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PRESENTATION MATRICES OF TORSION MODULES OVER POLYNOMIAL RINGS

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Abstract. Let R be a commutative local ring with unit. For $R[X]$-modules which are R-free of finite rank, we give presentation matrices which are square of minimal order. Some applications to modules over group rings are also given.

1. Introduction

All rings considered in this paper are supposed to be with unit. Let R be a commutative ring. An element of R is regular if it is a non-zero-divisor, and an ideal of R is not regular if it does not contain any regular element. A module M over the ring R is called a torsion module if all its elements are torsion elements, i.e., for each element $m \in M$ there exists a regular element $r \in R$ such that $rm = 0$.

The minimal number of generators of a finitely generated R-module M, which is denoted by $\mu_R(M)$, is the smallest cardinal of the generating families of M. If $M = (0)$, then we put $\mu_R(M) = 0$.

A generator system $\{x_1, x_2, ..., x_n\}$ of a finitely generated R-module M is called a CF-system if $M = \overset{n}{\underset{i=1}{\oplus}} (R/I_i)x_i$ with $I_i = \mathrm{Ann}_R x_i = \{r \in R \mid r.x_i = 0\}$ and $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \neq R$.

If an R-module M admits a finite projective resolution, the minimal length among all finite projective resolutions of M is called its projective dimension. If M does not admit a finite projective resolution, then by convention the projective dimension is said to be infinite. (See, for example, [11, 12]).

Let M be a finitely presented R-module. A presentation matrix of the module M associated to a generator system of M over R means relations matrix in the terminology of [2]. Let $\mathbb{M} \in M_{n,q}(R)$ be the presentation matrix of the module M associated to a generator system $\{x_1, x_2, ..., x_n\}$ of M over R. For a positive integer $k \in \{0, 1, ..., n-1\}$, the k^{th} Fitting ideal of M is defined to be the ideal $F_k(M)$ generated by the determinants of all $(n-k) \times (n-k)$-submatrices of the matrix $\mathbb{M}$, if $(n-k) \leq q$, and otherwise $F_k(M) = 0$. For a positive integer $k \geq n$, we define $F_k(M)$ by $F_k(M) = R$. These ideals are independent of the choice of

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the generator system $\{x_1, x_2, ..., x_n\}$ of M. (See, for example, [2, 7, 8, 11], for more details about Fitting ideals).

An exact sequence of $R[X]$-modules given in [1, 3, 4], and called characteristic exact sequence, gives a presentation matrix of the $R[X]$-module M whenever M is R-free. This matrix is square, but its order is not necessarily minimal.

Now, let R be a commutative local ring with maximal ideal $\mathfrak{p}$ and residue field K. For a polynomial $P \in R[X]$, $\overline{P}$ will denote the reduction of P modulo $\mathfrak{p}$. Let M be an $R[X]$-module. Then, $M/\mathfrak{p}M$ is a $K[X]$-module. Note that an $R[X]$-module that is R-free of finite rank is necessarily of torsion. An $R[X]$-module M which is finitely generated is called of type $(s_1, s_2, ..., s_n)$, if

$$M/\mathfrak{p}M \cong \bigoplus_{i=1}^{n} K[X]/(\overline{\phi_i}),$$

where $\overline{\phi_i}$ are the invariant factors of $M/\mathfrak{p}M$ such that $\deg(\overline{\phi_i}) = s_i$, for all $i \in \{1, 2, ..., n\}$.

Let M be an $R[X]$-module which is R-free of finite rank and of type $(s_1, s_2, ..., s_n)$, and let $\{x_1, x_2, ..., x_n\}$ be a generator system of M over $R[X]$ such that $\{\overline{x_1}, \overline{x_2}, ..., \overline{x_n}\}$ is a CF-system of $M/\mathfrak{p}M$ over $K[X]$, where $\overline{x_i} = x_i + \mathfrak{p}M$ for all $i \in \{1, 2, ..., n\}$. In section 2, we show that $\{X^j x_i \mid 1 \le i \le n,\ 0 \le j < s_i\}$ is a basis of M over R. In section 3, we give a generator system of the kernel of the homomorphism of $R[X]$-modules defined from $(R[X])^n$ to M by $e_i \mapsto x_i$, where $\{e_1, e_2, ..., e_n\}$ is the canonical basis of $(R[X])^n$. As a result of the knowledge of this generator system we give a presentation matrix $\mathbb{M}$ of the $R[X]$-module M associated to the generator system $\{x_1, x_2, ..., x_n\}$. This presentation matrix $\mathbb{M}$ is square of order n. Also, we show that M admits a presentation $0 \to (R[X])^n \to (R[X])^n \to M \to 0$, where n is minimal (i.e., n is the smallest nonzero natural number k such that there exists an exact sequence of form $0 \to (R[X])^k \to (R[X])^k \to M \to 0$). Finally, we show that the square presentation matrix $\mathbb{M}$ is of minimal order. Section 4 is devoted to some applications: let K be a commutative field of characteristic $p > 0$, let $G = G_1 \times G_2$, where G_1 and G_2 are two finite cyclic p-groups, and let M be a finitely generated $K[G]$-module. We show that if M seen as $K[G_1]$-module is free, then for all $k \in \{0, 1, ..., \mu_{K[G]}(M) - 1\}$, the Fitting ideal $F_k(M)$ of M is not regular. This implies that M have projective dimension > 1.

