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EXISTENCE OF A RENORMALIZED SOLUTION FOR SOME NONLINEAR ANISOTROPIC ELLIPTIC PROBLEMS

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Abstract. In this paper, we study a general class of nonlinear anisotrpic elliptic problems associated with the differential inclusion $\beta(u) - div(a(x, D(u) + F(u)) \ni f$ in Ω, where $f \in L^{\infty}(\Omega)$. A vector field $a(.,.)$ is a Carathéodory function. Using trunction techniques and the generalized monotonicity method in the functional spaces we prove the existence of renormalized solutions for $L^{\infty} - data$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with Lipschitz boundary $\partial\Omega$ if $N \geq 2$. Our aim is to prove existence of renormalized solutions to the nonlinear elliptic equation

$$(E, f) \begin{cases} \beta(u) - div(a(x, Du) + F(u)) \ni f & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega. \end{cases}$$

with a right-hand side f which is assumed to belong to $L^{\infty}(\Omega)$ for (E, f). Furthermore, $F : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ is locally lipschitz continuous and $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a set valued, maximal monotone mapping such that $0 \in \beta(0)$.

$a : \Omega \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is a Carathéodory function satisfying the following assumptions:

($\mathbf{H}_1$) There exists a positive constant λ such that

$$\sum_{i=1}^{N} a_i(x, \xi).\xi_i \geq \lambda \sum_{i=1}^{N} |\xi_i|^{p_i}$$

holds for all $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.

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($\mathbf{H}_2$) $|a_i(x,\xi)| \leq \gamma(d_i(x)+|\xi_i^{p_i-1}|)$ for almost every $x \in \Omega$, γ is a positive constant for $i = 1, ..., N$, d_i is a positive function in $L^{p'_i}(\Omega)$ and every $\xi \in \mathbb{R}^N$.

($\mathbf{H}_3$) $(a(x,\xi) - a(x,\eta)).(\xi - \eta) \geq 0$ for almost every $x \in \Omega$ and for $\xi, \eta \in \mathbb{R}^N$.
In the present paper we use the framework of renormalized solutions. The concept of this notion was introduced by DiPerna and Lions [9]. It was then extended to the study of various problems of partial differential equations of parabolic, elliptic-parabolic and hyperbolic type. This problem has been studied in variable exponents spaces and Orlicz by Wittbold et al. [16, 11] and in the weighted Sobolev spaces by Akdim and Allalou [3]. Other works in this direction can be found in [5, 6, 2]. The interest in studying (E, f) is its various applications to electrorheological fluids (see [14]) which is an important class of non-Newtonian fluids. Our objective in this work is to prove an existence result of (E, f) in anisotropic Sobolev spaces. The main tools in our proofs are Poincaré inequality and the embedding for anisotropic Sobolev spaces. The paper is organized as follows. In section 2, we recall the standard framework of anisotropic Sobolev spaces. In section 3, we introduce the notion of renormalized solution for the problem (E, f) and we prove the existence of a renormalized solution for any L^∞-data.

2. Function spaces

2.1. **Anisotropic Sobolev spaces.** Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ and let $1 \leq p_1, ..., p_N < \infty$. The anisotropic spaces (see [15])

$$W^{1,\overrightarrow{p}}(\Omega) = \{u \in W^{1,1}(\Omega) : \partial_{x_i} u \in L^{p_i}(\Omega), i = 1, ..., N\}.$$

is a Banach space with respect to norm

$$\|u\|_{W^{1,\overrightarrow{p}}(\Omega)} = \|u\|_{L^1(\Omega)} + \sum_{i=1}^{N} \|\partial_{x_i} u\|_{L^{p_i}(\Omega)}$$

.

