Gulf Journal of Mathematics
Vol 6, Issue 4 (2018) 151-165

EXISTENCE OF RENORMALISED SOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEMS IN WEIGHTED VARIABLE -EXPONENT SPACE WITH L^1-DATA

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ABSTRACT. In this paper we study the existence of a renormalized solution for the nonlinear $p(x)$–elliptic problem in the Weighted–Variable–Exponent Soblev spaces, of the form: $-\text{div}(a(x,u,\nabla u)) + H(x,u,\nabla u) = f$ in Ω, where the right-hand side f belong to $L^1(\Omega)$ and $H(x,s,\xi)$ is the nonlinear term satisfying some growth condition, but no sign condition on s.

1. INTRODUCTION

Let Ω be a bounded open subset of $\mathbb{R}^N$ with N ≥ 2, $p \in C_+(\overline{\Omega})$, $p(x) > 1$. Let A be the nonlinear operator defined from $W_0^{1,p(x)}(\Omega,\omega)$ into its dual $W^{-1,p'(x)}(\Omega,\omega^*)$ by the formula $Au = -\text{div}(a(x,u,\nabla u))$. The objective of this article, is to study the existence of a renormalised solution for a class of nonlinear $p(x)$- elliptic problems of the type

$$\begin{cases} Au + H(x,u,\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where $f \in L^1(\Omega)$. The operator Au is a Weighted Leray–Lions operator (see assumption (3.1))-(3.3) of Section 3) which is coercive and where H is a non linear lower order term. Partial differential equations with nonlinearities involving non constant exponent have attracted an increasing amount of attention in recent years. The development, mainly by Ruzika [16] of a theory modeling the behavior of electrorheological fluids. Other applications relate to thermistor problem [18], or the problem of image recovery [10]. This notion was adapted yet to the study with $H = 0$ of some nonlinear elliptic problems on Sobolev space a exponent variable with Dirichlet boundary conditions by Boccardo-Diaz et al.[6]. Recently, Bendahmane and Wittbold in [5] proved the existence and uniqueness of renormalized solutions to nonlinear elliptic equations with variable exponent and L^1 data, besides, when $f \in L^1(\Omega) + W^{-1,p'(x)}(\Omega)$ E. Azroul et al. in [4] proved the entropy solutions for some $p(x)$ quasilinear problem with right-hand side mesure. In the case where $p(x)$ is a constant some results have been proved

Date: Received: Dec 2, 2018

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2010 *Mathematics Subject Classification.* 35J15, 35J70, 35J85.

Key words and phrases. weighted variable exponent Lebesgue Sobolev space, Young's Inequality, Renomalized Solution, elliptic problems.

by, L. Aharouch and Y. Akdim [1], to the case of parabolic equations, is studied by Y. Akdim et al. [2], where $H=0$ and $f\in L^{p'}(0,T;W^{-1,p'}(\Omega,\omega^*))$.

2. PRELIMINARIES

In this section, we state some elementary properties for the Weighted Variable Exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega,\omega)$ which will be used in the next sections. The basic properties of the variable exponent Lebesgue–Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$, that is, when $\omega(x)\equiv 1$ can be found from [11, 13].
Let Ω be a bounded open subsect of $\mathbb{R}^N(N\geq 2)$.

$$\text{Set} \qquad C_+(\overline{\Omega})=\{p\in C(\overline{\Omega}):\min_{x\in\overline{\Omega}}p(x)>1\}.$$

For any $p\in C_+(\overline{\Omega})$, we define $p^+=\max_{x\in\overline{\Omega}}p(x)$, $\quad p^-=\min_{x\in\overline{\Omega}}p(x)$.
For any $p\in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ that consists of all measurable real-valued functions u such that

$$L^{p(x)}(\Omega,\omega)=\{u:\Omega\to\mathbb{R}, measurable,\ \int_\Omega|u(x)|^{p(x)}\omega(x)dx<\infty\}.$$

Then, $L^{p(x)}(\Omega,\omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)}=\inf\{\lambda>0:\int_\Omega|\frac{u(x)}{\lambda}|^{p(x)}\omega(x)dx\leq 1\}$$

becomes a normed space. When $\omega(x)\equiv 1$, we have $L^{p(x)}(\Omega,\omega)\equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(\Omega,\omega)}$. The following Hölder type inequality is useful for the next sections. The weighted variable exponent Sobolev space $W^{1,p(x)}(\Omega,\omega)$ is defined by

$$W^{1,p(x)}(\Omega,\omega)=\{u\in L^{p(x)}(\Omega);|\nabla u|\in L^{p(x)}(\Omega,\omega)\},$$

where the norm is

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)}=|u|_{L^{p(x)}(\Omega)}+|\nabla u|_{L^{p(x)}(\Omega,\omega)}, \tag{2.1}$$

or, equivalently

$$\|u\|_{W^{1,p(x)}(\Omega,\omega)}=\inf\{\lambda>0:\int_\Omega|\frac{u(x)}{\lambda}|^{p(x)}+\omega(x)|\frac{\nabla u(x)}{\lambda}|^{p(x)}dx\leq 1\}$$

for all $u\in W^{1,p(x)}(\Omega,\omega)$.
It is significant that smooth functions are not dense in $W^{1,p(x)}(\Omega)$ without additional assumptions on the exponent $p(x)$. This feature was observed by Zhikov [18] in connection with the Lavrentiev phenomenon. However, if the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x)-p(y)|\leq\frac{C}{-\log|x-y|} \tag{2.2}$$

for every x,y with $|x-y|\leq\frac{1}{2}$, then smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values $W^{1,p(x)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega)}$ (see[11]).

