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SOLVABILITY OF NONLINEAR ANISOTROPIC ELLIPTIC UNILATERAL PROBLEMS WITH VARIABLE EXPONENT

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Abstract. In this paper, we study in the anisotropic Sobolev spaces with variable exponent $W_0^{1,\vec{p}(\cdot)}(\Omega)$, the existence result of unilateral problem associated to the equations of the form,

$$-\sum_{i=1}^{N} \partial_i a_i(x, u, \nabla u) = f,$$

where the right hand side f belongs to $L^1(\Omega)$.

1. Introduction

We consider in this paper the following nonlinear anisotropic elliptic Dirichlet problem

$$\begin{cases} Au = f \text{ in } \Omega \\ u = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Where Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 2)$, $p_i(.) \in C_+(\overline{\Omega})$, A is a Leray-Lions operator defined from the anisotropic Sobolev space with exponent variable $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $W^{-1,\vec{p}'(\cdot)}(\Omega)$ by

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u),$$

and f belongs to $L^1(\Omega)$.

In the anisotropic case, L. Boccardo et al. in [13] studied the problem (1.1) with

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)$$

and f is a bounded Radon measure on Ω. In the case where

$$Au = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right)$$

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and the right hand side $f=(f_1,...,f_m)^\top$ is vector-valued Radon measure on Ω of finite mass, existence solutions of (1.1) is proved by Bendhmane et al. in [9].

In [8] M.B. Benboubker et al have studied the nonlinear anisotropic elliptic equation of the type

$$-\text{div}a(x,u,\nabla u)+g(x,u,\nabla u)+|u|^{p_0(x)-2}u=f-\text{div}\phi(u).$$

Our aim in this paper is to prove the existence of entropy solution of the unilateral problem associated to problem (1.1), more precisely, this work deals with the existence of entropy solution of problem

$$\begin{cases} u\geq\psi \ a.e. \text{ in } \Omega. \\ T_k(u)\in W_0^{1,\vec{p}(\cdot)}(\Omega). \\ \displaystyle\sum_{i=1}^N\int_\Omega a_i(x,u,\nabla u)\partial_i T_k(u-v)\leq\int_\Omega fT_k(u-v), \\ \forall v\in K_\psi\cap L^\infty(\Omega) \text{ and } \forall k>0, \end{cases} \tag{1.2}$$

in anisotropic Sobolev space with variable exponent $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ with $f\in L^1(\Omega)$. This paper is organized as follows: section 2 is devoted to preliminaries. In section 3 we give some assumptions and Lemmas. The main existence results is stated and proved in section 4. In section 5, we prove that the operator of approximate problem is coercive and pseudo-monotone .

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N\geq 2)$ with Lipschitz continuous boundary and let $\vec{p}(\cdot)=(p_1(\cdot),...,p_N(\cdot))$ such that for any $i=1,...,N$, $p_i(.):\overline{\Omega}\to\mathbb{R}$ is a continuous function with $1<p_i^-:=ess\inf_{x\in\Omega}p_i(x)\leq ess\sup_{x\in\Omega}p_i(x):=p_i^+<+\infty$.
We denote by $p_M(x)=\max\{p_1(x),...,p_N(x)\}$, $p_m(x)=\min\{p_1(x),...,p_N(x)\}$ and $\partial_i=\dfrac{\partial}{\partial x_i}$.
Let $C_+(\overline{\Omega})=\{p\in C(\overline{\Omega}), 1<p(x)<+\infty, \text{ for any x } \in\overline{\Omega}\}$. For any $p\in C_+(\overline{\Omega})$, the variable exponent Lebesque Space is defined by

$$L^{p(\cdot)}(\Omega)=\{u:\Omega\to\mathbb{R} \text{ measurable such that } \rho_{p(.)}(u)<+\infty\}$$

where $\rho_{p(.)}(u)=\displaystyle\int_\Omega|u|^{p(x)}dx$ is a modular function.
The expression

$$|u|_{p(\cdot)}:=\inf\{\lambda>0:\int_\Omega|\frac{u(x)}{\lambda}|^{p(x)}dx\leq 1\}$$

is a norm on $L^{p(\cdot)}(\Omega)$, called Luxemberg norm.
We recall the Hölder-inequality, for any $u\in L^{p(\cdot)}$ and $v\in L^{q(\cdot)}$ with $\frac{1}{p(x)}+\frac{1}{q(x)}=1$ for all $x\in\Omega$, we have

$$\Big|\int_\Omega uvdx\Big|\leq(\frac{1}{p^-}+\frac{1}{q^-})|u|_{p(\cdot)}|v|_{q(\cdot)}$$

The variable exponent anisotropic Sobolev Space is defined by :

$$W^{1,\vec{p}(\cdot)}(\Omega)=\Big\{u\in L^{p_M(\cdot)}(\Omega):\partial_i u\in L^{p_i(\cdot)}(\Omega), i=1,2,...,N\Big\}.$$

The space $W^{1,\vec{p}(\cdot)}(\Omega)$ is a separable and reflexive Banach space with respect to norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}=|u|_{p_M(\cdot)}+\sum_{i=1}^{N}|\partial_i u|_{p_i(\cdot)}. \tag{2.1}$$

We denote by $W_0^{1,\vec{p}(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.1) and we define

$$\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)=W^{1,\vec{p}(\cdot)}(\Omega)\cap W_0^{1,1}(\Omega).$$

Since Ω is a bounded open set with Lipschitz boundary $\partial\Omega$, then

$$\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)=\{u\in W^{1,\vec{p}(\cdot)}(\Omega):u=0 \text{ on } \partial\Omega\}.$$