The space $W_0^{1,\overrightarrow{p}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to this norm.
The dual space of anisotropic Sobolev space $W_0^{1,\overrightarrow{p}}(\Omega)$ is equivalent to $W^{-1,\overrightarrow{p'}}(\Omega)$, where $\overrightarrow{p'} = (p'_1, ..., p'_N)$ and $p'_i = \dfrac{p_i}{p_i - 1}$ for all $i = 1, ..., , N$. We recall a Poincare-type inequality. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, then for every $q \geq 1$ there exists a constant C_p (depending on q and i (see [10]), such that

$$\|u\|_{L^q(\Omega)} \leq C_p \|\partial_{x_i} u\|_{L^{p_i}(\Omega)} \ \ for \ \ i = 1, ..., N. \tag{2.1}$$

Moreover a Sobolev-type inequality holds. Let us denote by $\overline{p}$ the harmonic mean of these numbers, i.e. $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N} \frac{1}{p_i}$. Let $u \in W_0^{1,\overrightarrow{p}}(\Omega)$, then there exists [15] a constant C_s such that

$$\|u\|_{L^q(\Omega)}\leq C_s\prod_{i=1}^{N}\|\partial_{x_i}u\|^{\frac{1}{N}}_{L^{p_i}(\Omega)}, \tag{2.2}$$

where $q=\overline{p}^*=\frac{N\overline{p}}{N-\overline{p}}$ if $\overline{p}<N$ or $q\in[1,+\infty[$ if $\overline{p}\geq N$. On the right-hand side of (2.2) it is possible to replace the geometric mean by the arithmetic mean: let $a_1,...,a_N$ be positive numbers, it holds

$$\prod_{i=1}^{N}a_i^{\frac{1}{N}}\leq\frac{1}{N}\sum_{i=1}^{N}a_i,$$

which implies by (2.2) that

$$\|u\|_{L^q(\Omega)}\leq\frac{C_s}{N}\sum_{i=1}^{N}\|\partial_{x_i}u\|_{L^{p_i}(\Omega)}. \tag{2.3}$$

Note that when the following inequality holds

$$\overline{p}<N, \tag{2.4}$$

inequality (2.3) implies the continuous embedding of the space $W_0^{1,\overrightarrow{p}}(\Omega)$ into $L^q(\Omega)$ for every $q\in[1,\overline{p}^*]$. On the other hand, the continuity of the embeding $W_0^{1,\overrightarrow{p}}(\Omega)\hookrightarrow L^{p_+}(\Omega)$ with $p_+:=max\{p_1,...,p_N\}$ relies on inequality (2.1). It may happen that $\overline{p}^*<p_+$ if the exponent p_i are closed enough, then $p_\infty:=max\{\overline{p}^*,p_+\}$ turns out to be the critical exponent in the anisotrpic Sobolev embedding (see [15]).

Proposition 2.1. *If the condition* (2.4) *holds, then for* $q\in[1,p_\infty]$ *there is a continuous embedding* $W_0^{1,\overrightarrow{p}}(\Omega)\hookrightarrow L^q(\Omega)$. *For* $q<p_\infty$ *the embedding is compact.*

$$W_0^{1,\overrightarrow{p}}(\Omega)\hookrightarrow\hookrightarrow L^q(\Omega). \tag{2.5}$$

2.2. **Notation and functions.** Before we discuss the concept of solution we introduce some notation and functions that will be frequently used.
We begin by introducing the truncature operator. For given constant $k>0$ we define the cut function $T_k:\mathbb{R}\to\mathbb{R}$ as

$$\begin{cases}-k,\text{if } r\leq-k,\\ r,\text{if } |r|<k,\\ k,\text{if } r\geq k,\end{cases}$$

and for $r\in\mathbb{R}$, let $r\to r^+:=max(r,0), r\to sign_0(r)$ the usual sign function which is equal

$$r\to sign_0(r):=\begin{cases}-1,\text{ on }]-\infty,0[,\\ 1,\text{ on }]0,\infty[,\\ 0,\text{ if } r=0.\end{cases}$$

$$r\to sign_0^+(r):=\begin{cases}1,\text{ if } r>0,\\ 0,\text{ if } r\leq 0.\end{cases}$$

Let $h_l : \mathbb{R} \to \mathbb{R}$ be defined by $h_l(r) := min((l+1-|r|)^+, 1)$ for each $r \in \mathbb{R}$.