$W_0^{1,p(x)}(\Omega,\omega)$ is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{W^{1,p(x)}(\Omega,\omega)}$.
Throughout the paper, we assume that $p \in C_+(\overline{\Omega})$ and ω is a measurable positive and a.e. finite function in Ω.

Lemma 2.1. *(See [11, 13]).(Generalised Hölder inequality).*

- **i)** *For any functions $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have* $|\int_\Omega uv dx| \leq (\frac{1}{p^-} + \frac{1}{p'^-})||u||_{p(.)}||v||_{p'(.)} \leq 2||u||_{p(x)}||v||_{p'(x)}$.
- **ii)** *For all p, $q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ a.e. in Ω, we have* $L^{q(x)} \hookrightarrow L^{p(x)}$ *and the embedding is continuous.*

Lemma 2.2. *(See [11].) Denote $\rho(u) = \int_\Omega \omega(x)|u(x)|^{p(x)}dx$ for all $u \in L^{p(x)}(\Omega,\omega)$. Then,*

$$|u|_{L^{p(x)}(\Omega,\omega)} < 1 (= 1; > 1) \text{ if and only if } \rho(u) < 1 (= 1; > 1), \tag{2.3}$$

$$if\ |u|_{L^{p(x)}(\Omega,\omega)} > 1 \ then\ |u|^{p^-}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega,\omega)}, \tag{2.4}$$

$$if\ |u|_{L^{p(x)}(\Omega,\omega)} < 1 \ then\ |u|^{p^+}_{L^{p(x)}(\Omega,\omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega,\omega)}. \tag{2.5}$$

Remark 2.3. (See [14].) If we set $I(u) = \int_\Omega |u(x)|^{p(x)} + \omega(x)|\nabla u(x)|^{p(x)}dx$. Then, following the same argument, we have

$$\min\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\} \leq I(u) \leq \max\{\|u\|^{p^-}_{W^{1,p(x)}(\Omega,\omega)}, \|u\|^{p^+}_{W^{1,p(x)}(\Omega,\omega)}\}.$$

Let ω be a weight function satisfying that

$$\omega \in L^1_{loc(\Omega)} \quad \text{and} \quad \omega^{-\frac{1}{(p(x)-1)}} \in L^1_{loc}(\Omega); \tag{2.6}$$

$$\omega^{-s(x)} \in L^1(\Omega,\omega) \quad \text{with} \quad s(x) \in (\frac{N}{p(x)}, \infty) \cap [\frac{1}{p(x)-1}, \infty). \tag{2.7}$$

The reasons that we assume (2.6) and (2.7) can be found in [14].

Remark 2.4. ([14].)
(i) If ω is a positive measurable and finite function, then $L^{p(x)}(\Omega,\omega)$ is a reflexive Banach space.
(ii) Moreover, if (2.6) holds, then $W^{1,p(x)}(\Omega,\omega)$ is a reflexive Banach space.

For $p, s \in C_+(\overline{\Omega})$, denote $p_s(x) := \frac{p(x)s(x)}{s(x)+1} < p(x)$, where $s(x)$ is given in (2.7). Assume that we fix the variable exponent restrictions

$$\begin{cases} p_s^*(x) := \frac{p(x)s(x)N}{(s(x)+1)N - p(x)s(x)} & \text{if } N > p_s(x), \\ p_s^*(x) \text{ arbitrary} & \text{if } N \leq p_s(x) \end{cases}$$

for almost all $x \in \Omega$. These definitions play a key role in our paper. We shall frequently make use of the following (compact) imbedding theorem for the weighted variable exponent Lebesgue–Sobolev space in the next sections.

Lemma 2.5. *([14].) Let $p, s \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.2) and let (2.6) and (2.7)be satisfied. If $r \in C_+(\overline{\Omega})$ and $1 < r(x) \leq p_s^*(x)$ then, we obtain the continuous imbedding*

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega).$$

Moreover, we have the compact imbedding

$$W^{1,p(x)}(\Omega, \omega) \hookrightarrow L^{r(x)}(\Omega),$$

provided that $1 < r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$.

From Lemma 2.5, we have Poincaré–type inequality immediately.

Corollary 2.6. *([14].) Let $p \in C_+(\overline{\Omega})$ satisfy the log–Hölder continuity condition (2.2). If (2.6) and (2.7) hold, then the estimate*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega,\omega)}$$

holds, for every $u \in C_0^\infty(\Omega)$ with a positive constant C independent of u.

