Gulf Journal of Mathematics
Vol 6, Issue 4 (2018) 166-169

UNDER TOTALLY GEODETIC VARIETY OF $Geod\mathbb{C}P^n$

ABDELHAKIM CHILLALI*

ABSTRACT. $\mathbb{C}P^n$ is a variety whose all geodesics are closed and periodic of the same period π. We can apply all studies made by M. SARIH in his thesis, see ([1]), in particular $dim(Geod\mathbb{C}P^n) = 4n - 2$. In this paper we will find some results concerning the open problem in [1], page 56.

1. INTRODUCTION

S^{2n+1} is provided with the canonical riemannian structure induced by the scalar product of $\mathbb{R}^{2n+2}$, S^1 is the Lie group; it is the unit circle of $\mathbb{R}^2$.
$u : S^1 \times S^{2n+1} \longrightarrow S^{2n+1}$ an action of S^1 on S^{2n+1}:

- u is differentiable
- $\forall z \in S^1$, $u_z : x \longmapsto u(z, x)$ is a diffeomorphism of S^{2n+1}
- $z \longmapsto u_z$ is a homomorphism of S^1 on $Diff(S^{2n+1})$.

The action u is free if: $\forall z \in S^1$, u_z is without fixed point.
We define the relation: $x \sim y \Leftrightarrow \exists z \in S^1 / y = u(z, x)$; $\sim$ is an equivalence relation and S^{2n+1}/S^1 is the quotient set of S^{2n+1} by $\sim$.
If u is free then: [2, 3]

- S^{2n+1}/S^1 is a $C^\infty - variety$
- $S^1 \hookrightarrow S^{2n+1} \longrightarrow S^{2n+1}/S^1$ is a principal bundle of fiber of type S^1

Theorem 1.1. *If S^1 Operate isometrically and freely on (S^{2n+1}, can), the quotient riemannized by submersion is isometric to $(\mathbb{C}P^n, can)$.*

For the proof of this theorem, see [1], page 5.

2. UNDER VARIETY OF $\mathbb{C}P^n$

Under the conditions of the theorem 1.1, we have (S^{2n+1}, can) is isomorphic to $(\mathbb{C}P^n, can)$. The geodesics of $(\mathbb{C}P^n, can)$ are simply closed and of the same length π.

Definition 2.1. Let M and N be two connected Riemannian manifolds. An application $f : M \longrightarrow N$ of class C^∞, is said to be totally geodesic, if for any geodesic γ of M, $f \circ \gamma$ is a geodesic of N.

Date: Received: Dec 2, 2018
*Corresponding author.
2010 *Mathematics Subject Classification.* 53C40, 53C55.
Key words and phrases. Totally geodetic, Variety, Geodesics.

Proposition 2.2. *The injection $i : S^p \longrightarrow S^q$ is totally geodesic.*

Proof. Let γ be a geodesic of S^p, γ is a big circle, so $i(\gamma)$ is a big circle of S^q. □

Proposition 2.3. *The injection $j : \mathbb{C}P^m \longrightarrow \mathbb{C}P^n$ is totally geodesic.*

Proof. We have (S^{2n+1}, can) is isomorphic to $(\mathbb{C}P^n, can)$ and (S^{2m+1}, can) is isomorphic to $(\mathbb{C}P^m, can)$. The injection $i : S^{2m+1} \longrightarrow S^{2n+1}$ is totally geodesic, hence the result. □

Definition 2.4. We call C_l-manifold, a manifold M such that there exists a metric g on M satisfying all geodesics of M are periodical and of same period l.

Let (M, g) be a C_l-manifold, $\zeta : \begin{array}{c} \mathbb{R} \times UM \longrightarrow UM \\ (t, v) \longmapsto \zeta^t(v) \end{array}$ is the geodesic flow.

Proposition 2.5. *$S^1 \hookrightarrow UM \longrightarrow GeodM$ is a principal bundle of fiber of type S^1, and $dim(GeodM) = 2n - 2$.*

Remark 2.6. $\mathbb{C}P^m$ is a C_π-manifold and $dim(Geod\mathbb{C}P^m) = 4n - 2$.

