v^\varepsilon))H_n(v^\varepsilon)\exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+$ as test function in (4.4), defining $\Gamma_{n,k}(s)=\displaystyle\int_0^s (k-T_k(r))H_n(r)\exp(-H(r))\,dr$, and $H(s)=\displaystyle\int_0^s \frac{h(r)}{\gamma\alpha}\frac{\max_{\{|r|\le 2n\}}h(r)}{\min_{\{|r|\le 2n\}}h(r)}\,dr$.
Integrating by parts, we obtain

$$
\begin{aligned}
&-\int_Q \Gamma_{n,k}(v^\varepsilon)\frac{d}{dt}(\psi_\delta^+\psi_\eta^+)\,dxdt+\int_Q (k-T_k(v^\varepsilon))\exp(-H(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^+)\,dxdt\\
&\quad+\int_Q a(x,t,\nabla u^\varepsilon)\nabla H_n(v^\varepsilon)\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt\\
&\quad-\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\exp(-H(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt\\
&\quad+\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon)\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt\\
&-\frac{\max_{\{|s|\le 2n\}}h(s)}{\gamma\alpha\min_{\{|s|\le 2n\}}h(s)}\int_Q h(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt\\
&\quad=\int_Q f^\varepsilon\exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt\\
&\quad-\int_0^T <\operatorname{div}(G^\varepsilon),\exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+>\,dt\\
&+\int_Q \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon-\int_Q \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_-^\varepsilon.
\end{aligned}
\tag{4.47}
$$

For $n>k$, we have

$$H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}=a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\ \text{a.e. in } Q,$$

then rearranging all terms of (4.47) and using assumptions (3.4) and (3.7), we obtain

$$
\begin{aligned}
&\int_Q b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}\exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt+\int_Q \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda\\
&\quad+\max_{\{|s|\le 2n\}}h(s)\int_Q |\nabla u^\varepsilon|^p\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt\\
&\le-\int_Q \Gamma_{n,k}(v^\varepsilon)\frac{d}{dt}(\psi_\delta^+\psi_\eta^+)\,dxdt+\frac{2k}{n}\int_{\{-2n<v^\varepsilon\le -n\}}a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dx\,dt\\
&\quad+\int_Q \exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^+)\,dxdt\\
&\quad-\int_Q f^\varepsilon\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt
\end{aligned}
\tag{4.48}
$$

$$+\max_{\{|s|\le 2n\}} h(s)\int_Q |\nabla u^\varepsilon|^p H_n(v^\varepsilon)\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt$$
$$+\int_{\{|u^\varepsilon|\ge 2n\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon)\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt$$
$$-\int_0^T <\operatorname{div}(G^\varepsilon),\exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+>\,dt+\int_Q(k-T_k(v^\varepsilon))\exp(-H(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta$$
$$+\int_Q h(v^\varepsilon)H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla g^\varepsilon\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt$$
$$+\int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\nabla g^\varepsilon\exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt. \tag{4.49}$$

$$\int_Q b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}\exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt$$
$$+\int_Q \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon$$
$$\le -\int_Q \Gamma_{n,k}(v^\varepsilon)\frac{d}{dt}(\psi_\delta^+\psi_\eta^+)\,dxdt+\frac{2k}{n}\int_{\{-2n<v^\varepsilon\le -n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\exp(-H(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dx\,dt$$
$$+\int_Q \exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^+)\,dxdt$$
$$-\int_Q f^\varepsilon(k-T_k(v^\varepsilon))\exp(-H(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt$$
$$+\int_{\{|u^\varepsilon|\ge 2n\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon)\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt$$
$$-\int_0^T <\operatorname{div}(G^\varepsilon),\exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+>\,dt$$
$$+\int_Q (k-T_k(v^\varepsilon))\exp(-H(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,d\lambda_-^\varepsilon$$
$$+\int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\nabla g^\varepsilon\exp(-H(v^\varepsilon))(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt$$
$$+\int_Q h(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla g^\varepsilon\exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt$$

Let us analyze term by term the right hand side of (4.48). Due to Proposition 4.2 we have $\Gamma_{n,k}(v^\varepsilon)$ converges to $\Gamma_{n,k}(v)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, and since $\Gamma_{n,k}(v)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(Q)$, we deduce

$$\int_Q \Gamma_{n,k}(v^\varepsilon)\frac{d}{dt}(\psi_\delta^+\psi_\eta^+)\,dxdt$$
$$=\int_Q \Gamma_{n,k}(v)\frac{d\psi_\delta^+}{dt}\psi_\eta^+\,dxdt+\int_Q \Gamma_{n,k}(v)\frac{d\psi_\eta^+}{dt}\psi_\delta^+\,dxdt+\omega(\varepsilon)=\omega(\varepsilon,\delta).$$

Since $(k - T_k(v^\varepsilon))H_n(v^\varepsilon)$ converges to $(k - T_k(v))H_n(v)$ a.e. and $*$- weakly in $L^\infty(Q)$, thanks to Proposition 2.2, Proposition 4.2 and Lemma 4.6, we deduce

$$\int_Q \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))H_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\psi_\eta^+)\,dxdt$$
$$= \int_Q \exp(-H(v^\varepsilon))(k - T_k(v))H_n(v)a(x,t,\nabla u)\nabla(\psi_\delta^+\psi_\eta^+)\,dxdt + \omega(\varepsilon)$$
$$= \omega(\varepsilon,\delta).$$

Moreover, $(k-T_k(v^\varepsilon))\exp(-H(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+$ weakly converges to $(k-T_k(v))H_n(v)\psi_\delta^+\psi_\eta^+$ in $L^p(0,T;W_0^{1,p}(\Omega))$, and $*$- weakly in $L^\infty(Q)$, thanks again to Lemma 4.6, we have

$$\int_0^T < \operatorname{div}(G^\varepsilon), \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+ > \,dt = \omega(\varepsilon,\delta),$$

and

$$\int_Q f^\varepsilon \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt = \omega(\varepsilon,\delta).$$