2. Finitely generated torsion $R[X]$-modules

We begin this section by recalling the concept of a CF-module.

Definition 2.1. Let R be a commutative ring. An R-module M is called a CF-module if $M \cong \bigoplus_{i=1}^{n} R/I_i$, where I_i are ideals of R such that $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \neq R$.

This notion of CF-module appears in [13] and [14], under the appellation "canonical form for a module".

Let R be a commutative ring. Suppose $I_1 \subseteq I_2 \subseteq ... \subseteq I_n$ and $J_1 \subseteq J_2 \subseteq ... \subseteq J_m$ are two sequences of ideals in R. We assume $I_n \neq R \neq J_m$. If $\bigoplus_{i=1}^{n} R/I_i \cong$

$\overset{m}{\underset{j=1}{\oplus}} R/J_j$ as R-modules, then $n = m$ and $I_i = J_i$ for all $i \in \{1, 2, ..., n\}$ (see [2, Lemma 15.13]).

Now we give some useful propositions.

Proposition 2.2. *Let R be a commutative ring and M a finitely generated R-module. Then, M is a CF-module if and only if M admits a generator system which is a CF-system.*

Proof. Assume that M is a CF-module. Then, there exist ideals $I_1, I_2, ..., I_n$ of R such that $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \neq R$, and $M \cong \overset{n}{\underset{i=1}{\oplus}} R/I_i$. Let φ be the isomorphism from M to $\overset{n}{\underset{i=1}{\oplus}} R/I_i$. Let $x_i = \varphi^{-1}(1 + I_i)$, for all $i \in \{1, 2, ..., n\}$. Then, it is not difficult to see that $M = \overset{n}{\underset{i=1}{\oplus}} (R/I_i)x_i$, and therefore, $\{x_1, x_2, ..., x_n\}$ is a generator system of M which is a CF-system.
Now, assume that M admits a generator system $\{x_1, x_2, ..., x_n\}$ which is a CF-system. We have $M = \overset{n}{\underset{i=1}{\oplus}} (R/I_i)x_i$, where I_1, I_2, ..., I_n are ideals of R such that $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \neq R$, and the homomorphism $\varphi : \overset{n}{\underset{i=1}{\oplus}} R/I_i \longrightarrow \overset{n}{\underset{i=1}{\oplus}} (R/I_i)x_i$ defined by $\varphi(\overline{r_1}, \overline{r_2}, ..., \overline{r_n}) = \sum_{i=1}^{n} r_i x_i$, where $\overline{r_i} = r_i + I_i$, for all $i \in \{1, 2, ..., n\}$, is an isomorphism. So, $M \cong \overset{n}{\underset{i=1}{\oplus}} R/I_i$. Therefore, M is a CF-module. □

From the proof of Proposition 2.2, we note that if $M = \overset{n}{\underset{i=1}{\oplus}} (R/I_i)x_i$, where x_1, x_2, ..., x_n are elements of M and I_1, I_2, ..., I_n are ideals of R such that $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \neq R$, then $M \cong \overset{n}{\underset{i=1}{\oplus}} R/I_i$.

Proposition 2.3. *Let R be a commutative local ring with maximal ideal $\mathfrak{p}$ and residue field K. Let M be a finitely generated torsion $R[X]$-module. Every CF-system of $M/\mathfrak{p}M$ over $K[X]$ induces a generator system of M over $R[X]$.*

Proof. Suppose that M is of type $(s_1, s_2, ..., s_n)$ and let $S = \{x_1, x_2, ..., x_n\}$ be a system of M such that $\{\overline{x_1}, \overline{x_2}, ..., \overline{x_n}\}$ is a CF-system of $M/\mathfrak{p}M$ over $K[X]$. Therefore, $\{\overline{X^j x_i} \mid 1 \leq i \leq n,\ 0 \leq j < s_i\}$ is a basis of $M/\mathfrak{p}M$ over K. So, by Nakayama's lemma, $\{X^j x_i \mid 1 \leq i \leq n,\ 0 \leq j < s_i\}$ is a generator system of M over R. Therefore, S is a generator system of M over $R[X]$. □

Proposition 2.4. *Let R be a commutative local ring with maximal ideal $\mathfrak{p}$ and residue field K. Let M be a finitely generated torsion $R[X]$-module. Then, $\mu_{R[X]}(M) = \mu_{K[X]}(M/\mathfrak{p}M)$.*

