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A THEORETICAL FRAMEWORK FOR TWO-PARAMETER SEMIGROUPS

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ABSTRACT. Here, we present a theoretical framework for two-parameter semigroups of bounded linear operators on a Banach space. Our approach relies on a new definition of the infinitesimal generator of two-parameter semigroups. This definition, in the case of C_0-two parameter semigroups, allows trajectory to be differentiable on the nonnegative cone of the plane, when the initial state is in the domain of this generator. We provide also the abstract Cauchy problem satisfied by these trajectories. We prove some theoretical and general results concerning relationships between this generator and the infinitesimal generators of the components. We investigate commutativity relations and precise the domains of their validity. We establish the extension of the Hille-Yoshida known for one-parameter semigroups. We provide some examples and we give an application to the product semigroup.

1. Introduction and preliminaries

Let E a Banach space on the field $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $\mathcal{B}(E)$ designate the Banach algebra of all bounded linear operators on E. We denote $\mathbb{R}_+$ the set of nonnegative real numbers. A map $T : \mathbb{R}_+^2 \longrightarrow \mathcal{B}(E)$ is called a two-parameter semigroup, if it satisfies the two following conditions

[i] $T(0,0) = I$, where I is the identity mappping of E, and

[ii] $T(s_1 + s_2, t_1 + t_2) = T(s_1, t_1)T(s_2, t_2)$, for all $(s_1, t_1), (s_2, t_2) \in \mathbb{R}_+^2$.

The two-parameter semigroup T on E will be denoted by $(T(s,t))_{s\geq 0, t\geq 0}$.

Let $U(s)_{s\geq 0}$ and $V(t)_{t\geq 0}$ be one-parameter semigroups on E satisfying: $U(s)V(t) = V(t)U(s)$ for all $s, t \in \mathbb{R}_+$. We set $W(s,t) = U(s)V(t)$, for all $(s,t) \in \mathbb{R}_+^2$. Then W is a two-parameter semigroup on E

Conversely, let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter semigroup T on E. We set $L(s) = T(s,0)$ and $R(t) = T(0,t)$ for all $(s,t) \in \mathbb{R}_+^2$. Then, we have

(i) $T(s,t) = L(s)R(t) = R(t)L(s)$, for all $(s,t) \in \mathbb{R}_+^2$.

(ii) $L(s)_{s\geq 0}$ and $R(t)_{t\geq 0}$ are one-parameter semigroups (called, respectively, the left and right components of T).

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Historically, n-parameter semigroups were introduced by E. Hille in 1944. Some of their theoretical aspects were studied in [7]. In 1946, N. Dunford N. and I. E. Segal applied them (see [6]) to prove the Weierestrass theorem. Other results on n-parameter semigroups were given in 1966 by O. A. Ivanova [8]. In 2013, an extension of the Lumer-Phillips theorem for two-parameter C_0-semigroups was given in [1] by R. Abazari and A. Niknam and M. Hassani.

One of the problems of interest for two-parameter semigroups of bounded linear operators on Banach spaces is to define the infinitesimal generator for them. That problem was handled by several authors by different manners. For more informations, see the papers: [7], [12], [2], [3], [9] and [4].

Let A_1 (resp. A_2) designate the infinitesimal generator of $L(s)_{s\geq 0}$ (resp. $R(t)_{t\geq 0}$). Many authors see ([9],) think of the pair (A_1, A_2) as the infinitesimal generator of the two-parameter semigroups $(T(s,t))_{s\geq 0, t\geq 0}$. One of the aims of this work is to give a precise definition to that infinitesimal generator and to provide a theoretical approach based on that definition.

For the sequel, $\mathbb{R}^2$ will be endowed with the norm $\|.\|$ defined for all $(u,v)\in\mathbb{R}^2$ by $\|(u,v)\| := \sqrt{u^2+v^2}$. The canonical basis of the real space $\mathbb{R}^2$ will be denoted by $\{e_1, e_2\}$, where $e_1 := (1,0)$ and $e_2 := (0,1)$.

We start by introducing the following definition.

Definition 1.1. Let $f : \mathbb{R}^2_+ \longrightarrow E$ a function. We say that f is quasi-differentiable at $(0,0)$, if there exist a linear mapping $L : \mathbb{R}^2 \longrightarrow E$ a positive number $\alpha > 0$ and a function $\varepsilon : [0,\alpha[\times[0,\alpha[\to E$ satisfying the following condition:

(i) for all $(h,k)\in[0,\alpha[\times[0,\alpha[$, we have

$$f(h,k) - f(0,0) - L(h,k) = \varepsilon(h,k)\|(h,k)\|,$$

(ii) $\lim_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$.

In this case, L is called the quasi-derivative at $(0,0)$ and is denoted by $D^+f(0,0)$.

We speak about quasi-differentiability rather than ordinary differentiability, because $(0,0)$ is not in the interior of the non-negative cone $\mathbb{R}^2_+$. Also, it is easy to see that the concept above is not depending on the norm chosen in $\mathbb{R}^2$.

Now, we propose the following definition for the infinitesimal generator of a two-parameter semigroup on E. This definition will be the basis of our approach in the study of two-parameter semigroups of bounded linear operators on Banach spaces.

Definition 1.2. Let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter semigroup on the Banach space E. For every $x\in E$, we denote $\varphi_x : \mathbb{R}^2_+ \longrightarrow E$ the mapping defined by $\varphi_x(s,t) = T(s,t)x$. We consider the linear operator $A : D(A)\subseteq E \longrightarrow E\times E$ defined on its domain:

$$D(A) = \{x\in E \,:\, \varphi_x \text{ is quasi-differentiable at } (0,0)\},$$

by setting

$$Ax := (D^+\varphi_x(0,0)e_1, D^+\varphi_x(0,0)e_2).$$

The operator A will be called the infinitesimal generator of the two-parameter semigroup $(T(s,t))_{s\geq 0, t\geq 0}$.

We point out that Definition 1.2 is close to the one given by Sh. Al-Sharif and R. Khalil in [4], but it is completely different.

Let $(T(s,t))_{s\geq 0,t\geq 0}$ be a two-parameter semigroup on the Banach space E and let $x \in D(A)$, then by using Definition 1.2 above, it is easy to see that $D(A) \subset D(A_1) \cap D(A_2)$ and that $Ax = (A_1x, A_2x)$ for all $x \in D(A)$.

As in the case of one-parameter semigroups, we introduce the following.

Definition 1.3. Let $(T(s,t))_{s\geq 0,t\geq 0}$ be a two-parameter semigroup on the Banach space E. Then,

(i) we say that $(T(s,t))_{s\geq 0,t\geq 0}$ is uniformly continuous, if we have:

$$\lim_{(s,t)\to(0,0)} \|T(s,t) - I\| = 0.$$

(ii) We say that $(T(s,t))_{s\geq 0,t\geq 0}$ is strongly continuous, if we have:

$$\lim_{(s,t)\to(0,0)} \|T(s,t)x - x\| = 0, \text{ for all } x \in E.$$

Throughout this paper, E is real or complex Banach space and $(T(s,t))_{s\geq 0,t\geq 0}$ will be a two-parameter semigroup on E having the one-parameter semigroup $(L(s)_{s\geq 0})$ (resp. $(R(t)_{t\geq 0})$) as left (resp. right) component. That is, we have: $T(s,t) = L(s)R(t)$ for all $s,t \geq 0$.