For $\sigma > 0$ we define $H_\sigma^+ : \mathbb{R} \to \mathbb{R}$ by

$$H_\sigma^+(r) := \begin{cases} 0, \text{if} \quad r < 0, \\ \frac{1}{\sigma} r, \text{if } 0 \leq r \leq \sigma, \\ 1, \text{if} \quad r > \sigma. \end{cases}$$

and $H_\sigma : \mathbb{R} \to \mathbb{R}$ by

$$H_\sigma(r) := \begin{cases} -1, \text{if} \quad r < -\sigma, \\ \frac{1}{\sigma} r, \text{if } -\sigma \leq r \leq \sigma, \\ 1, \text{if} \quad r > \sigma. \end{cases}$$

3. Main result

3.1. Existence of a renormalized solution.

Definition 3.1. A renormalized solution to (E, f) is a pair of functions (u, b) satisfying the following conditions:

($\mathbf{R}_1$) $u : \Omega \to \mathbb{R}$ is measurable, $b \in L^1(\Omega), u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for a.e. $x \in \Omega$.

($\mathbf{R}_2$) For each $k > 0, T_k(u) \in W_0^{1,\overrightarrow{p}}(\Omega)$ and

$$\int_\Omega b.h(u)\phi + \int_\Omega (a(x, D(u)) + F(u)).D(h(u)\phi) = \int_\Omega f h(u)\phi, \tag{3.1}$$

holds for all $h \in C_c^1(\mathbb{R})$ and all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$.

($\mathbf{R}_3$) $\int_{\{k<|u|<k+1\}} a(x, Du).Du \longrightarrow 0$ as $k \longrightarrow \infty$.

Theorem 3.2. *For $f \in L^\infty(\Omega)$, there exists at least one renormalized solution (u, b) of (E, f).*

3.2. **Approximate solution for L^∞- data.** At first we approximate (E, f) for $L^\infty(\Omega)$ by a problem for which existence can be proved by standard variational arguments. For $0 < \varepsilon \leq 1$, let $\beta_\varepsilon : \mathbb{R} \longmapsto \mathbb{R}$ be the Yosida approximation of β (see [8]). We introduce the operators

$$A_{1,\varepsilon} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega),$$
$$u \to \beta_\varepsilon(T_{1/\varepsilon}(u)) - diva(x, Du)$$

and

$$A_{2,\varepsilon} : W_0^{1,\overrightarrow{p}}(\Omega) \to W^{-1,\overrightarrow{p}'}(\Omega),$$
$$u \to -divF(T_{1/\varepsilon}(u)).$$

Because of $(\mathbf{H}_2)-(\mathbf{H}_3)$, $A_{1/\varepsilon}$ is well-defined and monotone (see [13] for instance). Since $\beta_\varepsilon \circ T_{1/\varepsilon}$ is bounded and continuous and thanks to the growth condition (H_2) on a, it follows that $A_{1,\varepsilon}$ is hermicontinuous (see [13]). From the continuity and boundedness of $F\circ T_{1/\varepsilon}$ it follows that $A_{2,\varepsilon}$ is strongly continuous. Therefore the operator $A_\varepsilon := A_{1,\varepsilon} + A_{2,\varepsilon}$ is pseudomonotone. Using the monotonicity of β_ε, the Gauss-Green Theorem for Sobolev functions and the boundary condition on the convection term $\int_\Omega F(T_{1/\varepsilon}(u)).Du$, we show by using similar arguments as in [5]) that A_ε is coercive and bounded. Then it follows from [13] that A_ε is surjective. i.e., for each $0<\varepsilon\leq 1$ and $f\in W^{-1,\overrightarrow{p}'}(\Omega)$ there exists a solution $u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega)$ to the problem

$$(E_\varepsilon, f)\begin{cases} \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) - div(a(x,Du_\varepsilon)+F((T_{1/\varepsilon}(u_\varepsilon)))) = f & in \quad \Omega,\\ u_\varepsilon = 0 & on \quad \partial\Omega,\end{cases}$$

such that the following inequality holds for all $\phi\in W_0^{1,\overrightarrow{p}}(\Omega)$

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))\phi + \int_\Omega (a(x,D(u_\varepsilon)) + F(T_{1/\varepsilon}(u_\varepsilon)).D\phi =< f,\phi > \tag{3.2}$$

where $<.,.>$ denotes the duality pairing between $W_0^{1,\overrightarrow{p}}(\Omega)$ and $W^{-1,\overrightarrow{p}'}(\Omega)$.