Using assumptions (3.7), Proposition 2.2, Proposition 4.2 and Lemma 4.6, we deduce

$$\int_Q h(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla g^\varepsilon \exp(-H(v^\varepsilon))H_n(v^\varepsilon)(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt = \omega(\varepsilon,\delta),$$

$$\int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\nabla g^\varepsilon \exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))H_n(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt = \omega(\varepsilon,\delta).$$

Thnks to (4.18) we obtain

$$\int_{\{|u^\varepsilon|\ge 2n\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon)\exp(-H(v^\varepsilon))(k - T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,dxdt = \omega(\varepsilon,n).$$

Using assumptions (3.2), (3.5), Young's inequality, since $0 \le \psi_\delta^+ \le 1$ and $\exp(-H(v^\varepsilon)) \le \exp(C\|h\|_{L^1(\mathbb{R})})$we obtain

$$\Big|\frac{1}{n}\int_{\{-2n<v^\varepsilon\le -n\}} \exp(-H(v^\varepsilon))a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\psi_\delta^+\psi_\eta^+\,dx\,dt\Big|$$
$$\le \frac{1}{n}\int_{\{-2n<v^\varepsilon\le -n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\psi_\eta^+\,dx\,dt + \frac{C}{n}\int_Q \left(|\nabla g^\varepsilon|^p + |L|^{p'}\right)dxdt,$$

applying Lemma 4.8 for $\varphi_+^\eta = \psi_\eta^+$, we obtain

$$\frac{1}{n}\int_{\{-2n<v^\varepsilon\le -n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\psi_\eta^+\,dx\,dt = \omega(\varepsilon,n,\eta).$$

Using (4.38) in Lemma 4.6, we have

$$\Big|\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_-^\varepsilon\Big| \le 2k\int_Q \psi_\delta^+\psi_\eta^+ d\lambda_-^\varepsilon = 2k\int_Q \psi_\delta^+\psi_\eta^+ d\mu_s^- + \omega(\varepsilon) = \omega(\varepsilon,\delta).$$

Collecting all we have shown above, we get

$$\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon + \int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt \le \omega(\varepsilon,\delta,n,\eta).$$

Since $\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon \geq 0$ and $\exp(-H(+\infty)) \leq \exp(-H(v^\varepsilon)) \leq \exp(-H(-\infty))$ we obtain

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\psi_\delta^+\psi_\eta^+\,dxdt \leq \omega(\varepsilon,\delta,\eta).$$

On the other hand, reasoning as before with $(k+T_k(v^\varepsilon))H_n(v^\varepsilon)\exp(H(v^\varepsilon))\psi_\delta^-\psi_\eta^-$ as test function we can obtain

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\psi_\delta^-\psi_\eta^-\,dxdt \leq \omega(\varepsilon,\delta,\eta),$$

therefore, we obtain (4.42) which yields (4.43).

Remark 4.9. As we have shown above we have

$$\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon + \int_Q b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\leq k\}}\nabla u^\varepsilon\psi_\delta^+\psi_\eta^+\,dxdt \leq \omega(\varepsilon,\delta,n,\eta),$$

by assumptions (3.2), (3.4), thanks to Proposition 4.2, 4.3 and Lemma 4.6 one obtains

$$\int_Q H_n(v^\varepsilon)(k-T_k(v^\varepsilon))\psi_\delta^+\psi_\eta^+\,d\lambda_+^\varepsilon = \omega(\varepsilon,\delta,n,\eta).$$

Analogously we obtain

$$\int_Q H_n(v^\varepsilon)(k+T_k(v^\varepsilon))\psi_\delta^-\psi_\eta^-\,d\lambda_-^\varepsilon = \omega(\varepsilon,\delta,n,\eta).$$

The two last results above show an interesting property of approximating renormalized, they expresse the fact that v^ε (and so the solution u^ε) is very large (greater than any $k>0$) on the set where the singular measure μ_s^+ is concentrated, and small (smaller than any $k<0$) on the set where the singular measure μ_s^- is concentrated.

⋆ **Step 3**. Far from E.
We first prove a result that will be essential to deal with the second term on the right hand side of (4.40).

Lemma 4.10. *Let $k \geq 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})$-function such that $S(r)=r$ for $|r|\leq k$ and supp S' is compact. Then*

$$\int_0^T\int_0^t \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \geq \omega(\varepsilon,\nu).$$

Proof. The proof of the above Lemma follows the arguments in [14], Lemma 1 and we just sketch the proof of it.
Let $k\geq 0$ be fixed. Since S is increasing and $S(r)=r$ for $|r|\leq k$,

$$T_k(S(v^\varepsilon)) = T_k(v^\varepsilon) \text{ and } T_k(S(v)) = T_k(v) \text{ a.e. in } Q.$$

As a consequence $T_k(S(v))_\nu = T_k(v)_\nu$ a.e. in Q, for any $\nu>0$.
It follows that under the notation $z^\varepsilon = S(v^\varepsilon)$ and $z = S(v)$, and thanks to

properties of $T_k(z)_\nu$ we have

$$\begin{aligned} &\int_0^T\int_0^t\langle\frac{\partial S(v^\varepsilon)}{\partial t},(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \\ &=\int_0^T\int_0^t\langle\frac{\partial z^\varepsilon}{\partial t},(T_k(z^\varepsilon)-T_k(z)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \\ &=\int_0^T\int_0^t\langle\frac{\partial (z^\varepsilon-T_k(z)_\nu)}{\partial t},(z^\varepsilon-T_k(z)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \\ &\quad-\int_0^T\int_0^t\langle\frac{\partial z^\varepsilon}{\partial t},(z^\varepsilon-T_k(z^\varepsilon))(1-\Phi_{\delta,\eta})\rangle\,dsdt \\ &\quad+\int_0^T\int_0^t\langle\frac{\partial T_k(z)_\nu}{\partial t},(z^\varepsilon-T_k(z)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt, \end{aligned} \tag{4.50}$$