Proof. Indeed, $M/\mathfrak{p}M \cong K[X] \otimes_{R[X]} M$ and therefore $\mu_{R[X]}(M) \geq \mu_{K[X]}(M/\mathfrak{p}M)$. By Proposition 2.3, every CF-system of $M/\mathfrak{p}M$ over $K[X]$ induces a generator system of M over $R[X]$ and therefore $\mu_{R[X]}(M) \leq \mu_{K[X]}(M/\mathfrak{p}M)$. In conclusion $\mu_{R[X]}(M) = \mu_{K[X]}(M/\mathfrak{p}M)$. □

From Theorems 5.10 of [2] we have the following lemma.

Lemma 2.5. *Let R be a commutative ring (not necessarily local), and M a free R-module of finite rank. Any minimal generator system of M is a basis of this module.*

Proposition 2.6. *Let R be a commutative local ring with maximal ideal $\mathfrak{p}$ and residue field K. Let M be a finitely generated torsion $R[X]$-module of type $(s_1, s_2, ..., s_n)$. If $\{x_1, x_2, ..., x_n\}$ is a generator system of M over $R[X]$ such that $\{\overline{x_1}, \overline{x_2}, ..., \overline{x_n}\}$ is a CF-system of $M/\mathfrak{p}M$ over $K[X]$, then $B = \{X^j x_i \mid 1 \leq i \leq n,\ 0 \leq j < s_i\}$ is a minimal generator system of M over R. If in addition we assume that M is R-free of finite rank, then B is a basis of M over R.*

Proof. Indeed, let $\{x_1, x_2, ..., x_n\}$ be a generator system of M over $R[X]$ such that $\{\overline{x_1}, \overline{x_2}, ..., \overline{x_n}\}$ is a CF-system of $M/\mathfrak{p}M$ over $K[X]$. Therefore, $\{\overline{X^j x_i} \mid 1 \leq i \leq n,\ 0 \leq j < s_i\}$ is a basis of $M/\mathfrak{p}M$ over K. So, by Nakayama's lemma, $B = \{X^j x_i \mid 1 \leq i \leq n,\ 0 \leq j < s_i\}$ is a minimal generator system of M over R. Now, assume that M is R-free of finite rank. Then, by Lemma 2.5, B is a basis of M over R. □

3. Presentation matrices of torsion $R[X]$-modules

Let R be a commutative local ring with maximal ideal $\mathfrak{p}$ and residue field K. Let M be a finitely generated torsion $R[X]$-module of type $(s_1, s_2, ..., s_n)$. Let $\{x_1, x_2, ..., x_n\}$ be a generator system of M over $R[X]$ such that $\{\overline{x_1}, \overline{x_2}, ..., \overline{x_n}\}$ is a CF-system of $M/\mathfrak{p}M$ over $K[X]$. Then, by Proposition 2.6, $\{X^j x_i \mid 1 \leq i \leq n,\ 0 \leq j < s_i\}$ is a generator system of M over R. For all $i \in \{1, 2, ..., n\}$, let P_i be a polynomial of degree s_i of $R[X]$ and of dominant coefficient invertible. There exists $\{V_{i,j} \mid 1 \leq i, j \leq n\}$ a set of elements of $R[X]$ with $\deg(V_{i,j}) < s_j$ for $i, j \in \{1, 2, ..., n\}$, such that, for each i, $P_i.x_i = \sum\limits_{j=1}^{n} V_{i,j} x_j$. Let $\{e_1, e_2, ..., e_n\}$ be the canonical basis of $(R[X])^n$. Let $\varphi : (R[X])^n \to M$ be the homomorphism of $R[X]$-modules defined by $\varphi(e_i) = x_i$, for all $i \in \{1, 2, ..., n\}$. The kernel of φ will be denoted by $\ker(\varphi)$. In the rest of this section we keep these notations, and unless otherwise stated we suppose that M is R-free of finite rank. In this case, by Proposition 2.6, $\{X^j x_i \mid 1 \leq i \leq n,\ 0 \leq j < s_i\}$ is a basis of M over R.

To give a presentation matrix of the module M we first show the following lemma.

Lemma 3.1. *Let $y_i = (V_{i,1}, ..., V_{i,i-1}, V_{i,i} - P_i, V_{i,i+1}, ..., V_{i,n})$, for all $i \in \{1, 2, ..., n\}$. Then, $\{y_1, y_2, ..., y_n\}$ is a generator system of* $\ker(\varphi)$.