Throughout this paper, let $p \in C_+(\overline{\Omega})$ satisfy the log-Hölder continuity condition (2.2) and $X := W_0^{1,p(x)}(\Omega, \omega)$ be the weighted variable exponent Sobolev space that consists of all real valued functions u from $W^{1,p(x)}(\Omega, \omega)$ which vanish on the boundary $\partial\Omega$, endowed with the norm

$$\|u\|_X = \inf\{\lambda > 0 : \int_\Omega \Big|\frac{\nabla u(x)}{\lambda}\Big|^{p(x)} \omega(x)dx \leq 1\},$$

which is equivalent to the norm (2.1) due to Corollary 2.6. The following proposition gives the characterization of the dual space $(W_0^{k,p(x)}(\Omega, \omega))^*$, which is analogous to [[13], Theorem 3.16]. We recall that the dual space of weighted Sobolev spaces $W_0^{1,p(x)}(\Omega, \omega)$ is equivalent to $W^{-1,p'(x)}(\Omega, \omega)$, where $\omega^* = \omega^{1-p'(x)}$.

Lemma 2.7. *([7].) Let $g \in L^{p(x)}(\Omega, \omega)$ and let $g_n \in L^{p(x)}(\Omega, \omega)$, with $\|g_n\|_{L^{p(x)}(\Omega,\omega)} \leq c$, $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{p(x)}(\Omega, \omega)$, where $\rightharpoonup$ denotes weak convergence and ω is a weight function on Ω.*

Lemma 2.8. *([3].) Assume (3.1)-(3.3) and let $(u_n)_n$ be a sequence in $W_0^{1,p(.)}(\Omega, \omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p(.)}(\Omega, \omega)$ and*

$$\int_\Omega \Big(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)\Big) \cdot \nabla(u_n - u)dx \to 0. \tag{2.8}$$

Then, $u_n \to u$ strongly in $W_0^{1,p(.)}(\Omega, \omega)$.

3. ESSENTIAL ASSUMPTION

Throughout the paper, we assume that the following assumption hold true.
ASSUMPTION (H1): Let Ω is a bounded open set of $\mathbb{R}^N (N \geq 2)$, $p \in C_+(\bar{\Omega})$, such that $1 < p^- \leq p^+ < \infty$ and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.
We consider a Leray–Lions operator defined by the formula: $Au = -div\ a(x, u, \nabla u)$.

The function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions

$$|a(x,s,\xi)| \leq \beta\omega^{1/p(x)}(x)\Big[k(x) + \omega^{1/p'(x)}|s|^{p(x)-1} + \omega^{1/p'(x)}(x)|\xi|^{p(x)-1}\Big], \tag{3.1}$$

$$\Big[a(x,s,\xi) - a(x,s,\eta)\Big] \cdot (\xi - \eta) > 0 \ \ \forall\, \xi \neq \eta \in \mathbb{R}^N, \tag{3.2}$$

$$a(x,s,\xi)\cdot\xi \geq \alpha\omega|\xi|^{p(x)}, \tag{3.3}$$

for all $(s,\eta,\xi) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$ and for almost every $x \in \Omega$, where $k(x)$ is a positive function lying in $L^{p'(x)}(\Omega)$ and α, β are a positive constants.
ASSUMPTION (H2): Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x,s,\xi)| \leq \gamma(x) + g(s)\omega|\xi|^{p(x)}, \tag{3.4}$$

is satisfied, where $g : \mathbb{R} \to \mathbb{R}^+$ is a bounded continuous positive function that belongs to $L^1(\mathbb{R})$ while $\gamma \in L^1(\Omega)$.

$$f \in L^1(\Omega). \tag{3.5}$$

Finally, we recall that for $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(.)$ defined by
$$T_k(s) = \begin{cases} s & if \quad |s| \leq k \\ k\frac{s}{|s|} & if \quad |s| > k. \end{cases}$$

Definition 3.1. Let $f \in L^1(\Omega)$. A real-valued function u defined on Ω is a renormalized solution of problem (1.1) if

$$T_k(u) \in W_0^{1,p(x)}(\Omega,\omega)) \quad \text{ for all } k \geq 0, \tag{3.6}$$

$$\int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\nabla u dx \to 0 \text{ as } \ m \to \infty, \tag{3.7}$$

$$\begin{aligned} -\mathrm{div}\Big(S'(u)a(x,u,\nabla u)\Big) &+ S''(u)a(x,u,\nabla u)\nabla u + H(x,u,\nabla u)S'(u) \\ &= fS'(u) \text{ in } D'(\Omega), \end{aligned} \tag{3.8}$$

for all $S \in W^{2,\infty}(\mathbb{R})$ which are piecewise C^1 and such that S' has a compact support in $\mathbb{R}$.

Remark 3.2. ([3].) *Equation* (3.8) *is formally obtained through pointwise multiplication of problem* (1.1) *by* $S'(u)$. *However, while* $a(x,u,\nabla u)$ *and* $H(x,u,\nabla u)$ *do not in general make sense in* (1.1), *all the terms in* (3.8) *have a meaning in* $D'(\Omega)$.