integrating by parts we have

$$\begin{aligned} &\int_0^T\int_0^t\langle\frac{\partial S(v^\varepsilon)}{\partial t},(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \\ &=\frac{1}{2}\int_0^T\int_0^t\int_\Omega(z^\varepsilon-T_k(z)_\nu)^2\frac{d\Phi_{\delta,\eta}}{dt}\,dxdsdt-\frac{1}{2}\int_0^T\int_\Omega(z^\varepsilon-T_k(z^\varepsilon))^2\frac{d\Phi_{\delta,\eta}}{dt}\,dxdsdt \\ &\quad+\frac{1}{2}\int_0^T\int_\Omega(z^\varepsilon-T_k(z)_\nu)^2\,dxdt-\frac{T}{2}\int_\Omega(z^\varepsilon-T_k(z)_\nu)^2(t=0)\,dx \\ &\quad-\frac{1}{2}\int_0^T\int_\Omega(z^\varepsilon-T_k(z^\varepsilon))^2\,dxdt+\frac{T}{2}\int_\Omega(z^\varepsilon-T_k(z^\varepsilon))^2(t=0)\,dx \\ &\quad+\int_0^T\int_0^t\int_\Omega\frac{\partial T_k(z)_\nu}{\partial t}(z^\varepsilon-T_k(z)_\nu)(1-\Phi_{\delta,\eta})\,dx\,ds\,dt, \end{aligned} \tag{4.51}$$

since $\int_0^r(s-T_k(s))\,ds=\frac{1}{2}(r-T_k(r))^2$.

Using the definition of z^ε and z, the fact that S is bounded and v^ε converges to v a.e. on Q, we have z^ε converges to z strongly in $L^2(Q)$ and in $L^\infty(Q)$ $*-$ weakly, the strong convergence of $b(u_0^\varepsilon)$ to $b(u_0)$ in $L^1(\Omega)$ implies that $z^\varepsilon(t=0)$ converges to $S(b(u_0))$ strongly in $L^2(\Omega)$.

Passing to the limit as ε tends to zero in (4.51) leads to

$$\begin{aligned} &\int_0^T\int_0^t\langle\frac{\partial S(v^\varepsilon)}{\partial t},(T_k(v^\varepsilon)-T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle\,dsdt \\ &=\frac{1}{2}\int_0^T\int_\Omega(z-T_k(z)_\nu)^2\frac{d\Phi_{\delta,\eta}}{dt}\,dxdt-\frac{1}{2}\int_0^T\int_\Omega(z-T_k(z))^2\frac{d\Phi_{\delta,\eta}}{dt}\,dxdt \\ &\quad+\frac{1}{2}\int_0^T\int_\Omega(z-T_k(z)_\nu)^2\,dxdt-\frac{T}{2}\int_\Omega(z-T_k(z)_\nu)^2(t=0)\,dx \\ &\quad-\frac{1}{2}\int_0^T\int_\Omega(z-T_k(z))^2\,dxdt+\frac{T}{2}\int_\Omega(z-T_k(z))^2(t=0)\,dx \\ &\quad+\int_0^T\int_0^t\int_\Omega\frac{\partial T_k(z)_\nu}{\partial t}(z-T_k(z)_\nu)(1-\Phi_{\delta,\eta})\,dx\,ds\,dt+\omega(\varepsilon), \end{aligned} \tag{4.52}$$

by rewriting the definition of $T_k(u)_\nu$ in terms of $T_k(z)$ we have

$$\frac{\partial T_k(z)_\nu}{\partial t} + \nu(T_k(z)_\nu - T_k(z)) = 0 \text{ in } D'(Q),$$

$$T_k(z)_\nu(t=0) = v_0^\nu \text{ in } \Omega.$$

By properties of $T_k(z)_\nu$ we obtain that $T_k(z)_\nu$ converges to $T_k(z)$ strongly in $L^2(Q)$ and $T_k(z)_\nu(t=0)$ converges to $T_k(S(b(u_0)))$ strongly in $L^2(\Omega)$ as ν tends to ∞. Passing to the limit-inf as ν tends to ∞ in (4.52) leads to

$$\begin{aligned}&\varliminf_{\nu\to\infty}\lim_{\varepsilon\to 0}\int_0^T\int_0^T \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle \; dsdt \\ &= \nu\int_0^T\int_0^t\int_\Omega (T_k(z) - T_k(z)_\nu)(z - T_k(z)_\nu)(1-\Phi_{\delta,\eta}) \; dxdsdt.\end{aligned}$$

Thanks to definition of $T_k(z)_\nu$ we have

$$\begin{aligned}&\int_0^T\int_0^t\int_\Omega (T_k(z) - T_k(z)_\nu)(z - T_k(z)_\nu)(1-\Phi_{\delta,\eta}) \; dxdsdt \\ &= \int_{\{|z|\le k\}} (z - T_k(z)_\nu)(z - T_k(z)_\nu)(1-\Phi_{\delta,\eta}) \; dxdsdt \\ &+ \int_{\{z>k\}} (k - T_k(z)_\nu)(z - T_k(z)_\nu)(1-\Phi_{\delta,\eta}) \; dxdsdt \\ &+ \int_{\{z<-k\}} (-k - T_k(z)_\nu)(z - T_k(z)_\nu)(1-\Phi_{\delta,\eta}) \; dxdsdt,\end{aligned}$$

and the three terms are all non negatives, then

$$\int_0^T\int_0^t \langle \frac{\partial S(v^\varepsilon)}{\partial t}, (T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})\rangle \; dt \ge \omega(\varepsilon,\nu)$$

□

Now, let us multiply by $H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1-\Phi_{\delta,\eta})$ the equation solved by u^ε and integrate to obtain

$$\begin{aligned}&\int_0^T \Big\langle \frac{\partial v^\varepsilon}{\partial t}, H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1-\Phi_{\delta,\eta})\Big\rangle \; dt \qquad (4.53)\\ &+ \int_Q a(x,t,\nabla u^\varepsilon).\nabla(T_k(v^\varepsilon) - T_k(v)_\nu)^+ H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\, dxdt \\ &+ \int_Q a(x,t,\nabla u^\varepsilon).\nabla H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1-\Phi_{\delta,\eta})\, dxdt \\ &- \int_Q a(x,t,\nabla u^\varepsilon).\nabla\Phi_{\delta,\eta} H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+\, dxdt \\ &+ \int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1-\Phi_{\delta,\eta})\, dxdt \\ &= \int_Q f^\varepsilon H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)(1-\Phi_{\delta,\eta})^+\, dxdt\end{aligned}$$

$$-\int_0^T < \operatorname{div}(G^\varepsilon), H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) > \, dt$$
$$+ \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) \, d\lambda_+^\varepsilon$$
$$- \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) \, d\lambda_-^\varepsilon.$$