Proof. It is obvious that $y_i \in \ker(\varphi)$ and the dominant coefficient of $V_{i,i} - P_i$ is invertible in R, for all $i \in \{1, 2, ..., n\}$. Let $(U_1, U_2, ..., U_n) \in \ker(\varphi)$. There exists $(Q_{i,1}, R_{i,1}) \in (R[X])^2$ with $\deg(R_{i,1}) < s_i$, for all $i \in \{1, 2, ..., n\}$, such that

$$(D_{i,1}): \quad U_i = Q_{i,1}(V_{i,i} - P_i) + R_{i,1}.$$

We have

$$\begin{aligned}(U_1, U_2, ..., U_n) \in \ker(\varphi) \;&\Leftrightarrow\; \sum_{i=1}^{n} ((Q_{i,1}(V_{i,i} - P_i))x_i + Q_{i,1} \sum_{\substack{j=1\\ j\neq i}}^{n} V_{i,j}x_j \\ &\quad -Q_{i,1} \sum_{\substack{j=1\\ j\neq i}}^{n} V_{i,j}x_j + R_{i,1}x_i) = 0 \\ &\Leftrightarrow\; \sum_{i=1}^{n} (Q_{i,1}((V_{i,i} - P_i)x_i + \sum_{\substack{j=1\\ j\neq i}}^{n} V_{i,j}x_j) \\ &\quad +(R_{i,1} - \sum_{\substack{j=1\\ j\neq i}}^{n} Q_{j,1}V_{j,i})x_i) = 0 \\ &\Leftrightarrow\; (E_1): \;\; \sum_{i=1}^{n} (R_{i,1} - \sum_{\substack{j=1\\ j\neq i}}^{n} Q_{j,1}V_{j,i})x_i = 0.\end{aligned}$$

There exists $(Q_{i,2}, R_{i,2}) \in (R[X])^2$ with $\deg(R_{i,2}) < s_i$, for all $i \in \{1, 2, ..., n\}$, such that

$$(E_{i,2}): \;\; R_{i,1} - \sum_{\substack{j=1\\ j\neq i}}^{n} Q_{j,1}V_{j,i} = Q_{i,2}(V_{i,i} - P_i) + R_{i,2}.$$

From the above we see that

$$(D_{i,2}): \;\; U_i = (Q_{i,1} + Q_{i,2})(V_{i,i} - P_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} Q_{j,1}V_{j,i} + R_{i,2},$$

and that

$$\sum_{i=1}^{n} (R_{i,1} - \sum_{\substack{j=1\\ j\neq i}}^{n} Q_{j,1}V_{j,i})x_i = 0 \Leftrightarrow (E_2): \;\; \sum_{i=1}^{n} (R_{i,2} - \sum_{\substack{j=1\\ j\neq i}}^{n} Q_{j,2}V_{j,i})x_i = 0.$$

By induction, for any nonzero natural number k, there exists $(Q_{i,k}, R_{i,k}) \in (R[X])^2$ with $\deg(R_{i,k}) < s_i$, for all $i \in \{1, 2, ..., n\}$, such that

$$(E_{i,k}): \;\; R_{i,k-1} - \sum_{\substack{j=1\\ j\neq i}}^{n} Q_{j,k-1}V_{j,i} = Q_{i,k}(V_{i,i} - P_i) + R_{i,k},$$

where $R_{i,0} = U_i$ and $Q_{i,0} = 0$, and we have

$$(D_{i,k}): \;\; U_i = (\sum_{r=1}^{k} Q_{i,r})(V_{i,i} - P_i) + \sum_{\substack{j=1\\ j\neq i}}^{n} (\sum_{r=1}^{k-1} Q_{j,r})V_{j,i} + R_{i,k},$$

and

$$(E_k): \;\; \sum_{i=1}^{n} (R_{i,k} - \sum_{\substack{j=1\\ j\neq i}}^{n} Q_{j,k}V_{j,i})x_i = 0.$$

Let $i \in \{1, 2, ..., n\}$ and let k be a nonzero natural number. In comparing the degrees in the equality $(E_{i,k+1})$, we see that if $Q_{j,k+1} \neq 0$, then there exist $j \in \{1, 2, ..., n\}$ and $j \neq i$ such that $Q_{j,k} \neq 0$. Now, if for all nonzero natural number k, there exists at least one $j \in \{1, 2, ..., n\}$ such that $Q_{j,k} \neq 0$, then we set $u_k = \max\{\deg(Q_{i,k}) \mid 1 \leq i \leq n, \; Q_{i,k} \neq 0\}$, and in comparing the degrees in the equalities $(E_{i,k+1})$, $i \in \{1, 2, ..., n\}$, we see that $u_k > u_{k+1}$. Therefore, we have a strictly decreasing sequence of nonzero natural numbers. Which is impossible. So, there exists a nonzero natural number k_0 such that $Q_{j,k_0} = 0$, for all $j \in \{1, 2, ..., n\}$. Then, we have

$$(D_{i,k_0}): \quad U_i = (\sum_{r=1}^{k_0-1} Q_{i,r})(V_{i,i} - P_i) + \sum_{\substack{j=1 \\ j\neq i}}^{n} (\sum_{r=1}^{k_0-1} Q_{j,r})V_{j,i} + R_{i,k_0}, \forall i \in \{1, 2, ..., n\}.$$