Remark 2.1. In general, we have $W_0^{1,\vec{p}(\cdot)}(\Omega)\subset\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ and the smooth functions are not dense in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, but if for each $i=1,...,N$, p_i is Log-Hölder continuous, that is, there exists a positive constant C such that

$$|p_i(x)-p_i(y)|\leq\frac{C}{-log|x-y|}$$

for every x,y with $|x-y|\leq\frac{1}{2}$, then $C_0^\infty(\Omega)$ is dense in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, thus

$$W_0^{1,\vec{p}(\cdot)}(\Omega)=\mathring{W}^{1,\vec{p}(\cdot)}(\Omega).$$

3. Assumptions and Lemmas :

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N\geq 2)$ with Lipschitz continuous boundary $\partial\Omega$.

The functions a: $\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^{\mathbb{N}}$ such that $a(x,s,\xi)=\Big(a_1(x,s,\xi),...,a_N(x,s,\xi)\Big)$ with a_i is Carathéodory functions satisfying the following conditions, for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N,\xi'\in\mathbb{R}^N$ and a.e. in Ω :

$$\sum_{i=1}^{N}a_i(x,s,\xi)\xi_i\geq\alpha\sum_{i=1}^{N}|\xi_i|^{p_i(x)}, \tag{3.1}$$

$$|a_i(x,s,\xi)|\leq\beta[j_i(x)+|s|^{p_i(x)-1}+|\xi_i|^{p_i(x)-1}], \tag{3.2}$$

$$(a_i(x,s,\xi)-a_i(x,s,\xi'))(\xi_i-\xi_i')>0 \quad \text{for } \xi_i\neq\xi_i', \tag{3.3}$$

where α,β are some positive constants, j_i is a positive function in $L^{p_i(\cdot)'}(\Omega)$. Moreover, we suppose that

$$f\in L^1(\Omega). \tag{3.4}$$

Let us define the convex set

$$K_\psi=\{u\in W_0^{1,\vec{p}(\cdot)}(\Omega),\ u\geq\psi\ a.e. \text{ in } \Omega\}$$

where ψ is a measurable function with values in $\overline{\mathbb{R}}$ such that

$$\psi^+ \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega). \tag{3.5}$$

Lemma 3.1. [21] *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and $\vec{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. If $q \in C_+(\overline{\Omega})$ and for all $x \in \overline{\Omega}$, $q(x) < \max(p_M(x), \overline{p}^\star(x))$. Then the embedding*

$$\mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega),$$

is compact.
Where

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^N \frac{1}{p_i(x)}} \text{ and } \overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N, \\ +\infty, & \text{for } \overline{p}(x) \geq N. \end{cases}$$

Lemma 3.2. [21] *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, and $\vec{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x \in \overline{\Omega},\ p_M(x) < \overline{p}^\star(x). \tag{3.6}$$

Then the following Poincaré-type inequality holds

$$|u|_{p_M(\cdot)} \leq C \sum_{i=1}^N |D_i u|_{p_i(\cdot)}, \quad \forall u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega), \tag{3.7}$$

where C is a positive constant independent of u.

Thanks to (3.7), the following norm

$$\|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} = \sum_{i=1}^N |\partial_i u|_{p_i(\cdot)(\Omega)} \tag{3.8}$$

is an equivalent norm on $W_0^{1,\vec{p}(\cdot)}(\Omega)$.
Under the norm (3.8), the space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is separable and reflexive Banach space and its dual is denoted by $(W_0^{1,\vec{p}(\cdot)}(\Omega))'$ which equivalent to $W^{-1,\vec{p}'(\cdot)}(\Omega)$.

Moreover, we consider

$$\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega) = \{u \text{ measurable in } \Omega : T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega), \forall k > 0\}.$$

where

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k \,\text{sign}(s) & \text{if } |s| > k \end{cases}$$

Lemma 3.3. *If $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. Then there exists a unique measurable function $v_i : \Omega \to \mathbb{R}$ such that*

$$\partial_i T_k(u) = v_i \chi_{\{|u| \leq k\}} \quad a.e. \text{ in } \Omega, \tag{3.9}$$

where χ_A denotes the characteristic function of a measurable set A. Moreover, if $u \in W_0^{1,1}(\Omega)$ then v coincides with the standard distributional gradient of u, that is $\partial_i u = v_i$.

Lemma 3.4. [5, 17] *If $u_n, u \in L^{p(\cdot)}(\Omega)$ and $p^+ < +\infty$, then the following properties hold:*

- $|u|_{p(\cdot)} < 1$ (resp. $= 1, > 1$) $\Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ (resp. $= 1, > 1$),
- $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq |u|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right)$,
- $\min\left(|u|^{p^-}_{p(\cdot)}, |u|^{p^+}_{p(\cdot)}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(|u|^{p^-}_{p(\cdot)}, |u|^{p^+}_{p(\cdot)}\right)$,
- $|u|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$,
- $|u_n - u|_{p(\cdot)} \to 0 \Longleftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0$.

Lemma 3.5. [7] *Let $g \in L^{r(\cdot)}(\Omega)$ and let $g_n \in L^{r(\cdot)}(\Omega)$, $\|g_n\|_{L^{r(\cdot)}(\Omega)} < c$, $1 < r < +\infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{r(\cdot)}(\Omega)$.*

Lemma 3.6. *Assume that (3.1)-(3.3) hold and let $(u_n)_n$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and*

$$\lim_{n\to+\infty} \int_\Omega \Big(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\Big)\nabla(u_n - u) = 0. \tag{3.10}$$

Then $u_n \to u$ strongly in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ for a subsequence.