Let us analyze term by term the identity (4.53), by Lemma 4.8 we have

$$\int_0^T \Big\langle \frac{\partial v^\varepsilon}{\partial t}, H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) \Big\rangle \, dt \geq \omega(\varepsilon, \nu).$$

By assumption (3.7) we have

$$\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) \, dxdt \geq 0$$

The almost everywhere and $*-$weak convergence of $H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+$ to $H_n(v)(T_k(v) - T_k(v)_\nu)^+$ in $L^\infty(Q)$, the properties of $T_k(v)_\nu$ and thanks to Propositions 2.2 and 4.2 we have

$$\int_Q a(x,t,\nabla u^\varepsilon).\nabla \Phi_{\delta,\eta} H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+ \, dxdt = \omega(\varepsilon, \nu).$$

Due the strong convergence of $\operatorname{div}(G^\varepsilon)$ to $\operatorname{div}(G)$ in $L^{p'}(0, T, W^{-1,p'}(\Omega))$, Proposition 4.2 and the properties of $T_k(v)_\nu$ one obtains

$$\int_0^T < \operatorname{div}(G^\varepsilon), H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) > \, dt = \omega(\varepsilon, \nu).$$

The weak convergence of f^ε to f in $L^1(Q)$, the almost everywhere and $*-$ weak convergence of $H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+$ to $H_n(v)(T_k(v) - T_k(v)_\nu)^+$ in $L^\infty(Q)$, Propositions 2.2, the properties of $T_k(v)_\nu$ and the Lebesgue's dominated convergence theorem leads to

$$\int_Q f^\varepsilon H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) \, dxdt = \omega(\varepsilon, \nu).$$

By Lemma 4.6 and the fact that $|H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+| \leq 2k$ we obtain

$$\Big| \int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) \, d\lambda_+^\varepsilon \Big|$$
$$\leq 2k \int_Q (1 - \psi_\delta^+ \psi_\eta^+) d\lambda_+^\varepsilon + 2k \int_Q \psi_\delta^- \psi_\eta^- d\lambda_+^\varepsilon,$$

and

$$\int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) \, d\lambda_+^\varepsilon = \omega(\varepsilon, \delta, \eta),$$

and similarly we get

$$\int_Q H_n(v^\varepsilon)(T_k(v^\varepsilon) - T_k(v)_\nu)^+(1 - \Phi_{\delta,\eta}) \, d\lambda_-^\varepsilon = \omega(\varepsilon, \delta, \eta).$$

It remains to prove that

$$\int_Q a(x,t,\nabla u^\varepsilon).\nabla H_n(v^\varepsilon)(T_k(v^\varepsilon)-T_k(v)_\nu)^+(1-\Phi_{\delta,\eta})\,dxdt = \omega(\varepsilon,n,\delta,\eta).$$

We have

$$\Big|\frac{1}{n}\int_{\{n\le|v^\varepsilon|<2n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon(T_k(v^\varepsilon)-T_k(v)_\nu)^+(1-\Phi_{\delta,\eta})\,dx\,dt\Big|$$

$$\le \frac{2k}{n}\int_{\{n\le|v^\varepsilon|<2n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon(1-\Phi_{\delta,\eta})\,dx\,dt$$

$$+\frac{C}{n}\int_Q\left(|\nabla g^\varepsilon|^p+|L|^{p'}\right)dxdt = I_1+I_2,$$

we have $I_2=\omega(\varepsilon,n)$, and we rewrite I_1 as follows:

$$I_1=\frac{2k}{n}\int_{\{n\le v^\varepsilon<2n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon(1-\psi_\delta^+\psi_\eta^+)\,dx\,dt,$$

$$-\frac{2k}{n}\int_{\{n\le v^\varepsilon<2n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\psi_\delta^-\psi_\eta^-\,dx\,dt,$$

$$+\frac{2k}{n}\int_{\{-2n<v^\varepsilon\le -n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon(1-\psi_\delta^-\psi_\eta^-)\,dx\,dt,$$

$$-\frac{2k}{n}\int_{\{-2n<v^\varepsilon\le -n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\psi_\delta^+\psi_\eta^+\,dx\,dt.$$

We can apply Lemma 4.8 for every term above. Indeed, if we define $\varphi_-^{\delta,\eta}=1-\psi_\delta^+\psi_\eta^+$, we have by Lemma 4.6,

$$\int_Q \varphi_-^{\delta,\eta}\,d\mu_s^+\le\eta+\delta,$$

then $\varphi_-^{\delta,\eta}$ satisfies (4.44), thanks to Lemma 4.8 we obtain

$$\frac{2k}{n}\int_{\{n\le v^\varepsilon<2n\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon(1-\psi_\delta^+\psi_\eta^+)\,dx\,dt\le\omega(\varepsilon,n)+\delta+\eta=\omega(\varepsilon,n,\delta,\eta).$$

In analogous way we obtain the same result for the others terms. Therefore, we obtain our estimate far from E

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla(T_k(v^\varepsilon)-T_k(v)_\nu)^+H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\,dxdt\le\omega(\varepsilon,\nu,n,\delta,\eta).\quad(4.54)$$

Similarly to (4.54), we take $(T_k(v^\varepsilon)-T_k(v)_\nu)^-H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})$ as test function in (4.4) we deduce

$$\int_Q a(x,t,\nabla u^\varepsilon)\nabla(T_k(v^\varepsilon)-T_k(v)_\nu)H_n(v^\varepsilon)(1-\Phi_{\delta,\eta})\,dxdt\le\omega(\varepsilon,\nu,n,\delta,\eta).\quad(4.55)$$