After this introduction, this paper will be organized as follows:

In the second section we prove some general properties. See Theorem 2.1 and Theorem 2.2.

In the third section, we focus on two-parameter C_0-semigroups. Through Theorem 3.1, Theorem 3.2 and Theorem 3.3 and some other technical lemmas, we establish the following results:

(1) If one of the one-parameter semigroups $(L(s)_{s\geq 0})$ or $(R(t)_{t\geq 0})$ is strongly continuous, then we have the equality: $D(A) = D(A_1) \cap D(A_2)$, where A_1 (resp. A_2) is the infinitesimal generator of $(L(s)_{s\geq 0})$ (resp. $(R(t)_{t\geq 0})$).

Suppose that $(T(s,t))_{s\geq 0,t\geq 0}$ is a two-parameter C_0-semigroup on the Banach space E and let A be its infinitesimal generator (as defined in Definition 1.2 above), with $A = (A_1, A_2)$. Then

(2) $D(A)$ is dense in E.

(3) $(T(s,t))_{s\geq 0,t\geq 0}$ is uniquely determined by its infinitesimal generator A.

(4) $D(A_1A_2) \cap D(A_2A_1)$ is a dense linear subspace of E and we have:

$$A_1A_2x = A_2A_1x, \text{ for all } x \in D(A_1A_2) \cap D(A_2A_1).$$

(5) There exists $\omega \geq 0$, such that $\{\lambda \in \mathbb{C} \,/\, Re(\lambda) > \omega\} \subseteq \rho(A_1) \cap \rho(A_2)$ and for all $\lambda, \mu \in \mathbb{C}$, whith $Re(\lambda), Re(\mu) > \omega$, we have:

$$R(\lambda, A_1) \circ R(\mu, A_2) = R(\mu, A_2) \circ R(\lambda, A_1),$$

where, for $i \in \{1, 2\}$, as usual, $R(\lambda, A_i) := (\lambda I - A_i)^{-1}$ are the associated resolvents defined for all $\forall \lambda \in \rho(A_i)$.

(6) For all $x \in D(A)$, we have the properties:

(6-i) $\forall t \geq 0,\ R(t)x \in D(A_1)$ and $A_1R(t)x = R(t)A_1x$;

(6-ii) $\forall s \geq 0,\ L(s)x \in D(A_2)$ and $A_2L(s)x = L(s)A_2x$;

(6-iii) for all $(s,t) \in \mathbb{R}_+^2$, $T(s,t)x \in D(A)$ and we have:

$$A_1 T(s,t)x = T(s,t)A_1 x \quad \text{and} \quad A_2 T(s,t)x = T(s,t)A_2 x.$$

(6-iv) The map $\varphi_x : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow E$ given by $\varphi_x(s,t) = T(s,t)x$ is differentiable on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ and satisfies the abstract Cauchy problem in two variables given in Theorem 3.1.

In the fourth section, we give an extension of the classical Hille-Yoshida (for one-parameter seimgroups of contractions) to the case of two-parameter semigroups of contractions.

In the fifth (and last) section of this paper, we apply our previous results to deal with the product of semigroups. More precisely, let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be C_0-semigroups. Let A (resp. B) the infinitesimal generator of $(S(t))_{t\geq 0}$ (resp. of $(T(t))_{t\geq 0}$). We denote $\Delta = D(A) \cap D(B)$ and for all $t \geq 0$, we set $U(t) := S(t)T(t)$ and $V(t) := T(t)S(t)$. We put $Q(s,t) := S(s)T(t)$, for all $s,t \in \mathbb{R}_+$.

Then we prove that the following assertions are equivalent:

(i) $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ are semigroups, Δ is dense in E and $U(t)(\Delta) \subset \Delta$, for all $t \geq 0$.

(ii) $S(s)T(t) = T(t)S(s)$, forall $s,t \geq 0$.

(iii) Q is a two-parameter C_0-semigroup.

At the end of this paper, we give illustrative examples.

We may consider this work as a complement and a continuation of several papers devoted to the study of two-parameter semigroups on Banach spaces, see for instance the works: [2] and [3] of N. H. Abdelaziz, the paper [4] of Sh. Al-Sharif and R. Khalil and the paper [5] of S. Arora and S. Sharda. For more informations, the reader is invited to see the references of those papers.

We hope that in a future, further applications of our results would be given for two-time dynamical systems.

2. Some general results

In this section, we investigate few general theoretical aspects of two-parameter semigroups. At many times we have used theoretical results known for the usual one-parameter semigroups. All informations concerning the usual one-parameter semigroups on E are taken from the books [7], [11], [10].

Let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter semigroup on E. From [7], we know the following results:

(i) If $(T(s,t))_{s\geq 0, t\geq 0}$ is uniformly continuous, then the map $T : \mathbb{R}_+^2 \longrightarrow \mathcal{B}(E)$ is continuous on $\mathbb{R}_+^2$.

(ii) If $(T(s,t))_{s\geq 0, t\geq 0}$ is strongly continuous, then for all $x \in E$, the map φ_x is continuous on $\mathbb{R}_+^2$.

The first result of this section reads as follows.

Theorem 2.1. *Let $(T(s,t))_{s\geq 0,t\geq 0}$ be a two-parameter semigroup on the Banach space E. Then we have*

(i) $(T(s,t))_{s\geq 0,t\geq 0}$ is uniformly continuous, if and only if $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$ are uniformly continuous.

(ii) $(T(s,t))_{s\geq 0,t\geq 0}$ is strongly continuous, if and only if $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$ are strongly continuous.

Proof. (i) Suppose that $(T(s,t))_{s\geq 0,t\geq 0}$ is uniformly continuous, then it is clear that $(L(s))_{s\geq 0}$ et $(R(t))_{t\geq 0}$ are uniformly continuous.

Conversely, suppose that $(L(s))_{s\geq 0}$ et $(R(t))_{t\geq 0}$ are uniformly continuous, then we have

$$\begin{aligned}\|T(s,t)-I\| &= \|L(s)R(t)-I\| \\ &= \|L(s)R(t)-L(s)+L(s)-I\| \\ &\leq \|L(s)\|\|\|R(t)-I\|+\|L(s)-I\|\end{aligned}$$

Since, $(L(s))_{s\geq 0}$ is uniformly continuous, then $(L(s))_{s\geq 0}$ is bounded on every bounded neighborhood of 0, therefore we have $\lim_{(s,t)\to(0,0)}\|T(s,t)-I\|=0$.

(ii) If $(T(s,t))_{s\geq 0,t\geq 0}$ is strongly continuous, then is obvious that $(L(s))_{s\geq 0}$ et $(R(t))_{t\geq 0}$ are strongly continuous.