Proof: Let $D_n = \Big[a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\Big]\nabla(u_n - u)$, by (3.3), D_n is a positive function and by (3.10) $D_n \to 0$ in $L^1(\Omega)$ as $n \to +\infty$. Since $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, using Lemma (3.2), we obtain $u_n \to u$ in $L^{p_m(\cdot)}(\Omega)$. Then $u_n \to u$ a.e. in Ω and $D_n \to 0$ a.e. in Ω for a subsequence. Then there exists a subset B of Ω, of zero measure, such that for $x \in \Omega \setminus B$, $u(x) < +\infty$, $|\nabla u(x)| < +\infty$, $|j_i(x)| < +\infty$, $u_n(x) \to u(x)$ and $D_n(x) \to 0$. We have

$$\begin{aligned}
D_n(x) &= \sum_{i=1}^{N} \Big[a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u)\Big][\partial_i u_n - \partial_i u] \\
&= \sum_{i=1}^{N} \Big[a_i(x, u_n, \nabla u_n)\partial_i u_n + a_i(x, u_n, \nabla u)\partial_i u - a_i(x, u_n, \nabla u_n)\partial_i u - a_i(x, u_n, \nabla u)\partial_i u_n\Big] \\
&\geq \alpha \sum_{i=1}^{N} |\partial_i u_n|^{p_i(x)} + \alpha \sum_{i=1}^{N} |\partial_i u|^{p_i(x)} - \beta \sum_{i=1}^{N} \Big[j_i(x) + |u_n|^{p_i(x)-1} + |\partial_i u_n|^{p_i(x)-1}\Big]|\partial_i u| \\
&\qquad - \beta \sum_{i=1}^{N} \Big[j_i(x) + |u_n|^{p_i(x)-1} + |\partial_i u|^{p_i(x)-1}\Big]|\partial_i u_n|. \\
&\geq \alpha \sum_{i=1}^{N} |\partial_i u_n|^{p_i(x)} - c(x)\Big[1 + \sum_{i=1}^{N} |\partial_i u_n|^{p_i(x)-1} + \sum_{i=1}^{N} |\partial_i u_n|\Big]. \\
&\geq \sum_{i=1}^{N} |\partial_i u_n|^{p_i(x)}\Big[\alpha - \frac{c(x)}{N|\partial_i u_n|^{p_i(x)}} - \frac{c(x)}{N|\partial_i u_n|} - \frac{c(x)}{N|\partial_i u_n|^{p_i(x)-1}}\Big],
\end{aligned}$$

where $c(x)$ a constant depend on x and not depend on n.
Since $D_n(x) \to 0$ a.e. in Ω, the last inequality implies that $(\partial_i u_n)_n$ is bounded uniformly a.e. in Ω for $i = 1, ..., N$. Letting ξ_i^* be an accumulation point of $(\partial_i u_n)_n$ for $i = 1, ..., N$, we have $|\xi_i^*| < +\infty$ and by the continuity of $a_i(x, ., .)$, we obtain

$$\Big(a_i(x, u, \xi^*) - a_i(x, u, \nabla u)\Big)\Big(\xi_i^* - \partial_i u\Big) = 0.$$

Using (3.3), we obtain $\xi_i^* = \partial_i u$ for $i = 1, ..., N$. The uniqueness of the accumulation point implies that $\nabla u_n \to \nabla u$ a. e. in Ω. Since the sequence $\Big(a_i(x, u_n, \nabla u_n)\Big)_n$ is bounded in $L^{p_i(\cdot)'}(\Omega)$ and $a_i(x, u_n, \nabla u_n) \to a_i(x, u, \nabla u)$ a.e. in Ω, Lemma (3.5) implies that $a_i(x, u_n, \nabla u_n) \rightharpoonup a_i(x, u, \nabla u)$ weakly in $L^{p_i(\cdot)'}(\Omega)$ and a.e. in Ω.
As in [15], we have

$$a_i(x, u_n(x), \nabla u_n(x))\partial_i u_n \rightharpoonup a_i(x, u, \nabla u)\partial_i u \text{ weakly in } L^1(\Omega).$$

We denote $y_n^i = \frac{1}{\alpha} a_i(x, u_n, \nabla u_n)\partial_i u_n$ and $y^i = \frac{1}{\alpha} a_i(x, u, \nabla u)\partial_i u$, using Fatou's lemma, we get

$$\int_\Omega 2y^i dx \le \liminf_{n\to+\infty} \int_\Omega \Big(y_n^i + y^i - \frac{1}{2^{p_i-1}} |\partial_i u_n - \partial_i u|^{p_i(x)}\Big) dx.$$

Then, we have $0 \le -\limsup_{n\to+\infty} \int_\Omega |\partial_i u_n - \partial_i u|^{p_i(x)} dx$.
Thus, since $0 \le \liminf_{n\to+\infty} \int_\Omega |\partial_i u_n - \partial_i u|^{p_i(x)} dx \le \limsup_{n\to+\infty} \int_\Omega |\partial_i u_n - \partial_i u|^{p_i(x)} dx \le 0$.
We deduce $\int_\Omega |\partial_i u_n - \partial_i u|^{p_i(x)} dx \to 0$ as $n \to +\infty$.
Then, we obtain $\partial_i u_n \to \partial_i u$ in $L^{p_i(\cdot)}(\Omega)$ for $i = 1, ..., N$.
Consequently, we conclude that $u_n \to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. The proof is complete.