If M is a Riemannian manifold having closed geodesics of the same length l and N is a totally geodesic sub manifold whose all geodesics are also closed and of the same length l, then $GeodN$ is plunged into $GeodM$, as a submanifold totally geodesic.

Corollary 2.7. *$Geod\mathbb{K}P^m$ is plunged as a totally geodesic submanifold into $Geod\mathbb{C}P^n$, with $m \leq n$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

3. SOME RESULTS CONCERNING THE OPEN PROBLEM

Open problem:
Is there any geodesic subvariety of $Geod\mathbb{C}P^n$ other than $Geod\mathbb{C}P^m$, with $m \leq n$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$?
As an important class of homogeneous spaces, the Grassmann manifolds $G_{\mathbb{C}}(2, n-1)$, have been extensively studied by group-theoretic methods. In this section, we show how their differential geometry. The geodesic in $G_{\mathbb{C}}(2, n-1)$, can be characterized by proprieties of their images in the Euclidean space. We have:

$$GeodS^2 \longrightarrow_i Geod\mathbb{C}P^n \longrightarrow_\pi G_{\mathbb{C}}(2, n-1)$$

is a bundle of fiber of type S^2. The following two lemmas are results of a more general theorem see [4] page 38.

Lemma 3.1. *Let f be a continuous application of M in $Geod\mathbb{C}P^n$, it exists:*

- *A locally trivial bundle of the type fiber S^1; $E \rightarrow_r M$*
- *A continuous application F of E in $U\mathbb{C}P^n$ verifying:*

$$\begin{array}{ccc} E & \xrightarrow{F} & U\mathbb{C}P^n \\ {\scriptstyle r}\downarrow & \searrow & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & Geod\mathbb{C}P^n \end{array} \tag{3.1}$$

Lemma 3.2. *Let h be a continuous application of N in $G_{\mathbb{C}}(2, n-1)$, it exists:*

- *A locally trivial fiber of the type fiber S^2; $M \to_p N$*
- *A continuous application H of M in $Geod\mathbb{C}P^n$ verifying:*

$$\begin{array}{ccc} M & \xrightarrow{H} & Geod\mathbb{C}P^n \\ {\scriptstyle p}\downarrow & \searrow & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow[h]{} & G_{\mathbb{C}}(2, n-1) \end{array} \tag{3.2}$$

Remark 3.3.
- If M is a part of $Geod\mathbb{C}P^n$, and f is the canonical injection of M into $Geod\mathbb{C}P^n$, then E is a part of $U\mathbb{C}P^n$.
- If N is a part of $G_{\mathbb{C}}(2, n-1)$, and h is the canonical injection of N into $G_{\mathbb{C}}(2, n-1)$, then M is a part of $Geod\mathbb{C}P^n$.

Lemma 3.4. *If f is an injective application, totally geodesic of M in $Geod\mathbb{C}P^n$, then F transforms a horizontal geodesic into a horizontal geodesic.*

Proof. f transforms a geodesic into a geodesic. Let γ be a horizontal geodesic of E, such that $\gamma(0) = x$ and $\gamma'(0) = u \in Hor(x)$, then $f \circ r \circ \gamma$ is a horizontal geodesic of $Geod\mathbb{C}P^n$ of initial condition $f(x)$ and $(f \circ r \circ \gamma)'(0) \in Hor(f(x))$. So, $p \circ F \circ \gamma(t) = p(F(\gamma(t)))$, consequently $F \circ \gamma$ is a horizontal geodesic. □

Lemma 3.5. *If h is an injective application, totally geodesic of N in $G_{\mathbb{C}}(2, n-1)$, then H is an p-injective application, totally geodesic of M in $Geod\mathbb{C}P^n$.*