4. EXISTENCE RESULTS

Theorem 4.1. *Let* $f \in L^1(\Omega)$, $p \in C_+(\bar{\Omega})$. *Assume that* $(H1)$ *and* $(H2)$ *hold true. Then, there exists a renormalized solution* u *of problem* (1.1) *in the sense of Definition 3.1.*

Proof. Step 1: Approximate problem . For $n > 0$, let us define the following respective approximation of H and f;

$$\begin{aligned} H_n(x,s,\xi) &= \frac{H(x,s,\xi)}{1+\frac{1}{n}|H(x,s,\xi)|}. \\ \text{Note that} \quad |H_n(x,s,\xi)| &\leq |H(x,s,\xi)| \\ \text{and} \quad |H_n(x,s,\xi)| &\leq n \text{ for all } (s,\xi) \in \mathbb{R}\times\mathbb{R}^N. \end{aligned}$$

Let $(f_n)_n$ be sequence of smooth functions such that

$$f_n \to f \text{ a.e. in } \Omega, \text{ strongly in } L^1(\Omega) \text{ as } n \to \infty \text{ and } \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}. \tag{4.1}$$

We consider the sequence of approximate problems

$$\begin{cases} u_n \in W_0^{1,p(x)}(\Omega,\omega) \\ -\operatorname{div}(a(x,u_n,\nabla u_n)) + H_n(x,u_n,\nabla u_n) = f_n & \text{in } D'(\Omega) \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Let $f_n \in W^{-1,p'(x)}(\Omega,\omega^*)$, $p \in C_+(\bar{\Omega})$ for fixed n, the approximate problem (4.2) has at least one weak solution $u_n \in W_0^{1,p(x)}(\Omega,\omega)$. See [13].
STEP 2: A Priori Estimates.

Proposition 4.2. *Let u_n a solution of the approximate problem (4.2). Then, there exists a constant C(which does not depend on the n and k) such that*

$$\|T_k(u_n)\|_{W_0^{1,p(x)}(\Omega,\omega)} \leq Ck \quad \forall k > 0.$$

Proof. Let $\varphi \in W_0^{1,p(x)}(\Omega,\omega) \cap L^\infty(\Omega)$, with $\varphi > 0$. Choosing $v = \exp(G(u_n))\varphi$ as a test function in (4.2), where

$$G(s) = \int_0^s \frac{g(r)}{\alpha} dr.$$

(the function g appears in (3.4), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\Big(\exp(G(u_n))\varphi\Big)dx + \int_\Omega H_n(x,u_n,\nabla u_n)\exp(G(u_n))\varphi dx = \int_\Omega f_n \exp(G(u_n))\varphi dx.$$

In view of (3.4), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(G(u_n))\varphi dx + \int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dx$$
$$\leq \int_\Omega \gamma(x)\exp(G(u_n))\varphi dx + \int_\Omega f_n \exp(G(u_n))\varphi dx + \int_\Omega g(u_n)|\nabla u_n|^{p(x)}\omega(x)\exp(G(u_n))\varphi dx,$$

by using (3.3), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla\varphi dx \leq \int_\Omega \Big(\gamma(x) + f_n\Big)\exp(G(u_n))\varphi dx \tag{4.3}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega,\omega) \cap L^\infty(\Omega)$, with $\varphi > 0$.
On the other hand, taking $v = \exp(-G(u_n))\varphi$ as a test function in (4.2), we deduce as in (4.3) that

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(-G(u_n))\nabla\varphi dx + \int_\Omega \gamma(x)\exp(-G(u_n))\varphi dx \geq \int_\Omega f_n \exp(-G(u_n))\varphi dx \tag{4.4}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega,\omega) \cap L^\infty(\Omega)$, with $\varphi > 0$. Letting $\varphi = T_k(u_n)^+$ in (4.3), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla T_k(u_n)^+ dx \leq \int_\Omega \Big[\gamma(x) + f_n\Big]\exp(G(u_n))T_k(u_n)^+ dx \tag{4.5}$$

since $G(u_n) \leq \frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}$, we obtain

$$\int_\Omega a(x,u_n,\nabla T_k(u_n)^+)\exp(G(u_n))\nabla T_k(u_n)^+ dx \leq k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|\gamma\|_{L^1(\Omega)} + \|f_n\|_{L^1(\Omega)}\Big] \leq C_1 k$$

by using (3.3), we have

$$\int_\Omega |\nabla T_k(u_n)^+|^{p(x)}\omega(x)\exp(G(u_n))dx \leq C_1 k. \tag{4.6}$$

Similarly to (4.6), we take $\varphi = T_k(u_n)^-$ in (4.4), we deduce that

$$\int_\Omega |\nabla T_k(u_n)^-|^{p(x)}\omega(x)dx \leq C_2 k. \tag{4.7}$$

Combining (4.6), (4.7) and Remark 2.3, we conclude that

$$\min\Big\{\|T_k(u_n)\|^{p^+}_{W_0^{1,p(x)}(\Omega,\omega)}, \|T_k(u_n)\|^{p^-}_{W_0^{1,p(x)}(\Omega,\omega)}\Big\} \leq \rho(\nabla T_k(u_n)) \leq C_3 k.$$
$$\|T_k(u_n)\|_{W_0^{1,p(x)}(\Omega,\omega)} \leq C_4 k, \tag{4.8}$$

where C_1, C_2, C_3 and C_4 are constants independent of n. Thus, $T_k(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega,\omega)$ independently of n for any $k > 0$. Now we turn to proving the almost everywhere convergence of u_n. Consider a non decreasing function $g_k \in C^2(\mathbb{R})$ such that

$$g_k(s) = \begin{cases} s & if\ |s| \leq \frac{k}{2} \\ k & if\ |s| \geq k. \end{cases}$$