And from $(E_{k_0}): \quad \sum_{i=1}^{n} (R_{i,k_0} - \sum_{\substack{j=1 \\ j\neq i}}^{n} Q_{j,k_0} V_{j,i})x_i = 0$ we see that $R_{i,k_0} = 0$, for all $i \in \{1, 2, ..., n\}$. Therefore,

$$U_i = (\sum_{r=1}^{k_0-1} Q_{i,r})(V_{i,i} - P_i) + \sum_{\substack{j=1 \\ j\neq i}}^{n} (\sum_{r=1}^{k_0-1} Q_{j,r})V_{j,i}, \forall i \in \{1, 2, ..., n\}.$$

So,

$$\begin{aligned}
(U_1, U_2 ..., U_n) &= \sum_{i=1}^{n} ((\sum_{r=1}^{k_0-1} Q_{i,r})(V_{i,i} - P_i))e_i + \sum_{i=1}^{n} (\sum_{\substack{j=1 \\ j\neq i}}^{n} (\sum_{r=1}^{k_0-1} Q_{j,r})V_{j,i})e_i \\
&= \sum_{i=1}^{n} ((\sum_{r=1}^{k_0-1} Q_{i,r})(V_{i,i} - P_i))e_i + \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i\neq j}}^{n} (\sum_{r=1}^{k_0-1} Q_{j,r})V_{j,i}e_i \\
&= \sum_{i=1}^{n} ((\sum_{r=1}^{k_0-1} Q_{i,r})(V_{i,i} - P_i))e_i + \sum_{j=1}^{n} (\sum_{r=1}^{k_0-1} Q_{j,r}) \sum_{\substack{i=1 \\ i\neq j}}^{n} V_{j,i}e_i \\
&= \sum_{i=1}^{n} ((\sum_{r=1}^{k_0-1} Q_{i,r})(V_{i,i} - P_i))e_i + \sum_{i=1}^{n} (\sum_{r=1}^{k_0-1} Q_{i,r}) \sum_{\substack{j=1 \\ j\neq i}}^{n} V_{i,j}e_j \\
&= \sum_{i=1}^{n} (\sum_{r=1}^{k_0-1} Q_{i,r})y_i.
\end{aligned}$$

In conclusion $\{y_1, y_2, ..., y_n\}$ is a generator system of $\ker(\varphi)$. □

Now we can give a presentation matrix of the module M.

Theorem 3.2. *The square matrix*

$$\begin{bmatrix}
V_{1,1} - P_1 & V_{1,2} & V_{1,3} & \cdots & V_{1,n} \\
V_{2,1} & V_{2,2} - P_2 & V_{2,3} & \cdots & V_{2,n} \\
V_{3,1} & V_{3,2} & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & V_{n-1,n} \\
V_{n,1} & V_{n,2} & \cdots & V_{n,n-1} & V_{n,n} - P_n
\end{bmatrix}$$

is a presentation matrix of the module M associated to its generator system $\{x_1, x_2, ..., x_n\}$ over $R[X]$.

Proof. Obvious by Lemma 3.1. □

Corollary 3.3. *The square matrix*

$$\begin{bmatrix} \overline{V_{1,1}-P_1} & \overline{V_{1,2}} & \overline{V_{1,3}} & \cdots & \overline{V_{1,n}} \\ \overline{V_{2,1}} & \overline{V_{2,2}-P_2} & \overline{V_{2,3}} & \cdots & \overline{V_{2,n}} \\ \overline{V_{3,1}} & \overline{V_{3,2}} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \overline{V_{n-1,n}} \\ \overline{V_{n,1}} & \overline{V_{n,2}} & \cdots & \overline{V_{n,n-1}} & \overline{V_{n,n}-P_n} \end{bmatrix}$$

is a presentation matrix of the module $M/\mathfrak{p}M$ associated to its generator system $\{\overline{x_1}, \overline{x_2}, ..., \overline{x_n}\}$ over $K[X]$.