□

⋆ **Step 4**. Strong convergence of truncates.
Collecting together (4.40), (4.41) and (4.55), we have by taking again $n > k$,

$$\overline{\lim_{\varepsilon\to 0}} \int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\,dxdt \le \int_Q a(x,t,\nabla u)\nabla T_k(v)\,dxdt. \tag{4.56}$$

Now, we prove that

$$\lim_{\varepsilon\to 0}\int_Q b'(u^\varepsilon)\Big[a(x,t,\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}})-a(x,t,\nabla u\chi_{\{|v|\le k\}})\Big]\Big[\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}-\nabla u\chi_{\{|v|\le k\}}\Big]\,dx\,dt=0. \tag{4.57}$$

We set

$$A^\varepsilon=\int_Q b'(u^\varepsilon)\Big[a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}-a(x,t,\nabla u)\chi_{\{|v|\le k\}}\Big][\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}-\nabla u\chi_{\{|v|\le k\}}]\,dxdt.$$

We split (4.57), into $A^\varepsilon = A_1^\varepsilon + A_2^\varepsilon + A_3^\varepsilon$, where

$$A_1^\varepsilon=\int_Q b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}\,dxdt,$$

$$A_2^\varepsilon=-\int_Q b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u\chi_{\{|v^\varepsilon|\le k\}}\chi_{\{|v|\le k\}}\,dxdt,$$

$$A_3^\varepsilon=-\int_Q b'(u^\varepsilon)a(x,t,\nabla u)(\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}})-\nabla u\chi_{\{|v|\le k\}}))\,dxdt.$$

We pass to the limit as ε tends to 0 in A_1^ε, A_2^ε and A_3^ε. Let us remark that we have $b'(u^\varepsilon)\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}} = \nabla T_k(v^\varepsilon) + \nabla g^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}$ a.e in Q, and we have also $\chi_{\{|v^\varepsilon|\le k\}}$ almost everywhere converges to $\chi_{\{|v|\le k\}}$ in Q (see [7]), we obtain:

$$\lim_{\varepsilon\to 0}A_1^\varepsilon=\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u^\varepsilon)\nabla T_k(v^\varepsilon)\,dxdt+\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}}\nabla g^\varepsilon\,dx\,dt \tag{4.58}$$
$$\le\int_Q a(x,t,\nabla u)\nabla T_k(v)\,dx\,dt+\int_Q a(x,t,\nabla u)\nabla g\chi_{\{|v|\le k\}}\,dx\,dt.$$

As a consequence of Proposition 4.2, we deduce that

$$\lim_{\varepsilon\to 0}A_2^\varepsilon=-\int_Q a(x,t,\nabla u)(\nabla T_k(v)+\nabla g)\,dx\,dt, \tag{4.59}$$

and

$$\lim_{\varepsilon\to 0}A_3^\varepsilon=-\lim_{\varepsilon\to 0}\int_Q a(x,t,\nabla u)\Big(\nabla T_k(v^\varepsilon)+\nabla g^\varepsilon)\chi_{\{|v^\varepsilon|\le k\}} \tag{4.60}$$
$$-b'(u^\varepsilon)b'(u)^{-1}\Big(\nabla T_k(v)+\nabla g\chi_{\{|v|\le k\}}\Big)\Big)\,dx\,dt=0.$$

Therefore collecting (4.58), (4.59) and (4.60) yield (4.57). Through the monotonicity argument which relies on (3.6) (see [12], Lemma 5), we can deduce from (4.56) and Proposition 4.3, that $a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\chi_{\{|v^\varepsilon|\le k\}}$ converges to $a(x,t,\nabla u)\nabla u\chi_{\{|v|\le k\}}$ weakly in $L^1(Q)$, by coercivity argument we have that $|\nabla u^\varepsilon|^p\chi_{\{|v^\varepsilon|\le k\}}$ is equi-integrable, as a consequence of Vitali's theorem and since g^ε strongly converges in $L^p(0,T;W_0^{1,p}(\Omega))$ yields

$$T_k(v^\varepsilon)\to T_k(v)\text{ strongly in }L^p(0,T;W_0^{1,p}(\Omega)),$$

the proof of Theorem 4.7 is complete. □

⋆ **Equi − integrability of the nonlinearity sequence**. We shall prove now that $h(u^\varepsilon)|\nabla u^\varepsilon|^p$ converge strongly to $h(u)|\nabla u|^p$ in $L^1(Q)$, by Proposition 4.3 we have $h(u^\varepsilon)|\nabla u^\varepsilon|^p$ converge to $h(u)|\nabla u|^p$ a.e. in Q and (4.18) implies that

$$\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p|\chi_{\{|u^\varepsilon|>k\}}\,dxdt = \omega(k). \tag{4.61}$$

Let E be a subset of Q we have

$$\int_E h(u^\varepsilon)|\nabla u^\varepsilon|^p\,dxdt = \int_{E\cap\{|v^\varepsilon|<k+1,|u^\varepsilon|<k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p\,dxdt + \int_{E\cap\{|v^\varepsilon|<k+1,|u^\varepsilon|>k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p\,dxdt$$
$$+\int_{E\cap\{|v^\varepsilon|>k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p\Phi_{\delta,\eta}\,dxdt,$$
$$+\int_{E\cap\{|v^\varepsilon|>k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p(1-\Phi_{\delta,\eta})\,dxdt.$$