Multiplying the approximate equation by $g'_k(u_n)$, we get

$$\begin{aligned} -\mathrm{div}(a(x,u_n,\nabla u_n)g'_k(u_n)) + a(x,u_n,\nabla u_n)g''_k(u_n)\nabla u_n \\ + H_n(x,u_n,\nabla u_n)g'_k(u_n) = f_n g'_k(u_n). \end{aligned} \tag{4.9}$$

in the sense of distributions. This implies, thanks to the fact g'_k has compact support, that $g_k(u_n)$ is bounded in $W_0^{1,p(x)}(\Omega,\omega)$, while it's time derivative $\frac{\partial g_k(u_n)}{\partial t}$ is bounded in $L^1(Q) + W_0^{1,p(x)}(\Omega,\omega)$. Due to the choice of g_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Ω, which implies that the sequence u_n converge almost everywhere to some measurable function

v in Ω. Thus by using the same argument as in [7], [8], [9], we can show the following lemma.

Lemma 4.3. *Let u_n be a solution of the approximate problem (4.2). Then,*

$$u_n \quad \to \quad u \qquad a.e. \text{ in } \Omega.$$

We can deduce from ((4.8)) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad in \quad W_0^{1,p(x)}(\Omega,\omega),$$

which implies by using (3.3) that for all $k>0$, there exists $\varphi_k \in (L^{p'(x)}(\Omega,\omega^*))^N$ such that

$$a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi_k \quad in \quad \left(L^{p'(x)}(\Omega,\omega^*)\right)^N. \tag{4.10}$$

Lemma 4.4. *Let u_n be a solution of the approximate problem (4.2). Then,*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{m\le|u_n|\le m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx = 0. \tag{4.11}$$

Proof. See Appendix. □

Step 3:Almost everywhere convergence of the gradients :

We will use the following function of one real variable s, which is defined as follows

$$h_m(s)=\begin{cases}1 & if \quad |s|\le m\\ 0 & if \quad |s|\ge m+1\\ m+1-s & if \quad m\le s\le m+1\\ m+1+s & if \quad -m-1\le s\le -m,\end{cases}$$

where m is a nonnegative real parameter.

To prove the strong convergence of truncation $T_k(u_n)$, we have to prove the following assertions:

Proposition 4.5. The subsequence of u_n solution of problem (4.2) satisfies, for any $k\ge 0$ following assertion:

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup\int_\Omega a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)\Big(1-h_m(u_n)\Big)dx=0. \tag{4.12}$$

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup\int_\Omega \Big[a\Big(x,T_k(u_n,\nabla T_k(u_n)\Big)-a\Big(x,T_k(u_n,\nabla T_k(u)\Big)\Big]\times \Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]h_m(u_n)dx=0. \tag{4.13}$$

$$\lim_{n\to\infty}\int_\Omega \Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)-a\Big(x,T_k(u_n),\nabla T_k(u)\Big)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dx=0. \tag{4.14}$$

The proof of the above proposition is shown in the appendix. Thanks to (4.14) and Lemma 2.8, we have

$$T_k(u_n)\to T_k(u) \quad \text{strongly in} \quad W_0^{1,p(x)}(\Omega,\omega) \quad \forall k \tag{4.15}$$

and

$$\nabla u_n \to \nabla u. \quad \text{a.e. in} \quad \Omega,$$

which implies that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x, T_k(u), \nabla T_k(u)) \quad in \quad (L^{p'(x)}(\Omega, \omega^*))^N. \tag{4.16}$$

Step 4:Equi-integrability of the non linearity sequence :

We shall now prove that: $H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u)$ strongly in $L^1(\Omega)$, by using Vitali's theorem. Since $H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u)$ *a.e. in* Ω.

Considering now $\varphi = \rho_h(u_n) = \int_0^{u_n} g(s)\chi_{\{s>h\}}ds$ as a test function in (4.3), we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla u_n g(u_n)\chi_{\{u_n>h\}} \exp(G(u_n))dx$$
$$\leq \Big(\int_h^\infty g(s)\chi_{\{s>h\}}ds\Big) \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)}\Big],$$

using (3.3), we have

$$\alpha\int_{\{u_n>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)\exp(G(u_n))dx$$
$$\leq \Big(\int_h^\infty g(s)ds\Big) \exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\Big[\|f_n\|_{L^1(\Omega)} + \|\gamma\|_{L^1(\Omega)}\Big]$$

and since $g \in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n>h\}} |\nabla u_n|^{p(x)}w(x)g(u_n)dx = 0.$$

Similarly, taking $\varphi = \rho_h(u_n) = \int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$ as a test function in (4.4), we conclude that