3.3. **A priori estimates.**

Lemma 3.3. *For $0<\varepsilon\leq 1$ and $f\in L^\infty(\Omega)$ let $u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega))$ be a solution of (E_ε, f). Then*
(i) There exists a constant $C_1 = C_1(||f||_\infty, \lambda, p, N) > 0$, not depending on ε, such that

$$|||u_\varepsilon||| \leq C_1. \tag{3.3}$$

$(ii)$for all $0<\varepsilon\leq 1$, we have

$$||\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))||_\infty \leq ||f||_\infty \tag{3.4}$$

(iii) for all $0<\varepsilon\leq 1$ and all $l,k>0$, we have

$$\int_{\{l\leq|u|\leq k+l\}} a(x,Du_\varepsilon).Du_\varepsilon \leq k\int_{\{|u_\varepsilon|>l\}}|f|. \tag{3.5}$$

Proof. (i) Taking u_ε as a test function in (3.2) we obtain

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))u_\varepsilon + \int_\Omega a(x,Du_\varepsilon) + \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).Du_\varepsilon = \int_\Omega fu_\varepsilon dx$$

As the first term on the left-hand side is nonnegative and the integral over the convextion term vanishes by $mathbf(H_1)$, we have

$$\lambda\sum_{i=1}^N\int_\Omega |\frac{\partial u_\varepsilon}{\partial x_i}|^{p_i}dx \leq \sum_{i=1}^N\int_\Omega a_i(x,Du_\varepsilon).\frac{\partial u_\varepsilon}{\partial x_i}dx \leq \int_\Omega fu_\varepsilon dx \leq C||f||_\infty(\sum_{i=1}^N\int_\Omega|\frac{\partial u_\varepsilon}{\partial x_i}|^{p_i}dx)^{1/p_i}.$$

due to the Hölder inequality. Thus $|||u_\varepsilon|||^{p_i} \leq C_2|||u_\varepsilon|||$, where C_2 is a positive constant. Then we can deduce that u_ε remains bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ i.e.,

$$|||u_\varepsilon||| \leq C_1.$$

(ii) Taking $\frac{1}{\sigma}[T_{k+\sigma}(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))) - T_k(\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)))]$ as a test function in (3.2), passing to the limit with $\sigma \to 0$ and choosing $k > ||f||_\infty$, we obtain (ii).
(iii) For $k, l > 0$ fixed we take $T_k(u_\varepsilon - T_l(u_\varepsilon))$ as a test function in (3.2).
Using $\int_\Omega a(x, Du_\varepsilon).DT_k(u_\varepsilon - T(u_\varepsilon))dx = \int_{\{l<|u_\varepsilon|<l+k\}} a(x, Du_\varepsilon).Du_\varepsilon dx$, and as the first term on the left-hand side is nonnegative and the convection term vanishes, we get

$$\int_{\{l<|u_\varepsilon|<k+l\}} a(x, Du_\varepsilon).Du_\varepsilon \leq \int_\Omega fT_k(u_\varepsilon - T_l(u_\varepsilon))dx \leq \int_{\{|u_\varepsilon|>l\}} |f|dx. \tag{3.6}$$

□

Remark 3.4. For $k > 0$, from lemma $(3.4.1, iii)$, we deduce

$$\int_{\{l\leq|u_\varepsilon|\leq k+l\}} a(x, Du_\varepsilon).Du_\varepsilon \leq k||f||_\infty|\{|u_\varepsilon| > l\}| \leq \frac{C_2(k)}{l^{1-\frac{1}{p}}} \tag{3.7}$$

for any $0 < \varepsilon \leq 1$ and a constant $C_2(k) > 0$ not depending on ε.