Proof. Let $\{\overline{e_1}, \overline{e_2}, ..., \overline{e_n}\}$ be the canonical basis of $(K[X])^n$. Let $\overline{\varphi} : (K[X])^n \to M/\mathfrak{p}M$ be the homomorphism of $K[X]$-modules defined by $\varphi(\overline{e_i}) = \overline{x_i}$. Let $\overline{y_i} = (\overline{V_{i,1}}, ..., \overline{V_{i,i-1}}, \overline{V_{i,i} - P_i}, \overline{V_{i,i+1}}, ..., \overline{V_{i,n}})$. It is obvious that $\overline{y_i} \in \ker(\overline{\varphi})$, for all $i \in \{1, 2, ..., n\}$. Let $(\overline{U_1}, \overline{U_2}, ..., \overline{U_n}) \in \ker(\overline{\varphi})$.

$$\begin{aligned} \overline{\varphi}((\overline{U_1}, \overline{U_2}, ..., \overline{U_n})) = \overline{0} \quad &\Leftrightarrow \quad \sum_{i=1}^{n} \overline{U_i}\overline{x_i} = \overline{0} \\ &\Leftrightarrow \quad \sum_{i=1}^{n} \overline{U_i x_i} = \overline{0} \\ &\Leftrightarrow \quad \sum_{i=1}^{n} U_i x_i \in \mathfrak{p}M. \end{aligned}$$

So, there exist $p_1, p_2, ..., p_n \in \mathfrak{p}$ such that $\sum_{i=1}^{n} U_i x_i = \sum_{i=1}^{n} p_i x_i$. Therefore, $(U_1 - p_1, U_2 - p_2, ..., U_n - p_n) \in \ker(\varphi)$. By Lemma 3.1, there exist $q_1, q_2, ..., q_n \in R[X]$ such that $(U_1 - p_1, U_2 - p_2, ..., U_n - p_n) = \sum_{i=1}^{n} q_i y_i$. As for all $i \in \{1, 2, ..., n\}$ $p_i \in \mathfrak{p}$, then $(\overline{U_1}, \overline{U_2}, ..., \overline{U_n}) = \sum_{i=1}^{n} \overline{q_i}.\overline{y_i}$. So, $\{\overline{y_1}, \overline{y_2}, ..., \overline{y_n}\}$ is a generator system of $\ker(\overline{\varphi})$. Now, the rest of the proof is obvious. □

Remark 3.4. For all $i \in \{1, 2, ..., n\}$, let ϕ_i be a monic polynomial of $R[X]$ such that $\overline{\phi_i}$ are the invariant factors of $M/\mathfrak{p}M$. If for all $i \in \{1, 2, ..., n\}$, we take $P_i = -\phi_i$, then we have $\overline{0} = \overline{-\phi_i}.\overline{x_i} = \sum_{j=1}^{n} \overline{-V_{i,j}}\overline{x_j}$. So, $\overline{V_{i,j}} = \overline{0}$, for all $i, j \in \{1, 2, ..., n\}$. Therefore, the square matrix

$$\begin{bmatrix} \overline{\phi_1} & & & \overline{0} \\ & \overline{\phi_2} & & \\ & & \ddots & \\ \overline{0} & & & \overline{\phi_n} \end{bmatrix}$$

is a presentation matrix of the module $M/\mathfrak{p}M$ associated to its generator system $\{\overline{x_1}, \overline{x_2}, ..., \overline{x_n}\}$ over $K[X]$. This is an expected result since

$$M/\mathfrak{p}M \cong \bigoplus_{i=1}^{n} K[X]/(\overline{\phi_i}).$$

Assume that M is annihilated by a polynomial ϕ. Then, we can see M as $R[X]/(\phi)$-module.

Corollary 3.5. *Let $\mathbb{M} = (a_{ij}) \in M_{n\times n}(R[X]/(\phi))$, where $a_{ii} = V_{i,i} - P_i + \phi R[X]$ and $a_{ij} = V_{i,j} + \phi R[X]$ if $i \neq j$. Then, $\mathbb{M}$ is a presentation matrix of the module M associated to the generator system $\{x_1, x_2, ..., x_n\}$ of M over $R[X]/(\phi)$.*

Proof. Obvious by Thorem 3.2. □

We use the foregoing notations, but we not assume that M is R-free. Then, we have the following lemma.

Lemma 3.6. *Let $y_i = (V_{i,1}, ..., V_{i,i-1}, V_{i,i}-P_i, V_{i,i+1}, ..., V_{i,n})$, for all $i \in \{1, 2, ..., n\}$. Then, $\{y_1, y_2, ..., y_n\}$ is a family of elements of the $R[X]$-module $\ker(\varphi)$ which is free.*

Proof. For all $i \in \{1, 2, ..., n\}$, $y_i \in \ker(\varphi)$ and the dominant coefficient of $V_{i,i} - P_i$ is invertible in R. Let

$$\mathbb{M} = \begin{bmatrix} V_{1,1} - P_1 & V_{1,2} & V_{1,3} & \cdots & V_{1,n} \\ V_{2,1} & V_{2,2} - P_2 & V_{2,3} & \cdots & V_{2,n} \\ V_{3,1} & V_{3,2} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & V_{n-1,n} \\ V_{n,1} & V_{n,2} & \cdots & V_{n,n-1} & V_{n,n} - P_n \end{bmatrix}.$$

Then, Leibniz formula for determinants shows that $\det(\mathbb{M})$ is a polynomial of degree $\sum_{i=1}^{n} s_i$, and of dominant coefficient invertible. So, $\det(\mathbb{M})$ is a regular element. Therefore, $\{y_1, y_2, ..., y_n\}$ is free. □

Now, we return to the case where R is a commutative local ring and M is an $R[X]$-module which is R-free of finite rank. We have the following Theorem.