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dx = 0.$$

Consequently,

$$\lim_{h\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dx = 0.$$

Which implies, for h large enough and for a subset E of Ω,

$$\lim_{measE\to 0}\int_E |\nabla u_n|^{p(x)}\omega(x)g(u_n)dxdt \leq \|g\|_\infty \lim_{measE\to 0}\int_E |\nabla T_h u_n|^{p(x)}\omega(x)dx$$
$$+ \int_{\{|u_n|>h\}} |\nabla u_n|^{p(x)}\omega(x)g(u_n)dx,$$

so $g(u_n)|\nabla u_n|^{p(x)}\omega(x)$ is equi-integrable. Thus, we have shown that

$$g(u_n)|\nabla u_n|^{p(x)}(x)\omega(x) \to g(u)|\nabla u|^{p(x)}(x)\omega(x) \quad \text{stongly in} \quad L^1(\Omega).$$

consequently, by using (3.4), we conclude that

$$H_n(x, u_n, \nabla u_n) \to H(x, u, \nabla u) \quad strongly \quad in \quad L^1(\Omega). \tag{4.17}$$

Step 5: Passing to the limit:

a) Proof that u **satisfies** (3.7). For any fixed $m \geq 0$, one has

$$\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx = \int_\Omega a(x,u_n,\nabla u_n)\Big[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)\Big]dx$$

$$= \int_\Omega a(x,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)dx - \int_\Omega a(x,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dx.$$

According to (4.15) and (4.16), one can pass to the limit as $n \to \infty$ for fixed $m \geq 0$ to obtain

$$\lim_{n\to\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx = \int_\Omega a(x,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)$$
$$- \int_\Omega a(x,T_m(u),\nabla T_m(u))\nabla T_m(u)dx = \int_{\{m\leq|u|\leq m+1\}} a(x,u,\nabla u)\nabla u dx \quad (4.18)$$

Taking the limit as $m \to \infty$ in (4.18) and using the estimate (4.11), shows that u satisfies (3.7).

b) Proof that u satisfies (3.8)

Let S be a function in $W^{2,\infty}(\mathbb{R})$ with S' has a compact support in $\mathbb{R}$. Let $M > 0$ such that supp$(S')\subset [-M,M]$. Pointwise multiplication of the approximate problem (4.2) by $S(u_n)$ leads to :

$$-\text{div}\Big[S'(u_n)a(x,u_n,\nabla u_n)\Big] + S''(u_n)a(x,u_n,\nabla u_n)\nabla u_n$$
$$+H_n(x,u_n,\nabla u_n)S'(u_n) = f_n S'(u_n) \ \text{ in } \ D'(\Omega). \quad (4.19)$$

In was follows we pass to the limit in (4.19) as n tends to ∞.

• Limit of $\frac{\partial S(u_n)}{\partial t}$.

Since S is bounded and continuous, $u_n \to u$ a.e. in Ω implies that $S(u_n)$ converges to $S(u)$ a.e. in Ω and L^∞ weakly.

$$\text{Then, } \quad \frac{\partial S(u_n)}{\partial t} \to \frac{\partial S(u)}{\partial t} \text{ in } D'(\Omega) \text{ as } n \to \infty.$$

• Limit of $-\text{div}\Big[S'(u_n)a(x,u_n,\nabla u_n)\Big]$.

Since supp$(S')\subset [-M,M]$, we have for $n \geq M$

$$S'(u_n)a(x,u_n,\nabla u_n) = S'(u_n)a(x,T_M(u_n),\nabla T_M(u_n)) \ a.e. \ in \ \Omega.$$

The pointwise convergence of u_n to u and (4.16) and the bounded character of S' permit us to conclude that as $n \to \infty$,

$$S'(u_n)a(x,u_n,\nabla u_n) \rightharpoonup S'(u)a(x,T_M(u),\nabla T_M(u)) \ in \ (L^{p'(x)}(\Omega,\omega^*))^N. \quad (4.20)$$

$S'(u)a(x,T_M(u),\nabla T_M(u))$ has been denoted by $S'(u)a(x,u,\nabla u)$ in equation (3.8).

• Limit of $S''(u_n)a(x,u_n,\nabla u_n)\nabla u_n$.

Consider the "energy" term

$S''(u_n)a(x,u_n,\nabla u_n)\nabla u_n = S''(u_n)a(x,T_M(u_n),\nabla T_M(u_n))\nabla T_M(u_n)$ a.e. in Ω.

The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (4.15) and (4.16) as $n \to \infty$ and the bounded character of S'' permet us to conclude that

$$S''(u_n)a(x,u_n,\nabla u_n)\nabla u_n \rightharpoonup S''(u)a(x,T_M(u),\nabla T_M(u))\nabla T_M(u) \text{ in } L^1(\Omega). \quad (4.21)$$

Recall that $S''(u)a(x, T_M(u), \nabla T_M(u))\nabla T_M((u)) = S''(u)a(x, u, \nabla u)\nabla u$ a.e. in Ω.

• Limit of $S'(u_n)H_n(x, u_n, \nabla u_n)$.

From $\text{supp}(S') \subset [-M, M]$ and (4.17), we have

$$S'(u_n)H_n(x, u_n, \nabla u_n) \to S'(u)H(x, u, \nabla u) \text{ strongly in } L^1(\Omega) \text{ as } n \to \infty. \quad (4.22)$$

• Limit of $S'(u_n)f_n$.

Since $u_n \to u$ a.e. in Ω, and (4.1), we have

$S'(u_n)f_n \to S'(u)f$ strongly in $L^1(\Omega)$, *as* $n \to \infty$.