Let $\beta_k(s) = B_{k,k+1}(s^+)$, we can choose $\beta_k(v^\varepsilon)\psi_\eta^-$ as test function in (4.4) and rearranging conveniently all terms we obtain

$$\int_{\{v^\varepsilon>k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p\psi_\eta^-\,dxdt + \int_{\{k\le v^\varepsilon<k+1\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon\psi_\eta^-\,dx\,dt + \int_Q \beta_k(v^\varepsilon)\psi_\eta^-\,d\lambda_-^\varepsilon$$
$$\le C\int_{\{v^\varepsilon>k\}}\left(|\nabla g^\varepsilon|^p + |L|^{p'} + |G^\varepsilon|^{p'} + |f^\varepsilon|\right)dx\,dt$$
$$+\int_Q \overline{\beta_k}(v^\varepsilon)\frac{d\psi_\eta^-}{dt}\,dx\,dt - \int_Q a(x,t,\nabla u^\varepsilon).\nabla\psi_\eta^-\beta_k(v^\varepsilon)\,dx\,dt$$
$$+\int_Q G^\varepsilon.\nabla\psi_\eta^-\beta_k(v^\varepsilon)\,dx\,dt + \int_Q \beta_k(v^\varepsilon)\psi_\eta^-\,d\lambda_+^\varepsilon.$$

By following the same proof as in the Lemma 4.8 we obtain

$$\int_{\{v^\varepsilon>k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p\psi_\eta^-\,dxdt = \omega(\varepsilon,k,\eta).$$

Similarly as above we can choose $\beta_k(s) = B_{k,k+1}(s^-)$ and $\beta_k(v^\varepsilon)\psi_\eta^+$ as test function in (4.4) imply that

$$\int_{\{|v^\varepsilon|>k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p\Phi_{\delta,\eta}\,dxdt = \omega(\varepsilon,k,\eta). \tag{4.62}$$

By taking $T_1(v^\varepsilon - T_k(v^\varepsilon))^+(1-\Phi_{\delta,\eta})$ as test function in (4.4) we obtain

$$\int_Q \overline{\Theta_k}(v^\varepsilon)\frac{d\Phi_{\delta,\eta}}{dt}\,dx\,dt + \int_{\{v^\varepsilon>k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p(1-\Phi_{\delta,\eta})\,dxdt + \int_{\{k\le v^\varepsilon<k+1\}} b'(u^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla u^\varepsilon(1-\Phi_{\delta,\eta})\,d$$
$$\le C\int_{\{v^\varepsilon>k\}}\left(|\nabla g^\varepsilon|^p + |L|^{p'} + |G^\varepsilon|^{p'} + |f^\varepsilon|\right)dx\,dt - \int_Q G^\varepsilon.\nabla\psi_\eta^- T_1(v^\varepsilon - T_k(v^\varepsilon))^+\,dx\,dt$$
$$+\int_Q a(x,t,\nabla u^\varepsilon).\nabla\psi_\eta^- T_1(v^\varepsilon - T_k(v^\varepsilon))^+\,dx\,dt + \int_Q T_1(v^\varepsilon - T_k(v^\varepsilon))^+(1-\Phi_{\delta,\eta})\,d\lambda_+^\varepsilon$$

$$- \int_Q T_1(v^\varepsilon - T_k(v^\varepsilon))^+ (1 - \Phi_{\delta,\eta})\, d\lambda_-^\varepsilon$$

$$+ \int_Q \overline{\Theta_k}(v_0^\varepsilon)\, dx,$$

where $\overline{\Theta_k}(s) = \displaystyle\int_0^s T_1(r - T_k(r))^+\, dr$.

Then we obtain $\int_{\{v^\varepsilon > k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p (1-\Phi_{\delta,\eta})\, dxdt = \omega(\varepsilon,k,\delta,\eta)$, and similarly by using $T_1(v^\varepsilon - T_k(v^\varepsilon))^-$ as test function in (4.4)

$$\int_{\{|v^\varepsilon|>k+1\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p (1-\Phi_{\delta,\eta})\, dxdt = \omega(\varepsilon,k,\delta,\eta). \tag{4.63}$$

By assumption (3.2) and (3.7) we have

$$\int_{E\cap\{|v^\varepsilon|<k+1, |u^\varepsilon|<k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p\, dxdt \le \max_{\{|s|\le k\}} h(s) \int_{E\cap\{|v^\varepsilon|<k+1\}} |\nabla u^\varepsilon|^p\, dxdt$$

$$\le \max_{\{|s|\le k\}} h(s) \frac{1}{\gamma}\Big(\int_E |\nabla T_k(v^\varepsilon)|^p\, dxdt + \int_E |\nabla g^\varepsilon|^p\, dxdt\Big)$$

and

$$\int_{E\cap\{|v^\varepsilon|<k+1, |u^\varepsilon|>k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p\, dxdt \le \int_{E\cap\{|u^\varepsilon|>k\}} h(u^\varepsilon)|\nabla u^\varepsilon|^p\, dxdt$$

Then(4.18), (4.61), (4.62), (4.63) and thanks to Theorem 4.7 we deduce that $h(u^\varepsilon)|\nabla u^\varepsilon|^p$ is equi-integrable and by Vitali's Theorem we deduce that

$$h(u^\varepsilon)|\nabla u^\varepsilon|^p \to h(u)|\nabla u|^p \text{ in } L^1(Q). \tag{4.64}$$

Proof. (*Proof of Theorem 4.1*). Now we are able to prove that Problem (1.1)-(1.3) has a renormalized solutions.

Let S in $W^{2,\infty}(\mathbb{R})$, such that S' has a compact support as in Definition 3.1, and let $\varphi \in C_c^\infty(Q)$, then the approximating solutions u^ε (and v^ε) satisfy

$$\begin{aligned} -\int_0^T \langle \varphi_t, S(v^\varepsilon)\rangle\, dt + \int_Q S'(v^\varepsilon) a(x,t,\nabla u^\varepsilon)\nabla\varphi\, dxdt + \int_Q S''(v^\varepsilon) a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon \varphi\, dxdt \\ + \int_Q S'(v^\varepsilon) h(u^\varepsilon)|\nabla u^\varepsilon|^p \varphi\, dxdt \\ = \int_Q f^\varepsilon S'(v^\varepsilon)\varphi\, dxdt + \int_Q G^\varepsilon S'(v^\varepsilon)\nabla\varphi\, dxdt + \int_Q S''(v^\varepsilon) G^\varepsilon \nabla v^\varepsilon \varphi\, dxdt \\ + \int_Q S'(v^\varepsilon)\varphi\, d\lambda_+^\varepsilon - \int_Q S'(v^\varepsilon)\varphi\, d\lambda_-^\varepsilon. \end{aligned} \tag{4.65}$$