Theorem 3.7. *There exists an exact sequence of $R[X]$-modules of the form*

$$0 \to (R[X])^n \to (R[X])^n \xrightarrow{\varphi} M \to 0,$$

where $n = \mu_{R[X]}(M)$, and the family $\{y_1, y_2, ..., y_n\}$ mentioned in Lemma 3.1 is a basis of the $R[X]$-module $\ker(\varphi)$.

Proof. It is obvious that there exists an exact sequence of $R[X]$-modules

$$0 \to \ker(\varphi) \hookrightarrow (R[X])^n \xrightarrow{\varphi} M \to 0.$$

Now, by Lemmas 3.1 and 3.6, $\{y_1, y_2, ..., y_n\}$ is a basis of the $R[X]$-module $\ker(\varphi)$. This completes the proof. □

Remark 3.8. It is obvious, from Proposition 2.4, that n in Theorem 3.7 is the smallest nonzero natural number k such that there exists an exact sequence of form $0 \to (R[X])^k \to (R[X])^k \to M \to 0$. By Proposition 2.4 and the fact that $\{y_1, y_2, ..., y_n\}$ is a basis of the $R[X]$-module $\ker(\varphi)$, the order of the square matrix given in Theorem 3.2 is minimal. If M is of rank m as R-free module, then the characteristic exact sequence gives an exact sequence of $R[X]$-modules (see [3, 4])

$$0 \to (R[X])^m \to (R[X])^m \to M \to 0.$$

We have

$$m = s_1 + s_2 + ... + s_n \geq n = \mu_{R[X]}(M).$$

4. Applications to modules over group rings

Let K be a commutative field of characteristic $p > 0$ and let $G = G_1 \times G_2$, where G_1 is a finite abelian p-group and G_2 is a finite cyclic p-group. (See, for example, [6, 10, 15], for more information on these rings and their modules). We have $K[G] \cong R[G_2]$, where $R = K[G_1]$. Assume that G_2 is of order p^s and generated by an element σ. We easily see that the homomorphism $\psi : R[X] \longrightarrow R[G_2]$ defined by $\psi(X) = \sigma$ induces an isomorphism of $R[X]/(X^{p^s} - 1)$ to $R[G_2]$. Every $K[G]$-module can be regarded as $R[X]$-module annihilated by $X^{p^s} - 1$. Note that as K has characteristic $p > 0$, then $R[X]/(X^{p^s} - 1) \cong R[X]/(X - 1)^{p^s}$. When a finitely generated $K[G]$-module M is considered as a module over the subalgebra $K[G_1]$ of $K[G]$, we write $M \downarrow_{G_1}$.

Proposition 4.1. *Let M be a finitely generated $K[G]$-module such that $M \downarrow_{G_1}$ is $K[G_1]$-free. Then, M has a presentation of the form $(K[G])^n \xrightarrow{u} (K[G])^n \to M \to O$, where $n = \mu_{K[G]}(M)$.*

Proof. We have

$$\begin{aligned} n &= \mu_{K[G]}(M) \\ &= \mu_{R[G_2]}(M) \ (M \text{ is seen as } R[G_2]\text{-module}) \\ &= \mu_{R[X]/(X^{p^s}-1)}(M) \ (M \text{ is seen as } R[X]/(X^{p^s} - 1)\text{-module}), \end{aligned}$$

where $R = K[G_1]$. As M is annihilated by $X^{p^s} - 1$, then $\mu_{R[X]/(X^{p^s}-1)}(M) = \mu_{R[X]}(M)$. By Proposition 2.4, we have $\mu_{R[X]}(M) = \mu_{K[X]}(M/\mathfrak{p}M)$. So, $n = \mu_{K[X]}(M/\mathfrak{p}M)$, and by [2, Lemma 15.12], $\mu_{K[X]}(M/\mathfrak{p}M)$ is the number of the invariant factors of $M/\mathfrak{p}M$. By Theorem 3.7, we have an exact sequence

$$O \to (R[X])^n \to (R[X])^n \to M \to O.$$

Let $I = (X - 1)^{p^s} R[X]$. Then, the sequence

$$(R[X])^n \otimes_{R[X]} R[X]/I \to (R[X])^n \otimes_{R[X]} R[X]/I \to M \otimes_{R[X]} R[X]/I \to O$$

is exact. As $R[X] \otimes_{R[X]} R[X]/I \cong R[X]/I$, then we have an exact sequence $(R[X]/I)^n \to (R[X]/I)^n \to M/IM \to O$, which gives an exact sequence $(R[X]/I)^n \to (R[X]/I)^n \to M \to O$ since $IM = 0$. Therefore, $(R[G_2])^n \to (R[G_2])^n \to M \to O$ is exact since $A/I \cong R[G_2]$. As $R[G_2] \cong K[G]$, we obtain an exact sequence $(K[G])^n \xrightarrow{u} (K[G])^n \to M \to O$. □