Conversely, suppose that $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$ are strongly continuous, then for all $x\in E$, we have

$$\begin{aligned}\|T(s,t)x-x\| &= \|L(s)R(t)x-L(s)x+L(s)x-x\| \\ &\leq \|L(s)\|\|R(t)x-x\|+\|L(s)x-x\|\end{aligned}$$

as $(L(s))_{s\geq 0}$ is strongly continuous, we know (see [11] or [10]), that there exist constants $M\geq 1$ and $\omega\geq 0$, such that $\forall s\in\mathbb{R}_+,\ \|L(s)\|\leq Me^{\omega s}$.
We deduce that $\lim\limits_{(s,t)\to(0,0)}\|T(s,t)-x\|=0$. This completes the proof. □

One of the main results of this paper is the next result, where we give a sufficient condition ensuring the equality between $D(A)$ and $D(A_1)\cap D(A_2)$. More precisely, we have the following theorem.

Theorem 2.2. *Let $(T(s,t))_{s\geq 0,t\geq 0}$ be a two-parameter semigroup on the Banach space E. If one of the one-parameter semigroups $(L(s)_{s\geq 0})$ or $(R(t)_{t\geq 0})$ is strongly continuous, then we have the equality: $D(A)=D(A_1)\cap D(A_2)$.*

Proof. (i) Let $x\in D(A)$. By definition, we have

$$\lim_{(h,k)\to(0^+,0^+)}\frac{\|\varphi_x(h,k)-x-D^+\varphi_x(0,0)(h,k)\|}{\|(h,k)\|}=0.$$

In particular,

$$\lim_{h\to 0^+}\frac{\|\varphi_x(h,0)-x-D^+\varphi_x(h,0)\|}{|h|}=0$$

and hence, we have

$$\lim_{h\to 0^+}\left\|\frac{L(h)x-x}{h}-D^+\varphi(0,0)e_1\right\|=0.$$

From which we deduce that $x \in D(A_1)$ with $A_1 x = D^+\varphi_x(0,0)e_1$.

By a similar manner, we show that $x \in D(A_2)$ and $A_2 x = D^+\varphi_x(0,0)e_2$.

(ii) Let $x \in D(A_1) \cap D(A_2)$. To prove that $x \in D(A)$, it is sufficient to prove the following:

$$\lim_{(h,k)\to(0^+,0^+)} \frac{\|\varphi_x(h,k) - x - hA_1x - kA_2x\|}{\|(h,k)\|} = 0,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$.

Suppose that $(L(s))_{s\geq 0}$ is strongly continuous, then for all $h, k > 0$, we have

$$\begin{aligned}
&\varphi_x(h,k) - x - hA_1x - kA_2x = T(s,t)x - x - hA_1x - kA_2x \\
&= L(h)R(k)x - x - hA_1x - kA_2x \\
&= L(h)(R(k)x - x - kA_2x) + L(h)x + kL(h)A_2x - x - hA_1x - kA_2x \\
&= L(h)(R(k)x - x - kA_2x) + k(L(h)A_2x - A_2x) + (L(h)x - x - hA_1x)
\end{aligned}$$

By using the evident inequalities: $\dfrac{h}{\|(h,k)\|} \leq 1$ and $\dfrac{k}{\|(h,k)\|} \leq 1$ valid for all $h, k > 0$, we obtain (for all $h, k > 0$) the following successive inequalities:

$$\begin{aligned}
&\frac{\|\varphi_x(h,k) - x - hA_1x - kA_2x\|}{\|(h,k)\|} \leq \\
&\frac{k\|L(h)\|}{\|(h,k)\|}\frac{\|R(k)x - x - kA_2x\|}{k} + \frac{k\|L(h)A_2x - A_2x\|}{\|(h,k)\|} + \frac{h}{\|(h,k)\|}\frac{\|L(h)x - x - hA_1x\|}{h} \\
&\leq \|L(h)\|\frac{\|R(k)x - x - kA_2x\|}{k} + \|L(h)A_2x - A_2x\| + \frac{\|L(h)x - x - hA_1x\|}{h}.
\end{aligned}$$

Now, since $x \in D(A_1) \cap D(A_2)$, then we have

$$\lim_{h\to 0^+} \frac{\|L(h)x - x - hA_1x\|}{h} = 0 \text{ et } \lim_{k\to 0^+} \frac{\|R(k)x - x - kA_2x\|}{k} = 0$$

As $(L(h))_{h\geq 0}$ is a C_0-semigroup, then we have $\lim\limits_{h\to 0^+} \|L(h)A_2x - A_2x\| = 0$, and there exist some constants $M \geq 1$ and $\omega \geq 0$, such that $\|L(s)\| \leq Me^{\omega s}$, for all $s \in \mathbb{R}_+$.

All is done to conclude that $\lim_{(h,k)\to(0^+,0^+)} \dfrac{\|\varphi_x(h,k) - x - hA_1x - kA_2x\|}{\|(h,k)\|} = 0$, which says that φ_x is quasi-differentiable, i.e., $x \in D(A)$. This ends the proof. □

3. Strongly continuous two-parameter semigroups

A two-parameter semigroup on E which is strongly continuous is called a C_0-semigroup.

Let $(T(s,t))_{s\geq 0, t\geq 0}$ a two-parameter C_0-semigroup, then $T(s,t) = L(s)R(t)$ for all $s, t \in \mathbb{R}^+$, where $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$ are one-parameter C_0-semigroups. By the general theory of one parameter semigroups (see [10]), we know that there exist $\omega_1, \omega_2 \geq 0$ and $M_1, M_2 \geq 1$, satisfying :

$$\|L(s)\| \leq M_1 e^{\omega_1 s} \text{ and } \|R(t)\| \leq M_2 e^{\omega_2 t}, \quad \forall s, t \in \mathbb{R}_+.$$

By setting $M = M_1M_2$, we obtain $\|T(s,t)\| \leq Me^{\omega_1 s}e^{\omega_2 t}$, for all $(s,t) \in \mathbb{R}_+^2$.

A two-parameter C_0-semigroup $(T(s,t))_{s,t\geq 0}$ is said to be of contractions on E, if

$$\forall (s,t) \in \mathbb{R}_+^2,\ \|T(s,t)\| \leq 1$$

It is easy to see that $(T(s,t))_{s,t\geq 0}$ is a two-parameter semigroup of contractions, if and only if, $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$ are both one parameter semigroups of contractions.

The first main result of this section reads as follows.