4. Main results

Definition 4.1. A function $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $u \ge \psi$ a.e. in Ω is an entropy solution of the problem (1.2) if

$$\sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)\partial_i T_k(u - \varphi) \le \int_\Omega f T_k(u - \varphi)$$

for all $\varphi \in K_\psi(\Omega) \cap L^\infty(\Omega)$.

Theorem 4.2. *Assume that (3.1)-(3.4) hold. Then there exists at least an entropy solution of problem (1.2).*

Proof of theorem 4.2
Approximate problems.
We consider the following approximate problems

$$\begin{cases} u \in K_\psi. \\ \displaystyle\sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)\partial_i(u - v) \le \int_\Omega f_n(u - v), \\ \forall v \in K_\psi \text{ and } \forall k > 0, \end{cases} \tag{4.1}$$

where $f_n(x) = \frac{f(x)}{1 + \frac{1}{n}|f(x)|}$.
We have $|f_n(x)| \le |f(x)|$ and $|f_n(x)| \le n$.

Lemma 4.3. *We consider the operator* $B : K_\psi \to W^{-1,\vec{p'}(.)}(\Omega)$ *defined by*
$< Bu, v >= \int_\Omega a(x, u, \nabla u)\nabla v dx.$
The operator B *is coercive and pseudo-monotone .*

For the proof of Lemma 4.3, (see Appendix).

Proposition 4.4. *Under the conditions (3.1)-(3.4), there exists at least one solution of the problem (4.1).*

Proof of proposition 4.4
Thank's to Lemma 4.3 and theorem 8.2 chapiter 2 in [25] , there exists at least one solution of the problem (4.1).
A priori estimate.

Proposition 4.5. *Assume that (3.1)- (3.5) hold and if* u_n *is a solution of the approximate problem (4.1). Then there exists a constant* C *such that*

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n)|^{p_i(x)} \le Ck \quad \forall k > 0.$$

Proof: Let $v = u_n - \eta T_k(u_n^+ - \psi^+)$ where $\eta \ge 0$. Since $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ and for all η small enough, we have $v \in K_\psi$. We take v as test function in problem (4.1), we have

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n)\partial_i T_k(u_n^+ - \psi^+)dx \le \int_\Omega f_n T_k(u_n^+ - \psi^+)$$

That implies

$$\sum_{i=1}^{N} \int_\Omega a_i(x, u_n, \nabla u_n)\partial_i T_k(u_n^+ - \psi^+) \le \int_\Omega f_n T_k(u_n^+ - \psi^+)$$

$$\le \|f\|_{L^1(\Omega)} k \le Ck. \tag{4.2}$$

Thus, we have

$$\sum_{i=1}^{N} \int_{\{|u_n^+ - \psi^+| \le k\}} a_i(x, u_n, \nabla u_n)\partial_i u_n^+ \le \sum_{i=1}^{N} \int_{\{|u_n^+ - \psi^+| \le k\}} a_i(x, u_n, \nabla u_n)\partial_i \psi^+ + Ck.$$

By (3.1) and Young's inequality, we have

$$\sum_{i=1}^{N} \int_{\{|u_n^+ - \psi^+| \le k\}} |\partial_i u_n^+|^{p_i(x)} dx \le Ck. \tag{4.3}$$

Since $\{x \in \Omega,\ |u^+| \le k\} \subset \{x \in \Omega,\ |u^+ - \psi^+| \le k + \|\psi^+\|_\infty\}$, then

$$\sum_{i=1}^{N} \int_\Omega |\partial_i T_k(u_n^+)|^{p_i(x)} dx = \sum_{i=1}^{N} \int_{\{|u^+| \le k\}} |\partial_i u_n^+|^{p_i(x)} dx$$

$$\leq \sum_{i=1}^{N} \int_{\{|u^+ - \psi^+| \leq k + \|\psi^+\|_\infty\}} |\partial_i u_n^+|^{p_i(x)} dx,$$

which implies, by (4.3), that

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i T_k(u_n^+)|^{p_i(x)} dx \leq Ck, \ \forall k > 0. \tag{4.4}$$

Similarly, taking $v = u_n + T_k(u_n^-)$ as test function in approximate problem (4.1), we obtain

$$-\sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^-) \leq -\int_{\Omega} f_n \exp(-G(u_n)) T_k(u_n^-).$$

Thus, we have

$$-\sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^-) \leq -\int_{\Omega} f_n T_k(u_n^-), \tag{4.5}$$

since $f \in L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$, we have

$$\sum_{i=1}^{N} \int_{\{u_n \leq 0\}} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n) \leq Ck$$

. By (3.1), we deduce that

$$\sum_{i=1}^{N} \int_{\{u_n \leq 0\}} |\partial_i T_k(u_n)|^{p_i(x)} \leq Ck. \tag{4.6}$$

Combining (4.4) and (4.6), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i T_k(u_n)|^{p_i(x)} \leq Ck.$$

It yields, by Lemma 3.4,

$$\|T_k(u_n)\|^{\gamma}_{W_0^{1,\vec{p}(\cdot)}(\Omega)} \leq Ck + N \leq C'k, \forall k > 1 \tag{4.7}$$

where

$$\gamma = \begin{cases} p^- & \text{if } |\partial_i T_k(u_n)|_{p_i(\cdot)} \geq 1 \\ p^+ & \text{if } |\partial_i T_k(u_n)|_{p_i(\cdot)} < 1 \end{cases}$$

Strong convergence of truncations:

Lemma 4.6. *There exists a measurable function u and a subsequence of u_n such that*

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Proof:
We will prove that $(u_n)_n$ is a Cauchy sequence in measure in Ω. For all $\lambda > 0$, we have

$$meas\{|u_n - u_m| > \lambda\}$$

$$\leq meas\{|u_n| > k\} + meas\{|u_m| > k\} + meas\{|T_k(u_n) - T_k(u_m)| > \lambda\}. \quad (4.8)$$

By Hölder's inequality and (4.7), we have

$$\begin{aligned} k meas\{|u_n| > k\} = \int_{\{|u_n|>k\}} |T_k(u_n)|dx &\leq \int_\Omega |T_k(u_n)|dx \\ &\leq (\frac{1}{(p_M^-)'} + \frac{1}{p_M^-})|1|_{p'_M(\cdot)}|T_k(u_n)|_{p_M(\cdot)} \\ &\leq 2(meas(\Omega))\|T_k(u_n)\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} \\ &\leq 2Ck^{\frac{1}{\gamma}}. \end{aligned}$$

Then $meas\{|u_n| > k\} \leq 2\frac{1}{k^{1-\frac{1}{\gamma}}} \to 0$ as $k \to +\infty$. Which implies that, for all $\varepsilon > 0$, there exists k_0 $\forall k > k_0$, we have

$$meas\{|u_n| > k\} \leq \frac{\varepsilon}{3} \text{ and } meas\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \quad (4.9)$$

On the other hand, since the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, there exists a subsequence $(T_k(u_n))_n$ such that $T_k(u_n)$ converges to η_k a.e. in Ω, weakly in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and strongly in $L^{p_m(\cdot)}(\Omega)$ as n tends to $+\infty$. Then the sequence $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω, thus $\forall\lambda > 0$, there exists n_0 such that

$$meas\{|T_k(u_n) - T_k(u_m)| > \lambda\} \leq \frac{\varepsilon}{3}, \quad \forall n, m \geq n_0. \quad (4.10)$$

Combining (4.8), (4.9) and (4.10), then for all $\lambda > 0$ and for all $\varepsilon > 0$, we have

$$meas\{|u_n - u_m| > \lambda\} \leq \varepsilon \quad \forall n, m \geq n_0.$$

Then $(u_n)_n$ is a Cauchy sequence in measure in Ω, then there exists a subsequence denoted by $(u_n)_n$ such that u_n converges to a measurable function u a.e. in Ω and

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ and a. e. in } \Omega \quad \forall k > 0. \quad (4.11)$$

Now , we prove that

$$\lim_{n\to\infty} \sum_{i=1}^N \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)\Big(\partial_i T_k(u_n) - \partial_i T_k(u)\Big) = 0$$

Let us take $v = u_n + T_1(u_n - T_m(u_n))^-$ in approximate problem (4.1), we obtain

$$-\sum_{i=1}^N \int_\Omega a_i(x, u_n, \nabla u_n)\partial_i T_1(u_n - T_m(u_n))^- \leq -\int_\Omega f_n T_1(u_n - T_m(u_n))^-$$

Then, we have

$$\sum_{i=1}^{N}\int_{\{-(m+1)\leq u_n\leq -m\}} a_i(x,u_n,\nabla u_n)\partial_i u_n \leq -\int_\Omega f_n T_1(u_n - T_m(u_n))^-. \quad (4.12)$$

By Lebesgue's theorem, we have
$\lim_{m\to\infty}\lim_{n\to\infty}\int_\Omega f_n T_1(u_n - T_m(u_n))^- = \lim_{m\to\infty}\int_\Omega f T_1(u - T_m(u))^- = 0.$
We deduce, from (4.12), that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup\sum_{i=1}^{N}\int_{\{-(m+1)\leq u_n\leq -m\}} a_i(x,u_n,\nabla u_n)\partial_i u_n = 0. \quad (4.13)$$

Similarly, taking $v = u_n - \eta T_1(u_n - T_m(u_n))^+$ as test function in approximate problem (4.1), we get

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup\sum_{i=1}^{N}\int_{\{m\leq u_n\leq m+1\}} a_i(x,u_n,\nabla u_n)\partial_i u_n = 0. \quad (4.14)$$

We consider the following function of one real variable:

$$h_m(s) = \begin{cases} 1 \quad \text{if } \ |s|\leq m \\ 0 \quad \text{if } \ |s|\geq m+1 \\ j+1-|s| \quad \text{if } \ m\leq |s|\leq m+1 \end{cases}$$

where $m > k$.
Let $\varphi = u_n - \eta(T_k(u_n - T_k(u))^+ h_m(u_n)$ as test function in approximating problem (4.1), we obtain

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_i(T_k(u_n)-T_k(u))^+ h_m(u_n)$$
$$+\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)(T_k(u_n)-T_k(u))^+\partial_i u_n h'_m(u_n)$$
$$\leq \int_\Omega f_n \exp(G(u_n))(T_k(u_n)-T_k(u))^+ h_m(u_n).$$

Which implies that

$$\sum_{i=1}^{N}\int_\Omega a_i(x,u_n,\nabla u_n)\partial_i(T_k(u_n)-T_k(u))^+ h_m(u_n)$$
$$\leq \sum_{i=1}^{N}\int_{\{m\leq u_n\leq m+1\}} a_i(x,u_n,\nabla u_n)(T_k(u_n)-T_k(u))^+\partial_i u_n$$
$$+\int_\Omega f_n (T_k(u_n)-T_k(u))^+ h_m(u_n)$$