In the sequence $(K[G])^n \xrightarrow{u} (K[G])^n \to M \to O$ of Proposition 4.1, u can not be injective, otherwise the sequence is split as a sequence of K-vector spaces. Since $M \neq 0$, then by comparing the dimensions of K-vector spaces we see that this is impossible.

One of applications of Proposition 4.1 is that if we take M as in this proposition and if we have an exact sequence $0 \to L \to N \to M \to O$, then by [5, Lemma 2.5], $F_0(L).F_0(M) = F_0(N)$.

Let R be a commutative local ring with maximal ideal $\mathfrak{p}$. Let M be a torsion $R[X]$-module which is R-free of finite rank and of type $(s_1, s_2, ..., s_n)$. For all $i \in \{1, 2, ..., n\}$, let ϕ_i be a monic polynomial of $R[X]$ such that $\overline{\phi_i}$ are the invariant factors of $M/\mathfrak{p}M$. Assume that M is annihilated by a monic polynomial ϕ. Then, we can see M as $R[X]/(\phi)$-module. Assume that $\mathfrak{p}$ is nilpotent which has m as nilpotency order. We also assume that $\mathfrak{p}$ is generated by an element π.

Lemma 4.2. *For any $k \in \{0, 1, ..., n-1\}$, the Fitting ideal $F_k(M)$ of M seen as $R[X]/(\phi)$-module is not regular.*

Proof. In Corollary 3.5 we take $P_i = -\phi_i$, for all $i \in \{1, 2, ..., n\}$. The polynomial ϕ_1 is monic. We have $\overline{\phi_1}$ divide $\overline{\phi}$. Then, there exists $q \in R[X]$ such that $\overline{\phi} = \overline{q}\overline{\phi_1}$. So, there exists $u \in \mathfrak{p}R[X]$ such that $\phi = q\phi_1 + u$. q is necessarily monic. For all $i \in \{1, 2, ..., n\}$, there exists $q_i \in R[X]$ such that $\overline{\phi_i} = \overline{q_i}\overline{\phi_1}$. So, there exists $u_i \in \mathfrak{p}R[X]$ such that $\phi_i = q_i\phi_1 + u_i$. We set $v = \pi^{m-1}q + \phi R[X]$. As q is not zero and $\deg(q) < \deg(\phi)$ and ϕ is monic, then v is a nonzero element of $R[X]/(\phi)$. Therefore,

$$\begin{aligned} v.(\phi_i + \phi R[X]) &= \pi^{m-1}q_i(\phi - u) + \pi^{m-1}qu_i + \phi R[X] \\ &= \pi^{m-1}q_i\phi + \phi R[X] \\ &= \phi R[X]. \end{aligned}$$

From Remark 3.4 we have $V_{i,j} \in \mathfrak{p}R[X]$, for all $i, j \in \{1, 2, ..., n\}$. So, $\pi^{m-1}V_{i,j} = 0$, for all $i, j \in \{1, 2, ..., n\}$. As the Fitting ideals $F_k(M)$ $k \in \{0, 1, ..., n-1\}$ of M are generated by sums or sums of products whose terms are $\phi_i + \phi R[X]$ or $V_{i,j} + \phi R[X]$, then these ideals are not regular. □

Theorem 4.3. *Let M be a finitely generated $K[G]$-module such that $M \downarrow_{G_1}$ is $K[G_1]$-free. For any $k \in \{0, 1, ..., \mu_{K[G]}(M) - 1\}$, the Fitting ideal $F_k(M)$ of M is not regular.*

Proof. Note that R is local and its maximal ideal $\mathfrak{p}$ is nilpotent (see [6, Corollary 2.5, p. 464]). The fact that $\mathfrak{p}$ is nilpotent can also be deduced from [9, Theorem]. So, by Lemma 4.2, and for any $k \in \{0, 1, ..., \mu_{K[G]}(M) - 1\}$, the Fitting ideal $F_k(M)$ of M is not regular. □

Corollary 4.4. *The projective dimension of any finitely generated $K[G]$-module M such that $M \downarrow_{G_1}$ is $K[G_1]$-free is > 1.*

Proof. By Theorem 4.3, for all $k \in \{0, 1, ..., \mu_{K[G]}(M) - 1\}$, the Fitting ideal $F_k(M)$ of M is not regular. As M is a torsion module, then necessarily, by [2, Theorem 13.53], it's projective dimension is > 1. □

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