Theorem 3.1. *Let $(T(s,t))_{s\geq 0,t\geq 0}$ be a two-parameter C_0-semigroup and let A be its infinitesimal generator. Then, for all $x \in D(A)$, we have*

(i) $\forall t \geq 0,\ R(t)x \in D(A_1)$ and $A_1R(t)x = R(t)A_1x$;

(ii) $\forall s \geq 0,\ L(s)x \in D(A_2)$ and $A_2L(s)x = L(s)A_2x$;

(iii) for all $(s,t) \in \mathbb{R}_+^2$, $T(s,t)x \in D(A)$ and we have

$$A_1T(s,t)x = T(s,t)A_1x \quad \text{and} \quad A_2T(s,t)x = T(s,t)A_2x.$$

(iv) The map $\varphi_x : \mathbb{R}_+^* \times \mathbb{R}_+^* \longrightarrow E$ given by $\varphi_x(s,t) = T(s,t)x$ is differentiable on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ and satisfies the following abstract Cauchy problem in two variables:

$$(ACP) \qquad \begin{cases} \dfrac{\partial \varphi_x}{\partial s}(s,t) = A_1\varphi_x(s,t),\ \forall (s,t) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, \\ \\ \dfrac{\partial \varphi_x}{\partial t}(s,t) = A_2\varphi_x(s,t),\ \forall (s,t) \in \mathbb{R}_+^* \times \mathbb{R}_+^*, \\ \\ \varphi_x(0,0) = x. \end{cases}.$$

Proof. (i) Let $x \in D(A)$, then we have

$$\begin{aligned} \lim_{h\to 0} \frac{L(h)R(t)x - R(t)x}{h} &= R(t)\left(\lim_{h\to 0} \frac{L(h)x - x}{h}\right) \\ &= R(t)A_1x, \quad (\text{because } x \in D(A) \text{ and } D(A) \subseteq D(A_1)). \end{aligned}$$

It follows that $R(t)x \in D(A_1)$ and that $A_1R(t)x = R(t)A_1x$.

(ii) The statements in (ii) can be proved by a similar manner to (i).

(iii) Let $(s,t) \in \mathbb{R}_+^2$ and let $x \in D(A)$, then by virtue of (i), we know that $R(t)x \in D(A_1)$. Therefore, we have $L(s)R(t)x \in D(A_1)$ and

$$A_1L(s)R(t)x = L(s)A_1R(t)x = L(s)R(t)A_1x.$$

So, we have established that $T(s,t)x \in D(A_1)$ and $A_1T(s,t)x = T(s,t)A_1x$.

By a similar proof, one can show that $T(s,t)x \in D(A_2)$ and that the equality $A_2T(s,t)x = T(s,t)A_2x$, holds true for all $x \in D(A)$.

(iv) Let $(s,t) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ be fixed and let us show that

$$\lim_{(h,k)\to(0,0)} \frac{\|\varphi_x(s+h,t+k) - \varphi_x(s,t) - hA_1\varphi_x(s,t) - kA_2\varphi_x(s,t)\|}{\|(h,k)\|} = 0.$$

Since (h,k) is in a neighborhood of $(0,0)$, then we may suppose that

$$(h,k)\in W_{s,t},\ \text{where}\ W_{s,t}:=\{(u,v)\in\mathbb{R}^2:\max(|u|,|v|)\leq\min(s,t)\}.$$

It follows that

$$\varphi_x(s+h,t+k)-\varphi_x(s,t)-hA_1\varphi_x(s,t)-kA_2\varphi_x(s,t)=$$
$$T(s-|h|,t-|k|)\Big(\varphi(|h|+h,|k|+k)-\varphi_x(|h|,|k|)-hA_1\varphi_x(|h|,|k|)-kA_2\varphi_x(|h|,|k|)\Big)$$

Since $(T(s,t))_{s,t\geq 0}$ is a two-parameter C_0-semigroup, there exists a positive constant C, such that

$$\forall(h,k)\in\mathbb{R}^2,\ \max(|h|,|k|)\leq\min(s,t)\Longrightarrow\|\,T(s-|h|,t-|k|)\|\leq C.$$

We set $\alpha_h=h+|h|$. By taking account of the fact that $A_iT(s,t)x=T(s,t)A_ix$, for $i=1,2$ and all $(s,t)\in\mathbb{R}^2_+$, we obtain the following estimates:

$$\begin{aligned}
&\|\varphi_x(s+h,t+k)-\varphi_x(s,t)-hA_1\varphi_x(s,t)-kA_2\varphi_x(s,t)\|\\
&\leq C\|(\varphi_x(\alpha_h,\alpha_k)-x-\alpha_hA_1x-\alpha_kA_2x)-(\varphi_x(|h|,|k|)-x-|h|A_1x-|k|A_2x)\\
&\qquad -h(T(|h|,|k|)A_1x-A_1x)-k(T(|h|,|k|)A_2x-A_2x)\|\\
&\leq C\Big(\|\varphi_x(\alpha_h,\alpha_k)-x-\alpha_hA_1x-\alpha_kA_2x\|+\|\varphi_x(|h|,|k|)-x-|h|A_1x-|k|A_2x\|\\
&\qquad +|h|\|T(|h|,|k|)A_1x-A_1x\|+|k|\|T(|h|,|k|)A_2x-A_2x\|\Big).
\end{aligned}$$

By definition of α_h and α_k, we have $(\alpha_h,\alpha_k)\in\mathbb{R}^2_+$ and $\|(\alpha_h,\alpha_k)\|\leq 2\|(h,k)\|$, for all $(h,k)\in\mathbb{R}^2$.

This implies that for all $x\in D(A)$, we have

$$\lim_{(h,k)\to(0,0)}\frac{\|\varphi_x(\alpha_h,\alpha_k)-x-\alpha_hA_1x-\alpha_kA_2x\|}{\|(h,k)\|}=0.$$

Also, we have

$$\lim_{(h,k)\to(0,0)}\frac{\|\varphi_x(|h|,|k|)-x-|h|A_1x-|k|A_2x\|}{\|(h,k)\|}=0.$$

Since $(T(s,t))_{s,t\geq 0}$ is a two-parameter C_0-semigroup, we infer that

$$\lim_{(h,k)\to(0,0)}|\|T(|h|,|k|)A_1x-A_1x\|=0$$

and

$$\lim_{(h,k)\to(0,0)}\|T(|h|,|k|)A_2x-A_2x\|=0.$$

From the arguments above, we conclude that we have established the following equality:

$$\lim_{(h,k)\to(0,0)}\frac{\|\varphi_x(s+h,t+k)-\varphi_x(s,t)-hA_1\varphi_x(s,t)-kA_2\varphi_x(s,t)\|}{\|(h,k)\|}=0.$$

This completes the proof. □

The next lemma will help to prove the density of the domain of the infinitesimal generator of a two-parameter C_0-semigroup.