3.4. **Basic convergence result.**

Lemma 3.5. *For $0 < \varepsilon \leq 1$ and $f \in L^\infty(\Omega)$, let $u_\varepsilon \in W_0^{1,\overrightarrow{p}}(\Omega)$ be a solution of (E_ε, f). There exist $u \in W_0^{1,\overrightarrow{p}}(\Omega), b \in L^\infty(\Omega)$ such that for a not relabeled subsequence of $(u_\varepsilon)_{0<\varepsilon\leq 1}$ as $\varepsilon \mapsto 0$:*

$$u_\varepsilon \rightharpoonup u \qquad \text{in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ and a.e. in } \Omega, \tag{3.8}$$

$$T_k(u_\varepsilon) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,\overrightarrow{p}}(\Omega) \text{ and strongly in } L^q(\Omega), \tag{3.9}$$

$$\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \rightharpoonup b \quad \text{in } L^\infty(\Omega). \tag{3.10}$$

Moreover, for any $k > 0$,

$$DT_k(u_\varepsilon) \rightharpoonup DT_k(u) \quad \text{in } \prod_{i=1}^N L^{p_i}(\Omega), \tag{3.11}$$

$$a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u)) \quad \text{in } \prod_{i=1}^N L^{p_i'}(\Omega). \tag{3.12}$$

Proof. By combining Lemma 3.3 and Remark 3.4, we obtain (3.10). From (3.3) and (2.5), we deduce with a classical argument (see [1]) that for a subsequence still indexed by ε, (3.8) – (3.9) and (3.11) hold as ε tend 0, where u is mesurable function defined on Ω.

It is left to prove (3.12). For this, by (H_2) and (3.3) it follows that given any subsequence of $(a(x, DT_k(u_\varepsilon))_\varepsilon)$, there exists a subsequence, still denoted by $(a(x, DT_k(u_\varepsilon))_\varepsilon)$, such that $a(x, DT_k(u_\varepsilon)) \rightharpoonup \Phi_k$ in $\prod_{i=1}^N L^{p_i'}(\Omega)$. We will prove that $\Phi_k = a(x, DT_k(u))$ a.e. on Ω. The proof consists of three steps.

Step 1: For every $h \in W^{1,\infty}(\mathbb{R}), h \leq 0$ and $supp(h)$ compact, we will prove that

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x, DT_k(u_\varepsilon)).D[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx \leq 0. \tag{3.13}$$

Taking $h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (3.2), we have

$$\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon))h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) + \int_\Omega a(x, D(u_\varepsilon)).D[h_\varepsilon(T_k(u_\varepsilon) - T_k(u))] \tag{3.14}$$

$$+\int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).D[h_\varepsilon(T_k(u_\varepsilon)-T_k(u))] = \int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u)).$$

Using $|h_\varepsilon(T_k(u_\varepsilon) - T_k(u))| \leq 2k||h||_\infty$, by Lebesgue's dominated convergence theorem we find that $\lim_{\varepsilon\to 0}\int_\Omega fh(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) = 0$ and then $\lim_{\varepsilon\to 0}\int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).D[h_\varepsilon(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))] = 0$. By using the same arguments as in [4], we can prove that

$$\limsup_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)).[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx \geq 0.$$

Passing to the limit in (3.14) and using the above results, we obtain (3.13).

Step 2: We now prove that for every $k > 0$,

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x, DT_k(u_\varepsilon)).[D(T_k(u_\varepsilon) - DT_k(u))]dx \leq 0. \tag{3.15}$$

Indeed, for $k > l$, take $h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (3.2). Letting $\varepsilon \to 0$ and then $l \to \infty$, we obtain

$$\limsup_{\varepsilon\to 0}\int_\Omega a(x, DT_k(u_\varepsilon)).D[h_l(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))]dx = E_1 + E_2 + E_3.$$

where

$$E_1 = \int_{\{|u_\varepsilon|\leq k\}} h_l(u_\varepsilon)a(x, DT_k(u_\varepsilon)).[DT_k(u_\varepsilon) - DT_k(u)]dx$$

$$E_2 = \int_{\{|u_\varepsilon|>k\}} h_l(u_\varepsilon)a(x, DT_k(u_\varepsilon)).(-DT_k(u))]dx$$

$$E_3 = \int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon)).Du_\varepsilon dx$$