Proof. We have $h \circ p = \pi \circ H$. Let $x, y \in M$, then
$H(x) = H(y) \Longrightarrow h \circ p(x) = h \circ p(y) \Longrightarrow p(x) = p(y)$, so H is an p-injective application. Let γ be a geodesic of M, $\forall \epsilon > 0$, $\gamma(] - \epsilon, \epsilon[)$ is a sub manifold of M, so $H(\gamma(] - \epsilon, \epsilon[))$ is a sub manifold of $Geod\mathbb{C}P^n$.
We have: $\pi(H(\gamma(] - \epsilon, \epsilon[))) = h(p(\gamma(] - \epsilon, \epsilon[)))$, i.e, $\forall \epsilon > 0$, $H(\gamma(] - \epsilon, \epsilon[))$ is is a short way in $Geod\mathbb{C}P^n$, thereby H is totally geodesic. □

Lemma 3.6. *Let M is a totally geodesic sub manifold of N. If N is a totally geodesic sub manifold of P, then M is a totally geodesic sub manifold of P.*

Proof. We have:

$$M \xrightarrow[i]{} N \xrightarrow[j]{} P \tag{3.3}$$

$j \circ i$ is an injective application which turns a geodesic of M into a geodesic of P. □

Lemma 3.7. *M is generated by a totally geodesic sub manifold of $\mathbb{C}P^n$ if and only if E is a unit bundle.*

Proof. $\Longleftarrow$) If $E = U\mathbb{C}P^k$, then the following two fibrations:

$$\begin{array}{ccc} U\mathbb{C}P^k & \xrightarrow[F]{} & U\mathbb{C}P^n \\ {\scriptstyle r}\downarrow & \searrow & \downarrow{\scriptstyle p} \\ M & \xrightarrow{f} & Geod\mathbb{C}P^n \end{array} \tag{3.4}$$

$$\begin{array}{ccc} U\mathbb{C}P^k & \xrightarrow{F} & U\mathbb{C}P^n \\ \downarrow p & \searrow & \downarrow p \\ Geod\mathbb{C}P^k & \xrightarrow[h]{} & Geod\mathbb{C}P^n \end{array} \tag{3.5}$$

so, M is generated by a totally geodesic sub manifold of $\mathbb{C}P^n$.

$\Longrightarrow$) If M is generated by a totally geodesic sub manifold of $\mathbb{C}P^n$, then there exists k such that, $E = U\mathbb{C}P^k$. □

Lemma 3.8. *M is generated by a totally geodesic sub manifold of $\mathbb{C}P^n$ if and only if N is a Grassmann manifolds.*

Proof. $\Longleftarrow$) If $N = G_{\mathbb{C}}(2, m-1)$, then the following two fibrations:

$$\begin{array}{ccc} Geod\mathbb{C}P^m & \xrightarrow{H} & Geod\mathbb{C}P^n \\ \downarrow p & \searrow & \downarrow \pi \\ G_{\mathbb{C}}(2, m-1) & \xrightarrow[h]{} & G_{\mathbb{C}}(2, n-1) \end{array} \tag{3.6}$$

$$\begin{array}{ccc} M & \xrightarrow{H} & Geod\mathbb{C}P^n \\ \downarrow p & \searrow & \downarrow \pi \\ G_{\mathbb{C}}(2, m-1) & \xrightarrow[h]{} & G_{\mathbb{C}}(2, n-1) \end{array} \tag{3.7}$$

so, M is generated by a totally geodesic sub manifold of $\mathbb{C}P^n$.

$\Longrightarrow$) If M is generated by a totally geodesic sub manifold of $\mathbb{C}P^n$, then there exists m such that, $N = G_{\mathbb{C}}(2, m-1)$. □

References

[1] M. SARIH, Propriétés géodésiques et topologiques du projectif complexe $\mathbb{C}P^n$ et la varité de ses géodésiques, Thése de docteur de troixiéme cycle en Mathématiques, 1980.

[2] B. Glen E, Introduction to compact transformation groups, Academic Press New-york and London, (1972).

[3] A. M. Gleason, Spaces with compact Lie group of transformations, Proc. Amer. Math. Soc, 1 (1950) 35-43.

[4] C. Godbillon, Géométrie différentielle et Mécanique Analytique, Hermann, 1969.

Sidi Mohamed ben Abdullah University, FP, LSI, Taza, Morocco

E-mail address: abdelhakim.chillali@usmba.ac.ma