Thanks to Lebesgue's theorem, we have
$\lim_{m\to\infty}\lim_{n\to\infty}\int_\Omega f_n(x)T_1(u_n-T_m(u_n))^+h_m(u_n)=0$. Then

$$\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla u_n)\partial_i(T_k(u_n)-T_k(u))^+h_m(u_n)$$

$$\le\sum_{i=1}^N\int_{\{m\le u_n\le m+1\}}a_i(x,u_n,\nabla u_n)(T_k(u_n)-T_k(u))^+\partial_iu_n+\varepsilon(n,m),$$

using (4.14), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N\int_{\{m\le u_n\le m+1\}}a_i(x,u_n,\nabla u_n)(T_k(u_n)-T_k(u))^+\partial_iu_n=0.$$

It yields

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,u_n,\nabla u_n)\partial_i(T_k(u_n)-T_k(u))^+h_m(u_n)\le 0,$$

which implies that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N\int_{\{T_k(u_n)-T_k(u)\ge0,|u_n|\le k\}}a_i(x,u_n,\nabla u_n)\partial_i(T_k(u_n)-T_k(u))h_m(u_n)$$

$$-\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N\int_{\{T_k(u_n)-T_k(u)\ge0,|u_n|>k\}}a_i(x,u_n,\nabla u_n)\partial_i(T_k(u))^+h_m(u_n)\le 0,$$

since $h_m(u_n)=0$ if $|u_n|>m+1$, we have

$$\sum_{i=1}^N\int_{\{T_k(u_n)-T_k(u)\ge0,|u_n|>k\}}a_i(x,u_n,\nabla u_n)\partial_i(T_k(u))^+h_m(u_n)$$

$$=\sum_{i=1}^N\int_{\{T_k(u_n)-T_k(u)\ge0,|u_n|>k\}}a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\partial_i(T_k(u))^+h_m(u_n).$$

Moreover, $(a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))_n$ is bounded in $L^{p_i(\cdot)'}(\Omega)$, then
$a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\rightharpoonup X_m^i$ in $L^{p_i(\cdot)'}(\Omega)$. It yields

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^N\int_{\{T_k(u_n)-T_k(u)\ge0,|u_n|>k\}}a_i(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\partial_i(T_k(u))^+h_m(u_n)$$

$$=\lim_{m\to\infty}\sum_{i=1}^N\int_{\{|u|>k\}}X_m^i\partial_iT_k(u)h_m(u).$$

$$=0.$$

Using $a_i(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\to a_i(x,T_k(u),\nabla T_k(u))h_m(u)$ a.e. in Ω, the sequence
$\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\Big)_n$ is equi-integrable in $L^{p_i(\cdot)'}(\Omega)$ and Vitali's theorem implies that
$a_i(x,T_k(u_n),\nabla T_k(u_n))h_m(u_n)\to a_i(x,T_k(u),\nabla T_k(u))h_m(u)$ in $L^{p_i(\cdot)'}(\Omega)$. Since $\partial_iT_k(u_n)\rightharpoonup\partial_iT_k(u))$ weakly in $L^{p_i(\cdot)}(\Omega)$, we get

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a_i(x,T_k(u_n),\nabla T_k(u))\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0,$$ thus we conclude that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\geq 0\}}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0. \tag{4.15}$$

Similarly, we take $\varphi=u_n+(T_k(u_n)-T_k(u))^-h_m(u_n)$ as test function in approximating problem (4.1), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{T_k(u_n)-T_k(u)\leq 0\}}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0. \tag{4.16}$$

Combining (4.15) and (4.16), we deduce that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)$$
$$\partial_i(T_k(u_n)-T_k(u))h_m(u_n)=0. \tag{4.17}$$

Let $\varphi=u_n+T_k(u_n)^-(1-h_m(u_n))$ as test function in approximate problem (4.1). Using (3.2) , we get

$$\sum_{i=1}^{N}\int_{\{u_n\leq 0\}} a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n)(1-h_m(u_n))$$
$$\leq -\sum_{i=1}^{N}\int_{\{-(j+1)\leq u_n\leq -j\}} a_i(x,u_n,\nabla u_n)T_k(u_n)\partial_i u_n$$
$$-\sum_{i=1}^{N}\int_{\Omega} f_n(x)T_k(u_n)^-(1-h_m(u_n))$$

In view of (4.13) and Lebesgue's theorem, the second and the third integrals converge to zero as n and m tend to infinity. Then

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{u_n\leq 0\}} a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n)(1-h_m(u_n))=0. \tag{4.18}$$

On the other hand, we take $\varphi=u_n-\eta T_k(u_n^+-\psi^+)(1-h_m(u_n))$ as test function in approximating problem (4.1) and using (3.2), we get

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u_n,\nabla u_n)\partial_i T_k(u_n^+-\psi^+)(1-h_m(u_n))$$
$$\leq \sum_{i=1}^{N}\int_{\{-(j+1)\leq u_n\leq -j\}} a_i(x,u_n,\nabla u_n)T_k((u_n)^+-\psi^+)\partial_i u_n$$
$$+\sum_{i=1}^{N}\int_{\Omega} f_n(x)T_k(u_n^+-\psi^+)(1-h_m(u_n))$$
$$=\varepsilon_1(n,m).$$

By Lebesgue's theorem and (4.13) , we have $\varepsilon_1(n,m)$ converges to zero as n and m go to infinity. Then we deduce that

$$\sum_{i=1}^{N}\int_{\{|u_n^+-\psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i u_n^+(1-h_m(u_n))$$
$$\le \sum_{i=1}^{N}\int_{\{|u_n^+-\psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i \psi^+(1-h_m(u_n))+\varepsilon_1(n,m). \tag{4.19}$$