Since $l > k$, on the set $\{|u_\varepsilon| \leq k\}$ we have $h_l(u_\varepsilon) = 1$ so that we can write

$$\limsup_{\varepsilon\to 0} E_1 = \limsup_{\varepsilon\to 0}\int_\Omega a(x, DT_k(u_\varepsilon)).(DT_k(u_\varepsilon) - DT_k(u))dx.$$

For E_2, using Lebesgue's dominated convergence theorem, we get

$$\lim_{\varepsilon\to 0} E_2 = \int_{\{|u_\varepsilon|>k\}} h_l(u)\Phi_{l+1}.DT_k(u)dx = 0.$$

For E_3, we have

$$-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))a(x, DT_k(u_\varepsilon))Du_\varepsilon dx$$

$$\leq 2k \int_{\{l<|u_\varepsilon|\leq l+1\}} a(x, Du_\varepsilon) Du_\varepsilon dx.$$

Using (3.7), we deduce that

$$\limsup_{l\to\infty} \limsup_{\varepsilon\to 0} (-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) a(x, DT_k(u_\varepsilon)).Du_\varepsilon dx) \leq 0.$$

Applying (3.13) with h replaced by $h_l, l > k$, we get

$$\limsup_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)).[DT_k(u_\varepsilon) - DT_k(u)]dx$$

$$\leq \limsup_{\varepsilon\to 0} (-\int_\Omega h_l'(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u)) a(x, DT_k(u_\varepsilon)).Du_\varepsilon dx).$$

Now letting $l \to \infty$ yields (3.15).

Step 3: In this step, we prove by monotonicity arguments that for $k > 0$, $\Phi_k = a(x, DT_k(u))$ for almost every $x \in \Omega$. Let $\phi \in D(\Omega)$ and $\alpha \in \mathbb{R}$. Using (3.15), we have

$$\alpha \lim_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)).D\phi dx \geq \alpha \int_\Omega a(x, D(T_k(u) - \alpha\phi)).D\phi dx.$$

Dividing by $\alpha > 0$ and $\alpha < 0$ and letting $\alpha \to 0$, we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega a(x, DT_k(u_\varepsilon)).D\phi dx = \int_\Omega a(x, DT_k(u)).D\phi dx. \tag{3.16}$$

This means that for all $k > 0$, $\int_\Omega \Phi_k.D\phi dx = \int_\Omega a(x, DT_k(u))$. and so $\Phi_k = a(x, DT_k(u))$ in $D'(\Omega)$ for all $k > 0$. Hence $\Phi_k = a(x, DT_k(u))$ a.e. in Ω and so $a(x, DT_k(u_\varepsilon)) \rightharpoonup a(x, DT_k(u))$ weakly in $\prod_{i=1}^N L^{p_i'}(\Omega)$.

□

3.5. **Proof of the existence result.** We are now in position to conclude the proof of our main result presented in Theorem 3.2:

Proof. Let $h \in C_c^1(\mathbb{R})$ and $\varphi \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$. Taking $h_l(u_\varepsilon)h(u)(\varphi)$ as a test function in (3.2), we obtain

$$I_{\varepsilon,l}^1 + I_{\varepsilon,l}^2 + I_{\varepsilon,l}^3 = I_{\varepsilon,l}^4 \tag{3.17}$$

where

$$I_{\varepsilon,l}^1 = \int_\Omega \beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) h_l(u_\varepsilon) h(u)\varphi,$$

$$I_{\varepsilon,l}^2 = \int_\Omega a(x, Du_\varepsilon).D(h_l(u_\varepsilon)h(u)\varphi),$$

$$I_{\varepsilon,l}^3 = \int_\Omega F(T_{1/\varepsilon}(u_\varepsilon)).D(h_l(u_\varepsilon)h(u)\varphi),$$

$$I_{\varepsilon,l}^4 = \int_\Omega f h_l(u_\varepsilon) h(u)\varphi.$$

Step 1: Letting $\varepsilon \to 0$ using the convergence results (3.8), (3.10) from Lemma (3.5) we can immediately calculate the following limits:

$$\lim_{\varepsilon\to 0} I^1_{\varepsilon,l} = \int_\Omega b h_l(u)h(u)\varphi, \tag{3.18}$$

$$\lim_{\varepsilon\to 0} I^4_{\varepsilon,l} = \int_\Omega f h_l(u)h(u)\varphi. \tag{3.19}$$