Lemma 3.2. *Let $(T(s,t))_{s\geq 0,t\geq 0}$ be a two-parameter C_0-semigroup. Then,*

(a) for all $x \in E$, we have

$$\lim_{(h,k)\to(0,0)} \frac{1}{hk}\int_0^h \int_0^k T(u,v)x\, dudv = x.$$

(b) For all $x \in E$ and $h, k \geq 0$, we have

$$\int_0^h \int_0^k T(u,v)x\, dudv \in D(A).$$

Proof. (a) To prove (a), we start by the following equality:

$$\frac{1}{hk}\int_0^h \int_0^k T(u,v)x\, dudv - x = \frac{1}{hk}\int_0^h \int_0^k (T(u,v)x - x)\, dudv.$$

As $(T(s,t))_{s\geq 0,t\geq 0}$ is a two-parameter C_0-semigroup, then

$$\lim_{(u,v)\to(0,0)} \|T(u,v)x - x)\| = 0,$$

therefore, for each $\varepsilon > 0$, there exists $\alpha > 0$, such that for all $(u,v) \in [0,\alpha]^2$, we have

$$\|T(u,v)x - x)\| \leq \varepsilon.$$

Thus for all $(u,v) \in [0,\alpha]^2$, we have

$$\frac{1}{hk}\int_0^h \int_0^k \|(T(u,v)x - x)\|\, dudv \leq \varepsilon.$$

This means that

$$\lim_{(h,k)\to(0,0)} \frac{1}{hk}\int_0^h \int_0^k T(u,v)x\, du\, dv = x.$$

(b) We have

$$\int_0^h \int_0^k T(u,v)x\, du\, dv = \int_0^h L(u)\left(\int_0^k R(v)x\, dv\right) du.$$

Since $(L(s))_{s\geq 0}$ is a one-parameter C_0-semigroup, then by virtue of Theorem 2.4 of [11], we know that $\int\limits_0^k L(u)z\, dv \in D(A_1)$, forall $z \in E$. In particular, for $z := \int\limits_0^k R(v)x\, dv$, we have

$$\int_0^h L(u)\left(\int_0^k R(v)x\, dv\right) du \in D(A_1).$$

Also, we have

$$\int_0^h \int_0^k T(u,v)x\, dudv = \int_0^k R(v)\left(\int_0^h L(u)x\, du\right) dv.$$

By a similar manner, we have

$$\int_0^h L(u)\left(\int_0^k R(v)x\, dv\right) du \in D(A_2).$$

Since $D(A) = D(A_1) \cap D(A_2)$, we get the result. Thus the proof is finished. $\square$

In the next theorem, we show that the two-parameter semigroup $(T(s,t))_{s\geq 0, t\geq 0}$ is uniquely determined by its infinitesimal generator A and that $D(A)$ is dense in E.

Theorem 3.3. *Let $(T(s,t))_{s\geq 0,t\geq 0}$ be a two-parameter C_0-semigroup and let A be its infinitesimal generator, with $A = (A_1, A_2)$. Then,*

(i) $D(A)$ is dense in E.

(ii) $(T(s,t))_{s\geq 0,t\geq 0}$ is uniquely determined by its infinitesimal generator A.

Proof. (i) Let $x \in E$ and let $h, k \geq 0$. Consider $x_{h,k} \in E$ defined by

$$x_{h,k} = \frac{1}{hk}\int_0^h \int_0^k T(u,v)x\, dudv.$$

By the previous Lemma 3.2, we know that $x_{h,k} \in D(A)$ for all $h, k \geq 0$. Therefore, by taking the limit, we obtain $\lim_{(h,k)\to(0,0)} x_{h,k} = x$, which implies that $x \in \overline{D(A)}$.

(ii) Let $(W(s,t))_{s,t\geq 0}$ be another two parameter C_0-semigroup which has A as generator. We must prove that $T(s,t) = W(s,t)$ for all $s, t \geq 0$.

We know that we can write $T(s,t) = L(s)R(t)$ and $W(s,t) = U(s)V(t)$ for all $s, t \geq 0$, where $(L(s))_{s\geq 0}$, $(R(t))_{t\geq 0}$, $(U(s))_{s\geq 0}$ and $(V(t))_{t\geq 0}$ are one-parameter C_0-semigroups. Also, for all $x \in D(A)$, we have $Ax = (A_1x, A_2x) = (B_1x, B_2x)$, where A_1, A_2, B_1 and B_2 are respectively the infinitesimal generators of the one-parameter C_0-semigroups $(L(s))_{s\geq 0}$, $(R(t))_{t\geq 0}$, $(U(s))_{s\geq 0}$ and $(V(t))_{t\geq 0}$.

Let us start by proving that $L(s) = U(s)$, for all $s \geq 0$. To this end, let $x \in D(A)$ and $s_0 > 0$ be given. We consider the function $Q : [0, s_0] \longrightarrow E$ defined by $Q(s) = L(s_0 - s)U(s)x$, for all $s \in [0, s_0]$.

According to Theorem 2.2, we know that $D(A) = D(B_1) \cap D(B_2)$. Then $x \in D(B_1)$. By using theorem 4.1 of [11], we infer that $U(s)x \in D(B_1)$ and that the function: $s \longrightarrow U(s)x$ is differentiable on $[0, s_0]$.

By Theorem 3.1, we know that $U(s)x \in D(B_2)$. We deduce that $U(s)x \in D(B_1) \cap D(B_2)$. By Theorem 2.2, we know that $D(A) = D(A_1) \cap D(A_2) = D(B_1) \cap D(B_2)$, from which we infer that $U(s)x \in D(A_1)$.

By using a well known result concerning one-parameter C_0-semigroups (see for example Theorem 2.4 in [11], p. 5), we conclude that the function $s \longrightarrow Q(s)$ is differentiable on $[0, s_0]$.

Now by easy computations, we obtain

$$\begin{aligned}\frac{d}{ds}Q(s) &= -A_1L(s - s_0)U(s)x + L(s_0 - s)B_1U(s)x\\ &= -L(s - s_0)A_1U(s)x + L(s_0 - s)B_1U(s)x\\ &= 0, \quad \text{(because we have } A_1y = B_1y \text{ for all } y \in D(A)).\end{aligned}$$

Thus the map Q is constant on $[0, s_0]$ and therefore $Q(s) = Q(s_0)$, for each $s \in [0, s_0]$. In particular, we have $Q(0) = Q(s_0)$, then $L(s_0)x = U(s_0)x$, for every

$x \in D(A)$. Since $L(s_0)$ and $U(s_0)$ are bounded on E and since $D(A)$ is dense in E, then $L(s_0) = U(s_0)$ for all $s_0 > 0$. Hence, $L(s) = U(s)$ for all $s \geq 0$.

By a similar manner, we prove that $R(t) = V(t)$ for all $t > 0$. We conclude that $W(s,t) = T(s,t)$, for all $s, t \geq 0$. This ends the proof. □

In the next theorem, we investigate commutativity relations.

Theorem 3.4. *Let $(T(s,t))_{s\geq 0, t\geq 0}$ be a two-parameter C_0-semigroup and let A be its infinitesimal generator, with $A = (A_1, A_2)$. Then*

(i) $D(A_1A_2) \cap D(A_2A_1)$ is a dense linear subspace of E and we have:

$$A_1A_2x = A_2A_1x, \textit{ for all } x \in D(A_1A_2) \cap D(A_2A_1).$$

(ii) There exists $\omega \geq 0$, such that $\{\lambda \in \mathbb{C} \,/\, Re(\lambda) > \omega\} \subseteq \rho(A_1) \cap \rho(A_2)$ and for all $\lambda, \mu \in \mathbb{C}$, whith $Re(\lambda), Re(\mu) > \omega$, we have:

$$R(\lambda, A_1) \circ R(\mu, A_2) = R(\mu, A_2) \circ R(\lambda, A_1).$$

Proof. (i) We set $\Omega =]0, +\infty[\times]0, +\infty[$ and we consider the space $\mathcal{D}(\Omega)$, of C^∞ functions on Ω with compact support.