Thanks to Theorem 4.7 and (4.64), all terms in (4.65) easily pass to the limit on ε except the last two terms that give some problem. We can write following the arguments in [34]

$$\int_Q S'(v^\varepsilon)\varphi\, d\lambda_+^\varepsilon = \int_Q S'(v^\varepsilon)\varphi\psi_\delta^+\, d\lambda_+^\varepsilon + \int_Q S'(v^\varepsilon)\varphi(1-\psi_\delta^+)\, d\lambda_+^\varepsilon. \tag{4.66}$$

Let ψ_δ^+ be defined as in Lemma 4.6, then we have

$$\Big|\int_Q S'(v^\varepsilon)\varphi(1-\psi_\delta^+)\,d\lambda_+^\varepsilon\Big| \le C\int_Q (1-\psi_\delta^+)\,d\lambda_+^\varepsilon = \omega(\varepsilon,\delta),$$

while choosing $S'(v^\varepsilon)\varphi\psi_\delta^+$ in (4.4) one gets,

$$\begin{aligned}\int_Q S'(v^\varepsilon)\varphi\psi_\delta^+\,d\lambda_+^\varepsilon &= -\int_Q f^\varepsilon S'(v^\varepsilon)\varphi\psi_\delta^+\,dxdt - \int_Q G^\varepsilon S'(v^\varepsilon)\nabla(\varphi\psi_\delta^+)\,dxdt \quad (4.67)\\ &- \int_Q G^\varepsilon S''(v^\varepsilon)\nabla v^\varepsilon\varphi\psi_\delta^+\,dxdt + \int_Q S'(v^\varepsilon)\varphi\psi_\delta^+\,d\lambda_-^\varepsilon - \int_Q S(v^\varepsilon)(\varphi\psi_\delta^+)_t\,dxdt\\ &+ \int_Q S'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\varphi)\,dxdt + \int_Q S''(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\psi_\delta^+\varphi\,dxdt.\\ &+ \int_Q S'(v^\varepsilon)h(u^\varepsilon)|\nabla u^\varepsilon|^p\psi_\delta^+\varphi\,dxdt\end{aligned}$$

Now, thanks to Proposition 4.2 and the properties of ψ_δ^+, we have

$$\int_Q f^\varepsilon S'(v^\varepsilon)\varphi\psi_\delta^+\,dxdt = \omega(\varepsilon,\delta) \text{ and } \int_Q G^\varepsilon S'(v^\varepsilon)\nabla(\varphi\psi_\delta^+)\,dxdt = \omega(\varepsilon,\delta).$$

By Lemma 4.6, we deduce

$$\Big|\int_Q S'(v^\varepsilon)\varphi\psi_\delta^+\,d\lambda_-^\varepsilon\Big| \le C\int_Q \psi_\delta^+\,d\lambda_-^\varepsilon = \omega(\varepsilon,\delta).$$

Again by Lemma 4.4, and since $S(v) \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$,

$$\int_Q S(v^\varepsilon)(\varphi\psi_\delta^+)_t\,dxdt = \omega(\varepsilon,\delta).$$

By Theorem 4.7 and Lemma 4.6, we have

$$\int_Q S'(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\psi_\delta^+\varphi)\,dxdt = \omega(\varepsilon,\delta),$$

and

$$\int_Q S''(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\psi_\delta^+\varphi\,dxdt = \omega(\varepsilon,\delta).$$

and again by Lemma 4.6 and (4.64) we obtain

$$\int_Q S'(v^\varepsilon)h(u^\varepsilon)|\nabla u^\varepsilon|^p\psi_\delta^+\varphi\,dxdt = \omega(\varepsilon,\delta)$$

Therefore, from (4.66) we deduce

$$\int_Q S'(v^\varepsilon)\varphi\,d\lambda_+^\varepsilon = \omega(\varepsilon). \tag{4.68}$$

Similarly, we can prove that

$$\int_Q S'(v^\varepsilon)\varphi\,d\lambda_-^\varepsilon = \omega(\varepsilon). \tag{4.69}$$

As a consequence of the above convergence results, we are in a position to pass to the limit as ε tends to 0 in (4.65) and to conclude that u satisfies (3.11).

It remains to show that $S(v)$ satisfies the initial condition (3.12). To this end, firstly remark that $S(v^\varepsilon)$ being bounded in $L^\infty(Q)$, secondly, (4.65) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial S(v^\varepsilon)}{\partial t}$ is bounded in $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega))$.

As a consequence, an Aubin's type lemma (see e.g., [41], Corollary 4) implies that $S(v^\varepsilon)$ lies in a compact set of $\mathcal{C}([0,T];W^{-1,s}(\Omega))$ for any $s<\inf(p',\frac{N}{N-1})$. It follows that, on one hand, $S(v^\varepsilon)(t=0)$ converges to $S(v)(t=0)$ strongly in $W^{-1,s}(\Omega)$, On the other hand, the smoothness of S imply that $S(v^\varepsilon)(t=0)$ converges to $S(b(u))(t=0)$ strongly in $L^r(\Omega)$ for all $r<\infty$. Due to (4.2), we conclude that $S(v^\varepsilon)(t=0)=S(b(u_0^\varepsilon))$ converges to $S(b(u)(t=0)$ strongly in $L^r(\Omega)$. Then v satisfies (3.12).