Thanks to (3.2) and Young's inequality, we have

$$\sum_{i=1}^{N}\int_{\{|u_n^+-\psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i u_n^+(1-h_m(u_n))\le \varepsilon_2(n,m)$$

where $\varepsilon_2(n,m)$ converges to zero as n and m tend to infinity. It yields

$$\sum_{i=1}^{N}\int_{\{|u_n^+-\psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i u_n^+(1-h_m(u_n))\le \varepsilon_3(n,m).$$

Since $\{x\in\Omega,\ |u_n^+|\le k\}\subset\{x\in\Omega,\ |u_n^+-\psi^+|\le k+\|\psi^+\|_{L^\infty(\Omega)}\}$, hence

$$\sum_{i=1}^{N}\int_{\{|u_n^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i u_n^+(1-h_m(u_n)).$$
$$\le \sum_{i=1}^{N}\int_{\{|u_n^+-\psi^+|\le k\}} a_i(x,u_n,\nabla u_n)\partial_i u_n^+(1-h_m(u_n))\le \varepsilon_3(n,m).$$

Which implies that , for all $k>0$

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\{u_n\ge 0\}} a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_i T_k(u_n)(1-h_m(u_n))=0 \tag{4.20}$$

Combining (4.18) and (4.20), we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_i T_k(u_n)(1-h_m(u_n))=0. \tag{4.21}$$

Moreover, we have

$$\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\partial_i T_k(u_n)-\partial_i T_k(u)\Big)$$
$$=\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\partial_i T_k(u_n)-\partial_i T_k(u)\Big)h_m(u_n)$$
$$+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_i T_k(u_n)(1-h_m(u_n))$$
$$-\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))\partial_i T_k(u)(1-h_m(u_n))$$

$$-\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(\partial_i T_k(u_n)-\partial_i T_k(u))(1-h_m(u_n)).$$

By (4.17) and (4.20), the first and the second integrals of the right hand side converge to zero as $n,m\to+\infty$.

Since $\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))\Big)_n$ is bounded in $L^{p_i(\cdot)'}(\Omega)$ and $\partial_i T_k(u)(1-h_m(u_n))$ converges to zero in $L^{p_i(\cdot)}(\Omega)$, thus the third integral converges to zero. So the fourth integral converges to zero while $\partial_i T_k(u_n)\rightharpoonup\partial_i T_k(u)$ weakly in $L^{p_i(\cdot)}(\Omega)$ and $a_i(x,T_k(u_n),\nabla T_k(u_n))(1-h_m(u_n))$ converges to $a_i(x,T_k(u),\nabla T_k(u))(1-h_m(u))$ strongly in $L^{p_i(\cdot)'}(\Omega)$. We conclude that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)\Big(\partial_i T_k(u_n)-\partial_i T_k(u)\Big)=0 \tag{4.22}$$

Using (4.11), (4.22) and lemma 3.6, we deduce

$$T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ and a. e. in } \Omega \quad \forall k>0 \tag{4.23}$$

This implies that

$$\nabla u_n\to\nabla u \text{ a. e. in } \Omega, \tag{4.24}$$

which gives

$$a_i(x,u_n,\nabla u_n)\rightharpoonup a_i(x,u,\nabla u) \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega)$$

Passage to the limit:

Let $\varphi\in K_\psi\cap L^\infty(\Omega)$ and $v=u_n-T_k(u_n-\varphi)$ as test function in approximate problem (4.1). We get

$$\begin{cases} u_n\in K_\psi.\\ \displaystyle\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi))\le\int_\Omega f_nT_k(u_n-\varphi)\\ \forall\varphi\in K_\psi\cap L^\infty(\Omega) \text{ and } \forall k>0,\end{cases}$$

5. Appendix:

Proof of Lemma 4.3

Let $v_0\in K_\psi$, thanks to Hölder's inequality and (3.2), we have

$$\begin{aligned}|<Bv,v_0>|&\le\sum_{i=1}^{N}(\frac{1}{(p_i^-)'}+\frac{1}{p_i^-})|a_i(x,v,\nabla v)|_{p_i(.)'}|\partial_i v_0|_{p_i(\cdot)}\\ &\le 2\beta\sum_{i=1}^{N}\Big(|j_i|_{p_i(\cdot)'}+|v|_{p_i(\cdot)}+|\partial_i v|_{p_i(\cdot)}\Big)|\partial_i v_0|_{p_i(\cdot)}\\ &\le 2\beta\|v_0\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}\Big(NC+NC+\|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}\Big)\end{aligned}$$

Which implies, by (3.1), we have

$$\frac{< Bv, v - v_0 >}{\|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}} \geq \frac{\sum_{i=1}^{N} \int_\Omega a_i(x, v, \nabla v) \partial_i v}{\|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}} - \frac{2\beta \|v_0\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} \Big(NC + NC + \|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} \Big)}{\|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}}$$

$$\geq \alpha \frac{\sum_{i=1}^{N} \int_\Omega |\partial_i v|^{p_i(x)}}{\|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}} - 2\beta \|v_0\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} \frac{\Big(NC + NC + \|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)} \Big)}{\|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}}$$