We write $I^2_{\varepsilon,l} = I^{2,1}_{\varepsilon,l} + I^{2,2}_{\varepsilon,l}$ where
$I^{2,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon)a(x,Du_\varepsilon).Du_\varepsilon h(u)\varphi$, $\quad I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x,Du_\varepsilon).D(h(u)\varphi)$.
Using (3.7), we get the estimate

$$|\lim_{\varepsilon\to 0} I^{2,1}_{\varepsilon,l}| \leq ||h||_\infty ||\varphi||_\infty .C_2 l^{-(1-1/\bar{p})}. \tag{3.20}$$

By Lebesgue's dominated convergence theorem it follows that for any $i \in \{1, ..., N\}$, we have

$$h_l(u_\varepsilon)\frac{\partial}{\partial x_i}(h(u)\varphi) \to h_l(u)\frac{\partial}{\partial x_i}(h(u)\varphi) \quad in\ L^{p_i}\ as\ \varepsilon \to 0.$$

Keeping in mind that $I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)a(x, DT_{l+1}(u_\varepsilon)).D(h(u)\varphi)$. by (3.12), we get

$$\lim_{\varepsilon\to 0} I^{2,2}_{\varepsilon,l} = \int_\Omega h_l(u)a(x, DT_{l+1}(u)).D(h(u)\varphi). \tag{3.21}$$

Let us write $I^3_{\varepsilon,l} = I^{3,1}_{\varepsilon,l} + I^{3,2}_{\varepsilon,l}$, where

$$I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(u_\varepsilon)F(T_{1/\varepsilon}(u_\varepsilon)).Du_\varepsilon h(u)\varphi,$$

$$I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)F(T_{1/\varepsilon}(u_\varepsilon)).D(h(u)\varphi).$$

For any $l \in \mathbb{N}$, there exist $\varepsilon_0(l)$ such that for all $\varepsilon < \varepsilon_0(l)$,

$$I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(T_{l+1}(u_\varepsilon))F(T_{l+1}(u_\varepsilon)).h(u)\varphi. \tag{3.22}$$

Using the Gauss-Green Theorem for Sobolev functions in (3.22), we get for all $\varepsilon < \varepsilon_0(l)$,

$$I^{3,1}_{\varepsilon,l} = -\int_\Omega \int_0^{T_{l+1}(u_\varepsilon)} h'_l(r)F(r)dr.D(h(u)\varphi). \tag{3.23}$$

Now, using (3.8) and the Gauss-Green Theorem, after letting $\varepsilon \to 0$, we get

$$\lim_{\varepsilon\to} I^{3,1}_{\varepsilon,l} = \int_\Omega h'_l(u)F(u).Duh(u)\varphi. \tag{3.24}$$

Choosing ε small enough, we can write

$$I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u_\varepsilon)F(T_{l+1}(u_\varepsilon)).D(h(u)\varphi) \tag{3.25}$$

and conclude

$$\lim_{\varepsilon\to 0} I^{3,2}_{\varepsilon,l} = \int_\Omega h_l(u)F(u).D(h(u)\varphi). \tag{3.26}$$

Step 2: Passage to the limit with $l \to \infty$.
Combining (3.17) and (3.18) – (3.26) we find

$$I_l^1 + I_l^2 + I_l^3 + I_l^4 + I_l^5 = I_l^6 \tag{3.27}$$

where

$$I_l^1 = \int_\Omega b h_l(u) h(u)\varphi, \qquad I_l^2 = \int_\Omega h_l(u) a(x, DT_{l+1}(u)).D(h(u)\varphi),$$

$$|I_l^3| \le C_2 |l^{-(1-1/\bar{p})}||h||_\infty ||\varphi||_\infty, \qquad I_l^4 = \int_\Omega h_l(u) F(u).D(h(u)\varphi),$$

$$I_l^5 = \int_\Omega h_l'(u) F(u).Du h(u)\varphi, \qquad I_l^6 = \int_\Omega f h_l(u) h(u)\varphi.$$