Now choosing $\beta_n(v^\varepsilon)$ as test function in (4.4) where $\varphi\in C_c^\infty(Q)$, we obtain

$$-\int_0^T\langle\varphi_t,\overline{\beta_n}(v^\varepsilon)\rangle\,dt+\int_Q\beta_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla\varphi\,dxdt+\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}}a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\varphi\,dxdt \tag{4.70}$$
$$+\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\beta_n(v^\varepsilon)\varphi\,dxdt$$
$$=\int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi\,dxdt-\int_0^T<\operatorname{div}(G^\varepsilon),\beta_n(v^\varepsilon)\varphi>\,dxdt+\int_Q\beta_n(v^\varepsilon)\varphi\,d\lambda_+^\varepsilon-\int_Q\beta_n(v^\varepsilon)\varphi\,d\lambda_-^\varepsilon.$$

Reasoning as before (in particular as in the proof of Lemma 4.8) we obtain

$$\int_0^T\langle\varphi_t,\overline{\beta_n}(v^\varepsilon)\rangle\,dt=\omega(\varepsilon,n),\qquad\int_Q\beta_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon).\nabla\varphi\,dxdt=\omega(\varepsilon,n),$$
$$\int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi\,dxdt=\omega(\varepsilon,n),\qquad\int_0^T<\operatorname{div}(G^\varepsilon),\beta_n(v^\varepsilon)\varphi>\,dxdt=\omega(\varepsilon,n),$$

thanks to Theorem 4.7 we have

$$\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}}a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\varphi\,dxdt=\frac{1}{n}\int_{\{n\le v<2n\}}a(x,t,\nabla u)\nabla v\;\varphi\,dxdt+\omega(\varepsilon),$$

and (4.64) gives

$$\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\beta_n(v^\varepsilon)\varphi\,dxdt=\omega(\varepsilon,n).$$

Now we deal with the two last terms in the right hand side of (4.70) we can write

$$\int_Q\beta_n(v^\varepsilon)\varphi\,d\lambda_+^\varepsilon=-\int_Q h_n(v^\varepsilon)\varphi\,d\lambda_+^\varepsilon+\int_Q\varphi\,d\lambda_+^\varepsilon,$$

where $h_n(s)=H_n(s^+)$. By construction of λ_+^ε we have

$$\int_Q\varphi\,d\lambda_+^\varepsilon=\int_Q\varphi\,d\mu_s^++\omega(\varepsilon).$$

Following the same argument as in (4.64) and (4.65) by taking $h_n(v^\varepsilon)=S'(v^\varepsilon)$ we obtain

$$\int_Q h_n(v^\varepsilon)\varphi\,d\lambda_+^\varepsilon=\omega(\varepsilon).$$

If we prove that

$$\int_Q \beta_n(v^\varepsilon)\varphi\, d\lambda^\varepsilon_- = \omega(\varepsilon), \tag{4.71}$$

then, we obtain for every $\varphi \in C_c^\infty(Q)$

$$\lim_{n\to\infty} \frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v\ \varphi\, dxdt = \int_Q \varphi\, d\mu_s^+ \tag{4.72}$$

We can write

$$\int_Q \beta_n(v^\varepsilon)\varphi\, d\lambda^\varepsilon_- = \int_Q \beta_n(v^\varepsilon)\varphi\psi_\delta^-\, d\lambda^\varepsilon_- + \int_Q \beta_n(v^\varepsilon)\varphi(1-\psi_\delta^-)\, d\lambda^\varepsilon_-,$$

by Lemma 4.6, we obtain

$$\int_Q \beta_n(v^\varepsilon)\varphi(1-\psi_\delta^-)\, d\lambda^\varepsilon_- = \omega(\varepsilon,\delta).$$

Choosing $\beta_n(v^\varepsilon)\varphi\psi_\delta^-$ as a test function in the formulation of u^ε

$$\begin{aligned}\int_Q \beta_n(v^\varepsilon)\varphi\psi_\delta^-\, d\lambda^\varepsilon_- &= \int_0^T \langle(\varphi\psi_\delta^-)_t, \overline{\beta_n}(v^\varepsilon)\rangle\, dt - \int_Q \beta_n(v^\varepsilon)a(x,t,\nabla u^\varepsilon)\nabla(\varphi\psi_\delta^-)\, dxdt\\ &\quad -\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\varphi\psi_\delta^-\, dxdt\\ -\int_Q h(u^\varepsilon)|\nabla u^\varepsilon|^p\beta_n(v^\varepsilon)\psi_\delta^-\varphi\, dxdt &+ \int_Q f^\varepsilon\beta_n(v^\varepsilon)\varphi\psi_\delta^-\, dxdt + \int_Q G^\varepsilon\beta_n(v^\varepsilon)\nabla(\varphi\psi_\delta^-)\, dxdt\\ &\quad +\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} G^\varepsilon\nabla v^\varepsilon\varphi\psi_\delta^-\, dxdt + \int_Q \beta_n(v^\varepsilon)\varphi\psi_\delta^-\, d\lambda^\varepsilon_+.\end{aligned}$$

Using again Proposition 2.2, Proposition 4.2, Lemma 4.6, Lemma 4.8 and (4.64) yields (4.71), and therefore we obtain (4.72) for every $\varphi \in C_c^\infty(Q)$. Now if $\varphi \in C^\infty(\overline{Q})$, we can split

$$\begin{aligned}\frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v\ \varphi\, dxdt &= \frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v\ \varphi\psi_\delta^+\, dxdt\\ &\quad +\frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v\ \varphi(1-\psi_\delta^+)\, dxdt,.\end{aligned} \tag{4.73}$$

Thanks to (4.72), we have

$$\lim_{n\to\infty} \frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v\ \varphi\psi_\delta^+\, dxdt = \int_Q \varphi\, d\mu_s^+ + \omega(\delta),$$

By Lemma 4.8, we obtain

$$\frac{1}{n}\int_{\{n\le v^\varepsilon<2n\}} a(x,t,\nabla u^\varepsilon)\nabla v^\varepsilon\ \varphi(1-\psi_\delta^+)\, dxdt = \omega(\varepsilon,n,\delta).$$

Thanks to Theorem 4.7, we deduce

$$\frac{1}{n}\int_{\{n\le v<2n\}} a(x,t,\nabla u)\nabla v\ \varphi(1-\psi_\delta^+)\, dxdt = \omega(n,\delta).$$

Putting together all these facts above, from (4.73) we get (3.13) for every $\varphi \in C^\infty(\overline{Q})$, and by density argument (3.13) holds for every $\varphi \in C(\overline{Q})$. To obtain (3.14) we can reason as before using ψ_δ^+ in the place of ψ_δ^- and viceversa, and this conclude the proof of Theorem 4.1. □

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