For each $x \in E$ and for each $\varphi \in D(\Omega)$, we define $y(x, \varphi)$ by

$$y(x,\varphi) = \int_0^{+\infty}\int_0^{+\infty} \varphi(s,t)T(s,t)x \, dsdt.$$

Next we show that $y(x,\varphi) \in D(A_1A_2) \cap D(A_2A_1)$, for each $x \in E$ and for each $\varphi \in D(\Omega)$.

Let $x \in E$ be fixed and consider a function $\varphi \in \mathcal{D}(\Omega)$ with compact support satisfying $supp(\varphi) \subseteq [a,b] \times [c,d]$. Then for every $h > 0$, we have

$$\begin{aligned}
&\frac{L(h)y(x,\varphi) - y(x,\varphi)}{h} \\
&= \frac{1}{h}\int_0^{+\infty}\int_0^{+\infty} \varphi(s,t)T(s+h,t)x \, dsdt - \frac{1}{h}\int_0^{+\infty}\int_0^{+\infty} \varphi(s,t)T(s,t)x \, dsdt \\
&= \frac{1}{h}\int_h^{+\infty}\int_0^{+\infty} \varphi(s-h,t)T(s,t)x \, dsdt - \frac{1}{h}\int_0^{+\infty}\int_0^{+\infty} \varphi(s,t)T(s,t)x \, dsdt.
\end{aligned}$$

Then for all number h, with $0 < h < a$, we have

$$\frac{L(h)y(x,\varphi) - y(x,\varphi)}{h} = \int_0^{+\infty}\int_0^{+\infty} \frac{\varphi(s-h,t) - \varphi(s,t)}{h} T(s,t)x \, dsdt.$$

By assumption, φ has a compact support, then

$$\lim_{h\to 0} \frac{L(h)y(x,\varphi) - y(x,\varphi)}{h} = -\int_0^{+\infty}\int_0^{+\infty} \frac{\partial\varphi}{\partial s}(s,t)T(s,t)x \, dsdt,$$

which implies that that $y(x,\varphi) \in D(A_1)$ and that

$$A_1 y(x,\varphi) = -\int_0^{+\infty}\int_0^{+\infty} \frac{\partial\varphi}{\partial s}(s,t)T(s,t)x \, dsdt.$$

Now, the function $\dfrac{\partial\varphi}{\partial s}$ is still in the space $\mathcal{D}(\Omega)$, then by arguments as above, we show that $A_1y(x,\varphi) \in D(A_2)$ and that

$$A_2A_1y(x,\varphi) = \int_0^{+\infty}\int_0^{+\infty} \frac{\partial^2\varphi}{\partial t\partial s}(s,t)T(s,t)x\,dsdt. \tag{3.1}$$

Thus, we have proved that $y(x,\varphi) \in D(A_2A_1)$.

By similar arguments, one can prove that $y(x,\varphi) \in D(A_1A_2)$ and that

$$A_1A_2y(x,\varphi) = \int_0^{+\infty}\int_0^{+\infty} \frac{\partial^2\varphi}{\partial s\partial t}(s,t)T(s,t)x\,dsdt. \tag{3.2}$$

According to the equalities (3.1) and (3.2), we conclude by the usual Schwartz lemma that

$$A_2A_1y(x,\varphi) = A_1A_2y(x,\varphi), \quad \forall x \in E, \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{3.3}$$

Consider the set $Y := \{y(x,\varphi) \ : \ x \in E \text{ et } \varphi \in \mathcal{D}(\Omega)\}$ and let F be the linear span of Y. According to (3.3), we have $F \subseteq D(A_1A_2) \cap D(A_2A_1)$ and $A_1A_2u = A_2A_1u$, for all $u \in F$. So, to finish the proof, it is sufficient to show that F is dense in E. To get a contradiction, suppose that F is not dense in E. Then by virtue of the Hahn-Banach theorem, there exists a nonzero continuous linear functional x^* on E, such that x^* vanishes on F. Therefore, we have

$$x^*(y(x,\varphi)) = 0, \quad \forall x \in E, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

In particular, for all $x \in E$, we have

$$\int_0^{+\infty}\int_0^{+\infty} \varphi(s,t)x^*(T(s,t)x)\,dsdt = 0, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

For all $(s,t) \in \mathbb{R}_+^2$, we set $f(s,t) = x^*(T(s,t)x)$. The function $f : \mathbb{R}_+^2 \longrightarrow \mathbb{C}$ is continuous on $\mathbb{R}_+^2$ and has the following property:

$$\int_0^{+\infty}\int_0^{+\infty} \varphi(s,t)f(s,t)\,dsdt = 0, \quad \forall \varphi \in \mathcal{D}(\Omega).$$

By a density argument, it follows that $f(s,t) = 0$, for all $(s,t) \in \mathbb{R}_+^2$. In particular, we get $x^*(x) = f(0,0) = 0$, for every $x \in E$, which is a contradiction.

(ii) We know that $T(s,t) = L(s)R(t)$, where $(L(s))_{s\geq 0}$, $(R(t))_{t\geq 0}$ are one-parameter C_0-semigroups such that $L(s)R(t) = R(t)L(s)$, for all $s,t \geq 0$.

By (Theorem 1.10, p. 55 in) [10], there exist $\omega_1, \omega_2 \geq 0$, such that for all complex numbers λ and μ satisfying $Re(\lambda) > \omega_1$ and $Re(\mu) > \omega_2$, we have

$$R(\lambda, A_1) = \int_0^{\infty} \mathsf{e}^{-\lambda s}L(s)ds \ \text{ and } \ R(\mu, A_2) = \int_0^{\infty} \mathsf{e}^{-\mu t}R(t)dt.$$

Hence for $Re(\lambda), Re(\mu) > \omega$, where $\omega = \max(\omega_1, \omega_2)$, we have

$$\begin{aligned} R(\lambda, A_1)R(\mu, A_2) &= \left(\int_0^\infty \mathsf{e}^{-\lambda s} L(s)ds\right)\left(\int_0^\infty \mathsf{e}^{-\mu t} R(t)dt\right) \\ &= \int_0^\infty \left(\int_0^\infty \mathsf{e}^{-\lambda s} L(s)ds\right) R(t)dt \\ &= \int_0^\infty \left(\int_0^\infty \mathsf{e}^{-\lambda s-\mu t} L(s)R(t)ds\right) dt. \end{aligned}$$

By similar arguments, we obtain

$$R(\mu, A_2)R(\lambda, A_1) = \int_0^\infty \left(\int_0^\infty \mathsf{e}^{-\lambda s-\mu t} R(t)L(s)dt\right) ds.$$

According to the commutativity of $R(t)$ and $L(s)$ (for all $s, t \in \mathbb{R}_+$) and by using Fubini's theorem, we obtain $R(\lambda, A_1)R(\mu, A_2) = R(\mu, A_2)R(\lambda, A_1)$. This completes the proof. □

Remark. The results contained in (i) of Theorem 3.4 give some complements to a result obtained in [[9], Lemma 2.2].