As a consequence of the above convergence result, we are in a position to pass to the limit as $n \to \infty$ in equation (4.19) and to conclude that u satisfies (3.8). As a consequence, an Aubin's type Lemma (see, [17] implies that $S(u_n)$ converge to $S(u)$ strongly in $L^1(\Omega)$. As a conclusion, of steps 1 to step 5 the proof of Theorem 4.1 is complete. □

5. Appendix

Proof of lemma 4.4. Set $\varphi = T_1(u_n - T_m(u_n))^+$ in (4.3), this function is admissible since $\varphi \in W_0^{1,p(x)}(\Omega, \omega)$ and $\varphi \geq 0$. Then, we have

$$\begin{aligned}
\int_{\{m \leq u_n \leq m+1\}} & a(x, u_n, \nabla u_n) \exp(G(u_n))\nabla u_n dx \\
&\leq \int_\Omega \gamma(x) \exp(G(u_n))T_1(u_n - T_m(u_n))^+ dx \\
&\quad + \int_\Omega f_n \exp(G(u_n))T_1(u_n - T_m(u_n))^+ dx.
\end{aligned} \quad (5.1)$$

Since $f_n \to f$ in $L^1(\Omega)$ and

$f_n \exp(G(u_n))T_1(u_n - T_m(u_n))^+ \leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) f_n$, then by Lebesgue's theorem, we have

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_\Omega f_n \exp(G(u_n))T_1(u_n - T_m(u_n))^+ dx = 0. \quad (5.2)$$

Similarly, since $\gamma \in L^1(\Omega)$, we obtain

$$\lim_{m\to\infty} \lim_{n\to\infty} \int_\Omega \gamma \exp(G(u_n))T_1(u_n - T_m(u_n))^+ dx = 0. \quad (5.3)$$

Together (5.1), (5.2) and (5.3), we conclude that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{m \leq u_n \leq m+1\}} a(x, u_n, \nabla u_n)\nabla u_n dx = 0. \quad (5.4)$$

On the other hand, taking $\varphi = T_1(u_n - T_m(u_n))^-$ as a test function in (4.4) and reasoning as in the proof of (5.4), we deduce that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \int_{\{-(m+1) \leq u_n \leq -m\}} a(x, u_n, \nabla u_n)\nabla u_n dx = 0. \quad (5.5)$$

Thus, proof of Lemma 4.4 follows from (5.4) and (5.5).

Proof of Proposition 4.5.

Assertion (4.12). We take $\varphi = T_k(u_n)^+(1-h_m(u_n))$ (where h_m is defined in step 3) as test function in (4.3), we obtain

$$\int_{\{m\leq u_n\leq m+1\}} a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n T_k(u_n)^+dx$$
$$+\int_{\{T_k(u_n)\geq 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\exp(G(u_n))(1-h_m(u_n))dx$$
$$\leq \int_\Omega \Big(\gamma(x)+f_n\Big)\exp(G(u_n))T_k(u_n)^+(1-h_m(u_n))dx \tag{5.6}$$

because

$$\int_{\{m\leq u_n\leq m+1\}} a(x,u_n,\nabla u_n)\exp(G(u_n))\nabla u_n T_k(u_n)^+dx$$
$$\leq C\int_{\{m\leq u_n\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n dx.$$

By using (3.3) and (4.11), the first integral of (5.6) converges to 0 as firstly n goes to infinity and m tend to infinity. On the other hand, in view of Lebesgue's theorem, we can deduce that the right hand side of (5.6) goes to zero. Hence, by passing to the limit of (5.6), becomes

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{T_k(u_n)\geq 0\}} a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)\exp(G(u_n))(1-h_m(u_n))dx = 0$$

which gives

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int_{\{T_k(u_n)\geq 0\}} a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)(1-h_m(u_n))dx = 0. \tag{5.7}$$

On the other hand, let $\varphi = T_k(u_n)^-(1-h_m(u_n))$ in (4.4) as test the function, then similarly to (5.7), we deduce that

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)\leq 0\}} a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)\nabla T_k(u_n)(1-h_m(u_n))dx = 0. \tag{5.8}$$

Finally, combining (5.7) and (5.8), we conclude assertion (i).

Assertion (4.13). On one hand, let $\varphi = (T_k(u_n)-T_k(u))^+h_m(u_n) \in W_0^{1,p(x)}(\Omega,\omega)\cap L^\infty(\Omega)$, with h_m is defined in step 3 and $\varphi\geq 0$, then we take φ as test function in (4.3), we obtain

$$\int_{\{T_k(u_n)-T_k(u)\geq 0\}} a(x,u_n,\nabla u_n)\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)\exp(G(u_n))dx$$
$$-\int_{\{m\leq u_n\leq m+1\}} a(x,u_n,\nabla u_n)\nabla u_n\Big(T_k(u_n)-T_k(u)\Big)^+\exp(G(u_n))dx$$
$$\leq \int_\Omega\Big(\gamma(x)+f_n\Big)\Big(T_k(u_n)-T_k(u)\Big)^+h_m(u_n)\exp(G(u_n))dx.$$