Obviously, we have

$$\lim_{\varepsilon\to\infty} I_l^3 = 0. \tag{3.28}$$

Choosing $m > 0$ such that *supp* $h \subset [-m, m]$, we can replace u by $T_m(u)$ in $I_l^1, I_l^2, ..., I_l^6$, and

$$h_l'(u) = h_l'(T_m(u)) = 0 \ \ if \ \ l+1 > m, \qquad h_l(u) = h_l(T_m(u)) = 0 \ \ if \ \ l > m.$$

Therefore, letting $l \to \infty$ and combining (3.27) with (3.28) we obtain

$$\int_\Omega b h(u)\varphi + \int_\Omega (a(x, Du) + F(u)).D(h(u)\varphi) = \int_\Omega f h(u)\varphi \tag{3.29}$$

for all $h \in C_c^1(\mathbb{R})$ and all $\varphi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. Step 3: Subdifferential argument

It is left to prove that $u(x) \in D(\beta(x))$ and $b(x) \in \beta(u(x))$ for almost all $x \in \Omega$. Since β is a maximal monotone graph, there exist a convex, l.s.c and proper function $j : \mathbb{R} \to [0, \infty]$, such that

$$\beta(r) = \partial j(r) \quad for \ \ all \ \ r \in \mathbb{R}.$$

According to [8], for $0 < \varepsilon \le 1, j_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by $j_\varepsilon(r) = \int_0^r \beta_\varepsilon(s)ds$ has the following properties as in [16]

i) For any $0 < \varepsilon \le 1, j_\varepsilon$ is convex and differentiable for all $r \in \mathbb{R}$, such that

$$j_\varepsilon'(r) = \beta_\varepsilon(r) \text{ for all } r \in \mathbb{R} \text{ and any } 0 < \varepsilon \le 1$$

ii) $j_\varepsilon(r) \to j(r)$ for all $r \in \mathbb{R}$ as $\varepsilon \to 0$.

From *i*), it follows that for any $0 < \varepsilon \le 1$

$$j_\varepsilon(r) \ge j_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) + (r - T_{1/\varepsilon}(u_\varepsilon))\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \tag{3.30}$$

holds for all $r \in \mathbb{R}$ and almost everywhere in Ω. Let $E \cup \Omega$ be an arbitrary measurable set and χ_E its characteristic function. We fix $\varepsilon_0 > 0$. Multiplying (3.30) by $h_l(u_\varepsilon)\chi_E$, integrating over Ω and using *ii*), we obtain

$$j(r)\int_E h_l(u_\varepsilon) \ge \int_E j_{\varepsilon_0}(T_{l+1}(u_\varepsilon))h_l(u_\varepsilon) + (r - T_{l+1}h_l(u_\varepsilon)\beta_\varepsilon(T_{1/\varepsilon}(u_\varepsilon)) \tag{3.31}$$

for all $r \in \mathbb{R}$ and all $0 < \varepsilon < min(\varepsilon_0, \frac{1}{l})$. As $\varepsilon \to 0$, taking into account that E arbitrary we obtain from (3.31)

$$j(r)h_l(u) \geq j_{\varepsilon_0}(T_{l+1}(u))h_l(u) + bh_l(u)(r - T_{l+1}(u)) \tag{3.32}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω. Passing to the limit with $l \to \infty$ and then with $\varepsilon_0 \to 0$ in (3.32) finally yields

$$j(r) \geq j(u(x)) + b(x)(r - u(x)) \tag{3.33}$$

for all $r \in \mathbb{R}$ and almost everywhere in Ω, hence $u \in D(\beta)$ and $b \in \beta(u)$ for almost everywhere in Ω. With this last step the proof of Theorem 3.2 is concluded.

□

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