4. Hille-Yoshida theorem

In this section, we establish an extension of the classical Hille-Yoshida (for one-parameter semigroups of contractions) to the context of two-parameter semigroups of contractions. Our methods of proof are different from those used in [[9], Theorem 2.4]. Another extension of Hille-Yoshida theorem was established in [[4], Theorem 2.11].

Theorem 4.1. *Let $A : D(A) \subseteq E \longrightarrow E \times E$ be a linear operator. Then the following assertions are equivalent :*

(a) A is the infinitesimal generator of a two-parameter C_0-semigroup of contractions on E.

(b) There exists two linear operators $A_1 : D(A_1) \longrightarrow E$ and $A_2 : D(A_2) \longrightarrow E$ which are closed and densely defined, such that

(i) $D(A) = D(A_1) \cap D(A_2)$ and for all $x \in D(A)$, $Ax = (A_1x, A_2x)$.

(ii) $\mathbb{R}_+^* \subseteq \rho(A_1) \cap \rho(A_2)$.

(iii) Ther exists $\omega > 0$, such that $R(\lambda, A_1)$ and $R(\mu, A_2)$ commute for all $\lambda, \mu > \omega$.

(iv) For all $\lambda > 0$, we have $\|R(\lambda, A_1)\| \leq \dfrac{1}{\lambda}$ and $\|R(\lambda, A_2)\| \leq \dfrac{1}{\lambda}$

Proof. $(a) \Longrightarrow (b)$. We suppose that A is the infinitesimal generator of a two-parameter C_0-semigroup of contractions $(T(s,t))_{s,t\geq 0}$, with $T(s,t) = L(s)R(t)$, for all $s, t \geq 0$. Then $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$ are one-parameter C_0-semigroups of contractions on E. Let A_1 and A_2 be respectively the infinitesimal generators of $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$. Then the second part of (i) is abvious and the first part of (i) results from Theorem 3.1.

Hille-Yoshida theorem for one-parameter semigroups of contractions ensures the properties (ii) and (iv).

The property (iii) is implied by the statement (ii) of Theorem 3.4.

$(b) \Longrightarrow (a)$. We suppose that the conditions (i), (ii), (iii) and (iv) are satisfied. Then by the Hille-Yoshida theorem for one parameter semigroups of contractions, we know that both conditions (ii) and (iv) infer that A_1 and A_2 are infinitesimal generators of one-parameter semigroups of contractions (called) $(L(s))_{s\geq 0}$ and $(R(t))_{t\geq 0}$.

Now, by using Corollary 3.5 of [11], for all $s,t \geq 0$, we have the following equalities:

$$L(s)x = \lim_{\lambda\to\infty} \mathsf{e}^{sA_{1,\lambda}} x$$

and

$$R(t) = \lim_{\mu\to\infty} \mathsf{e}^{tA_{2,\mu}} x,$$

for all $x \in E$, where $A_{1,\lambda}$ and $A_{2,\mu}$ are respectively the Yoshida approximation of A_1 and A_2, which are given (respectively) by

$$A_{1,\lambda} = \lambda A_1 R(\lambda, A_1) = \lambda^2 R(\lambda, A_1) - \lambda I$$

and

$$A_{2,\mu} = \mu A_2 R(\mu, A_2) = \mu^2 R(\mu, A_2) - \mu I.$$

The condition (iii) ensures that $A_{1,\lambda}$ and $A_{2,\mu}$ commute for all $\lambda, \mu > \omega$. Therefore, the bounded linear operators $\mathsf{e}^{sA_{1,\lambda}}$ and $\mathsf{e}^{tA_{2,\mu}}$ are also commuting for all $\lambda, \mu > \omega$. Hence, for all $x \in E$, we have

$$\begin{aligned}
L(s)R(t)x &= \lim_{\lambda\to\infty} \mathsf{e}^{sA_{1,\lambda}} R(t)x \\
&= \lim_{\lambda\to\infty} \mathsf{e}^{sA_{1,\lambda}} \big(\lim_{\mu\to\infty} \mathsf{e}^{tA_{2,\mu}} x\big) \\
&= \lim_{\lambda\to\infty} \big(\lim_{\mu\to\infty} \mathsf{e}^{sA_{1,\lambda}} \mathsf{e}^{tA_{2,\mu}} x\big) \\
&= \lim_{\lambda\to\infty} \big(\lim_{\mu\to\infty} \mathsf{e}^{tA_{2,\mu}} \mathsf{e}^{sA_{1,\lambda}} x\big) \\
&= \lim_{\lambda\to\infty} R(t)\mathsf{e}^{sA_{1,\lambda}} x \\
&= R(t) \lim_{\lambda\to\infty} \mathsf{e}^{sA_{1,\lambda}} x = R(t)L(s)x
\end{aligned}$$

Finally, we set $T(s,t) := L(s)R(t)$, for all $s,t \geq 0$, then $(T(s,t))_{s,t\geq 0}$ is a two-parameter C_0-semigroup of contractions on E.

Let us denote B the infinitesimal generator of $(T(s,t))_{s,t\geq 0}$. We know, by Theorem 2.2, that $D(B) = D(A_1) \cap D(A_2)$. Thus, for all $x \in D(B)$, we have $Bx = (A_1 x, A_2 x)$. Hence, by using (i), we conclude that $B = A$ and this ends the proof. □

5. Applications and examples

5.1. **Product semigroup.** Let E a Banach space and let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be strongly one-parameter semigroups on E. We set $U(t) = S(t)T(t)$ for all $t \geq 0$. The questions are:

is U a C_0-one-parameter semigroup on E ? and then what is its infinitesimmal generator ?

These questions were studied by H. F. Trotter in his paper [12] published in 1959. Some supplementary informations and other details on this problem are given in the book [10] of R. Nagel and K.-J. Engel.

A third question may be adressed: When the semigroups $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ are commuting ?

Answers to this question were given by N. H. Abdelaziz in the papers [2] and [3].

Before stating our contribution devoted to all these questions, we need to recall the following important and technical lemma taken from the book [10].

Lemma 5.1 (B.16 Lemma, [10], p. 520)**.** *Let J be some real interval and $P, Q : J \to \mathcal{B}(X)$ be two strongly continuous operator-valued functions defined on J. Moreover, assume that $P(\cdot)x : J \to X$ and $Q(\cdot)x : J \to X$ are differentiable for all $x \in D$ for some subspace D of X, which is invariant under Q. Then $(PQ)(\cdot)x : J \to X$, defined by $(PQ)(t)x := P(t)Q(t)x$, is differentiable for every $x \in D$ and, for all $t_0 \in J$, we have*

$$\frac{d}{dt}\Big(P(\cdot)Q(\cdot)x\Big)(t_0) = \frac{d}{dt}\Big(P(\cdot)Q(t_0)x\Big)(t_0) + P(t_0)\left(\frac{d}{dt}Q(\cdot)x\right)(t_0).$$

Now, we are ready to state our contribution.