Observe that

$$|\int_{\{m\le u_n\le m+1\}} a(x,u_n,\nabla u_n)\nabla u_n\Big(T_k(u_n)-T_k(u)\Big)^+\exp(G(u_n))dx|$$
$$\le 2k\exp\Big(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\Big)\int_{\{m\le u_n\le m+1\}}|a(x,u_n,\nabla u_n)\nabla u_n|dx$$

thanks to (4.11) the second integral tend to zero as n and m tend to infinity, and by Lebesgue theorem, we deduce that the right hand side converge to zero as n and m goes to infinity.
In view of (4.1), the convergence f_n to f in $L^1(\Omega)$ and $\gamma\in L^1(\Omega)$.
Let $\epsilon_i(n,m)$ $i=1...,n$ various functions tend to zero as n and m tend to infinity, then we obtain

$$\int_{\{T_k(u_n)-T_k(u)\ge 0\}} a(x,u_n,\nabla u_n)\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)\exp(G(u_n))dx\le\epsilon_1(n,m). \tag{5.9}$$

Moreover, (5.9)

$$\int_{\{T_k(u_n)-T_k(u)\ge 0,|u_n|\le k\}} a(x,T_k(u_n),\nabla u_n)\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)\exp(G(u_n))dx$$
$$-\lim_{m\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\ge 0,|u_n|\ge k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)h_m(u_n)\exp(G(u_n))dx$$
$$\le\epsilon_1(n,m).$$

Since

$$|\int_{\{T_k(u_n)-T_k(u)\ge 0,|u_n|\ge k\}} a(x,u_n,\nabla u_n)\nabla T_k(u))h_m(u_n)\exp(G(u_n))dx|$$
$$\le C_1\int_{\{|u_n|\ge k\}}|a(x,u_n,\nabla u_n)||\nabla T_k|(u)h_m(u_n)dx\to 0\ as\ n\to\ \infty$$

which implies that

$$\int_{\{T_k(u_n)-T_k(u)\ge 0,|u_n|\le k\}}\exp(G(u_n))a(x,T_k(u_n),\nabla T_k(u_n))\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)dx$$
$$\le\epsilon_2(n,m)$$

and so that

$$\int_{\{T_k(u_n)-T_k(u)\ge 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla\Big(T_k(u_n)-T_k(u)\Big)h_m(u_n)dx$$
$$\le\epsilon_3(n,m). \tag{5.10}$$

Moreover

$$\int_{\{T_k(u_n)-T_k(u)\ge 0\}}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)]h_m(u_n)dx$$
$$-\int_{\{T_k(u_n)-T_k(u)\ge 0\}} a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]h_m(u_n)dx$$
$$\le\epsilon_3(n,m).$$

Since $a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varphi$ in $L^{p'(x)}(\Omega,\omega^*)$ and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p(x)}(\Omega,\omega)$ hence, the last integral tend to zero as $n \to +\infty$. Which yields,

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup\int_{\{T_k(u_n)-T_k(u)\geq 0\}}\Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]h_m(u_n)dx = 0. \tag{5.11}$$

On the other hand, take $\varphi = (T_k(u_n) - T_k(u))^- h_m(u_n)$ in (4.4). Similarly, we can deduce as in (5.11) that

$$\lim_{m\to\infty}\lim_{n\to\infty}\sup\int_{\{T_k(u_n)-T_k(u)\leq 0\}}\Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]h_m(u_n)dx = 0. \tag{5.12}$$

Combining (5.11) and (5.12), we obtain the desired assertion (ii).
Assertion (4.14). First, we have

$$\begin{aligned}&\int_\Omega\Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big) - a\Big(x,T_k(u_n),\nabla T_k(u)\Big)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dx\\ &\quad= \int_\Omega\Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big) - a\Big(x,T_k(u),\nabla T_k(u)\Big)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]h_m(u_n)dx\\ &\quad+\int_\Omega\Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big) - a\Big(x,T_k(u_n),\nabla T_k(u)\Big)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big](1-h_m(u_n))dx.\end{aligned}$$

By (4.13), the first integral of the right hand side converge to zero as n and m goes to infinity. For the second term, we have

$$\begin{aligned}&\int_\Omega\Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big) - a\Big(x,T_k(u_n),\nabla T_k(u)\Big)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big](1-h_m(u_n))dx\\ &= \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_m(u_n))dx - \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)(1-h_m(u))dx\\ &\quad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u))\nabla[T_k(u_n)-T_k(u)\Big](1-h_m(u))dx.\end{aligned}$$

From (4.12) the first term of the right hand side converge to 0 as $n,m\to+\infty$, and since $\Big(a(x,T_k(u_n),\nabla T_k(u_n))\Big)_n$ in is bounded $(L^{p'(x)}(\Omega,\omega^*))^N$ and $\nabla T_k(u_n)(1-h_m(u))$ converge to zero as n and m goes to infinity. For the third integral it's converge to zero because $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L^{p(x)}(\Omega,\omega))^N$ and by using Lemma 2.8, we deduce that

$$\lim_{n\to\infty}\int_\Omega\Big[a\Big(x,T_k(u_n),\nabla T_k(u_n)\Big) - a\Big(x,T_k(u_n),\nabla T_k(u)\Big)\Big]\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]dx = 0.$$

The proof of Proposition 4.5 is complete.

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