On the other hand, we have

$$\|v\|^\gamma_{W_0^{1,\vec{p}(\cdot)}(\Omega)} = \Big(\sum_{i=1}^{N} |\partial_i v|_{p_i(\cdot)} \Big)^\gamma \leq \sum_{i=1}^{N} |\partial_i v|^\gamma_{p_i(\cdot)} \leq \sum_{i=1}^{N} \int_\Omega |\partial_i v|^{p_i(x)}$$

where

$$\gamma = \begin{cases} p_m^- & \text{if } \ |\partial_i v|_{p_i(\cdot)} \geq 1 \\ p_M^+ & \text{if } \ |\partial_i v|_{p_i(\cdot)} < 1 \end{cases}$$

Then
$$\|v\|^{\gamma-1}_{W_0^{1,\vec{p}(\cdot)}(\Omega)} \leq \frac{\sum_{i=1}^{N} \int_\Omega |\partial_i v|^{p_i(x)}}{\|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}}$$

Thus
$$\frac{\sum_{i=1}^{N} \int_\Omega |\partial_i v|^{p_i}}{\|v\|_{W_0^{1,\vec{p}(\cdot)}}} \to +\infty \text{ as } \|v\|_{W_0^{1,\vec{p}(\cdot)}} \to +\infty.$$

Which implies that

$$\frac{< Bv, v - v_0 >}{\|v\|_{W_0^{1,\vec{p}(\cdot)}}} \to +\infty \text{ as } \|v\|_{W_0^{1,\vec{p}(\cdot)}} \to +\infty.$$

We deduce the operator B is coercive.

Now we prove that the operator B is pseudo-monotone. Let $(u_k)_k$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \quad \text{weakly in } \ W_0^{1,\vec{p}(\cdot)}(\Omega) \\ Bu_k \rightharpoonup \chi \quad \text{weakly in } \ W^{-1,\vec{p'}(\cdot)}(\Omega) \\ \limsup_{k \to +\infty} < Au_k, u_k > \ \leq \ < \chi, u >. \end{cases}$$

We will prove that $\chi = Bu$ and $< Bu_k, u_k > \to < \chi, u >$ as $k \to +\infty$. Since $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{p^-(\cdot)}(\Omega)$, then $u_k \to u$ strongly in $L^{p^-(\cdot)}(\Omega)$ and a.e. in Ω for a subsequence denoted again $(u_k)_k$. Since $(u_k)_k$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, by (3.2), we have $(a_i(x, u_k, \nabla u_k))_k$ is bounded in $L^{p_i'(\cdot)}(\Omega)$. Then there exists a function $\varphi_i \in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x, u_k, \nabla u_k) \rightharpoonup \varphi_i \text{ as } k \to +\infty. \tag{5.1}$$

For all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, using (5.1) , we obtain

$$\begin{aligned} < \chi, v > &= \lim_{k \to +\infty} < Au_k, v > \\ &= \lim_{k \to +\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, u_k, \nabla u_k) \partial_i v \end{aligned}$$

$$= \sum_{i=1}^{N} \int_{\Omega} \varphi_i \partial_i v.$$

Hence, we have

$$\begin{aligned} \limsup_{k\to+\infty} < Bu_k, u_k > &= \limsup_{k\to+\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k)\partial_i u_k \\ &\leq \quad < \chi, u > \\ &= \sum_{i=1}^{N} \int_{\Omega} \varphi_i \partial_i u, \end{aligned}$$

which implies that

$$\limsup_{k\to+\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k)\partial_i u_k \leq \sum_{i=1}^{N} \int_{\Omega} \varphi_i \partial_i u. \tag{5.2}$$

By (3.3), we have $\sum_{i=1}^{N} \int_{\Omega} \Big(a_i(x, u_k, \nabla u_k) - a_i(x, u_k, \nabla u)\Big)(\partial_i u_k - \partial_i u) > 0$. Then

$$\begin{aligned} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k)\partial_i u_k \geq &- \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u)\partial_i u + \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k)\partial_i u. \\ &+ \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u)\partial_i u_k. \end{aligned}$$

Using (5.1), we get

$$\liminf_{k\to+\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k)\partial_i u_k \geq \sum_{i=1}^{N} \int_{\Omega} \varphi_i \partial_i u. \tag{5.3}$$

Combining (5.2) and (5.3), we obtain

$$\lim_{k\to+\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k)\partial_i u_k = \sum_{i=1}^{N} \int_{\Omega} \varphi_i \partial_i u. \tag{5.4}$$

$$\begin{aligned} \lim_{k\to+\infty} < Bu_k, u_k > &= \lim_{k\to+\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k)\partial_i u_k \\ &= \sum_{i=1}^{N} \int_{\Omega} \varphi_i \partial_i u \\ &=< \chi, u > . \end{aligned}$$

In addition to this, since $a_i(x, u_k, \nabla u)$ converges to $a_i(x, u, \nabla u)$ strongly in $L^{p_i'(\cdot)}(\Omega)$, by (5.4), we obtain

$$\sum_{i=1}^{N} \int_{\Omega} \Big(a_i(x, u_k, \nabla u_k) - a_i(x, u_k, \nabla u)\Big)(\partial_i u_k - \partial_i u) = 0.$$

Using lemma (3.6), we get u_k converges to u strongly in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and a. e. in Ω, then $a_i(x, u_k, \nabla u)$ converges to $a_i(x, u, \nabla u)$ weakly in $L^{p_i'(\cdot)}(\Omega)$. Then for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}
< \chi, v > &= \lim_{k \to +\infty} < Au_k, v > \\
&= \lim_{k \to +\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, u_k, \nabla u_k) \partial_i v \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \partial_i v \\
&= < Au, v >.
\end{aligned}$$

Which implies that $Bu = \chi$.

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