Theorem 5.2. *Let E a real or complex Banach space and let $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ be one-parameter C_0-semigroups on E. Let A (resp. B) the infinitesimal generator of $(S(t))_{t\geq 0}$ (resp. of $(T(t))_{t\geq 0}$). We denote $\Delta = D(A) \cap D(B)$ and forall $t \geq 0$, we set $U(t) = S(t)T(t)$ and $V(t) := T(t)S(t)$.*

We define Q by setting: $Q(s,t) := S(s)T(t)$, for all $s, t \in \mathbb{R}_+$.

Then the following assertions are equivalent

(i) $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ are semigroups, Δ is dense in E and $U(t)(\Delta) \subset \Delta$, for all $t \geq 0$.

(ii) $S(s)T(t) = T(t)S(s)$, forall $s, t \geq 0$.

(iii) Q is a two parameter C_0-semigroup.

Proof. (i) $\Longrightarrow$ (ii): Since $(S(t))_{t\geq 0}$ and $(T(t))_{t\geq 0}$ are one-parameter C_0-semigroups on E and as $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ are supposed to be semigroups, then it easy to see that $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ are strongly continuous. Let Ω and Φ be respectively the infinitesimal generators of U and V. According to [10] (see 5.15 page 44), it is sufficient to prove that $U(t) = V(t)$, forall $t \geq 0$. To this end, we start by proving the following statements:

$$\Delta \subseteq D(\Omega) \cap D(\Phi) \quad \text{and} \quad \Omega x = \Phi x = Ax + Bx, \text{ for all } x \in \Delta.$$

Let $x \in \Delta$, then we have

$$\begin{aligned} \lim_{t\to 0} \frac{U(t)x - x}{t} &= \lim_{t\to 0} \frac{S(t)T(t)x - x}{t} \\ &= \lim_{t\to 0} S(t)\frac{T(t)x - x}{t} + \lim_{t\to 0} \frac{S(t)x - x}{t}. \end{aligned}$$

We know that

$$\lim_{t\to 0} \frac{T(t)x - x}{t} = Bx, \quad \text{and} \quad \lim_{t\to 0} \frac{S(t)x - x}{t} = Ax.$$

For all $t > 0$, we set $B(t) := \dfrac{T(t) - I}{t}$, then we have

$$\|S(t)B(t)x - Bx\| \leq \|S(t)\|\|B(t)x - Bx\| + \|S(t)Bx - Bx\|. \tag{5.1}$$

Since S is a C_0-semigroup, there exist a constant $M \geq 1$ and a real number ω such that $\|S(t)\| \leq Me^{t\omega}$ for all $t \geq 0$. By using this in the inequality (5.1), we obtain $\lim_{t\to 0} \|S(t)B(t)x - Bx\| = 0$. From which, we deduce that $x \in D(\Omega)$ and $\Omega x = Ax + Bx$.

By the same way, we show that $x \in D(\Phi)$ and $\Phi x = Ax + Bx$.

Now let $x \in \Delta$ and $t > 0$ and consider the application $\alpha : [0, t] \longrightarrow E$ defined by $\alpha(s) = U(t-s)V(s)x$. Then according to lemma B.16 of [10], for all $s_0 \in [0, t]$, we have

$$\frac{d\alpha}{ds}(s_0) = \frac{dU(t-s)}{ds}(s_0)V(s_0)x + U(t - s_0)\frac{dV(s)}{ds}(s_0)x.$$

As U and V are C_0-semigroups, then we have

$$\frac{dU(t-s)}{ds}(s_0)V(s_0)x = -\Omega U(t-s)V(s_0)x = -U(t-s)\Omega V(s_0)x$$

and

$$\frac{dV(s)}{ds}(s_0)x = \Phi V(s_0)x = V(s_0)\Phi x.$$

Therefore, we have

$$\frac{d\alpha}{ds}(s_0) = -U(t - s_0)\Omega V(s_0)x + U(t - s_0)\Phi V(s_0)x.$$

Since Δ is invariant under V, then $V(s_0)x \in \Delta$ and then $\Omega V(s_0)x = \Phi V(s_0)x$. Thus we have $\dfrac{d\alpha}{ds}(s_0) = 0$, for all $s_0 \in [0, t]$. This implies that α is constant on $[0, t]$. Therefore we have $\alpha(0) = \alpha(t)$ and consequently we get $U(t)x = V(t)x$, for all $x \in \Delta$. Since Δ is dense in E, then we obtain $U(t)x = V(t)x$, for all $x \in E$.

(ii) $\Longrightarrow$ (iii): This implication is a consequence from the statement (ii) of Theorem 2.1.

(iii) $\Longrightarrow$ (i): If Q is a two-parameter C_0-semigroup, then by Theorem 2.1, we know that U and V are C_0-semigroups. By virtue of Theorem 3.3, we know that Δ is dense in E. Also, by Theorem 3.1, we know that Δ is invariant under $U(t)$ (and $V(t)$) for all $t \geq 0$. This completes the proof. □

5.2. **Examples.**

Example 5.3. Let $(S(t))_{t\geq 0}$ be a one-parameter C_0-semigroup with generator B and let $\alpha, \beta \in \mathbb{R}^+$. For all $(s,t) \in \mathbb{R}^2$, we put $T(s,t) = S(\alpha t + \beta s)$, then it is easy to see that $(T(s,t))_{s,t\geq 0}$ is a two-parameter C_0-semigroup with generator A defined on its domain $D(A) = D(B)$ by $Ax = (Bx, Bx)$ for all $x \in D(A)$.

Example 5.4. Let $(S(t))_{t\geq 0}$ a one-parameter C_0-semigroup with generator B. Let F be another Banach-space and let $V : F \longrightarrow E$ be an isomorphism. For all $t \in \mathbb{R}^+$, we set $S_V(t) := V^{-1}S(t)V$. It is easy to see that $(S_V(t))_{t\geq 0}$ is a one-parameter C_0-semigroup on F with generator $B_V : V^{-1}BV$ defined on its domain $D(B_V)$ given by $D(B_V) := \{y \in F : Vy \in D(B)\}$.

Let $(T(s,t))_{s,t\geq 0}$ a two-parameter C_0-semigroup on E. We know that there exist two commuting one-parameter C_0-semigroups $(L(s))_{t\geq 0}$ and $(R(t))_{t\geq 0}$ such that $T(s,t)=L(s)R(t)$, for all $(s,t)\in\mathbb{R}^2_+$.

For all $(s,t)\in\mathbb{R}^2_+$, we set $T_V(s,t)=V^{-1}T(s,t)V$. Then $(T_V(s,t))_{s,t\geq 0}$ is a two-parameter C_0-semigroup on the Banach space F and we have $T_V(s,t)=L_V(s)R_V(t)$ for all $s,t\geq 0$. Let A_V be the infinitesimal generator of the two-parameter C_0-semigroup $(T_V(s,t))_{s,t\geq 0}$. Then $D(A_V)=D(A_{1,V})\cap D(A_{2,V})$, where $A_{i,V}:=V^{-1}A_iV$, for $i\in\{1,2\}$ and we have $A_Vx=(V^{-1}A_1Vx, V^{-1}A_2Vx)$ for all $x\in D(A_V)$.

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