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COSINE OPERATOR FUNCTIONS IN $\mathbb{R}^2$

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ABSTRACT. In this paper, we consider the topic from the theory of cosine operator functions in 2-dimensional real vector space, which is an interplay between functional analysis and matrix theory. For the various cases of a given real matrix $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, we find out the appropriate cosine operator function $C(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$ $(t \in \mathbb{R})$ in a real vector space $\mathbb{R}^2$, as the solutions of the Cauchy problem $C''(t) = AC(t)$, $C(0) = I$, $C'(0) = 0$.

1. Introduction and some preliminaries

The theory of cosine operator function and its application to Cauchy problem of the second order was first studied by Fattorini [2] and Sova [9] and further investigated by many authors [3, 5, 7, 8, 10]. The real functions $y(t)$ of a real variable t as the only solutions of the Cauchy problem

$$y''(t) = a \cdot y(t),\ y(0) = 1,\ y'(0) = 0 \quad (a \in \mathbb{R}), \tag{1.1}$$

are:

(1) for $a > 0$, $\quad y = \frac{1}{2}\left(e^{\sqrt{a}t} + e^{-\sqrt{a}t}\right) = \cosh(\sqrt{a}t)$;
(2) for $a < 0$, $\quad y = \frac{1}{2}\left(e^{i\sqrt{-a}t} + e^{-i\sqrt{-a}t}\right) = \cos(\sqrt{-a}t)$;
(3) for $a = 0$, $\quad y \equiv 1$.

These solutions, besides the condition $y(0) = 1$, satisfy also the D'Alambert functional equation

$$y(t+s) + y(t-s) = 2y(t)y(s), \tag{1.2}$$

which characterizes the family of cosine and hyperbolic cosine functions and the function $y \equiv 1$. According to that, many mathematicians have studied the general family of cosine operator functions in a vector space, Banach algebra or in Banach space X.

Definition 1.1. Let X be a Banach space. A cosine operator function is a family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself satisfying:

(1) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $t, s \in \mathbb{R}$;

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(2) $C(0) = I$, I the identity operator;
(3) $C(\cdot)x$ is continuous on $\mathbb{R}$, for all $x \in \mathbb{X}$.

In a paper [11] we have defined a family of cosine operator function and its generator on a different way, as a solution of certain functional equations. The associated sine operator function $S(\cdot)$ is defined by the formula

$$S(t)x := \int_0^t C(s)x\,ds, \text{ for all}\, t \in \mathbb{R}, \text{for all}\, x \in X. \tag{1.3}$$

For the cosine operator function there exists some $M \geq 1$, $\omega \in \mathbb{R}$, such that $\|C(t)\| \leq Me^{\omega t}$ for all $t \geq 0$. Further, the operator A on X defined by

$$Ax := \lim_{h \to 0} \frac{2\,(C(h)x - x)}{h^2}, \qquad \text{with the domain}$$

$D(A) = \left\{ x \in X : \lim_{h \to 0} \frac{2\,(C(h)x - x)}{h^2} \text{exists} \right\}$ is called the generator of $(C(t))_{t \in \mathbb{R}}$.
The operator A is closed and $D(A)$ is dense in X, i.e. $\overline{D(A)} = X$. Throughout the building of the theory of cosine operator functions, it has been found that the family of operators $C(t)\,(t \in \mathbb{R})$ in Banach space X satisfies the following Cauchy problem (equivalent with (1.1)),

$$C''(t) = AC(t), \quad C(0) = I, \quad C'(0) = 0, \tag{1.4}$$

where A is a generator of the family of operators $C(t)\,(t \in \mathbb{R})$ (see for example [2, 7, 11]). It also holds vice versa, for a given linear and closed, everywhere dense defined operator A in X, there exists a cosine operator function $C(t)\,(t \in \mathbb{R})$ with generator A and which is the solution of the Cauchy problem (1.4). As the solution of Cauchy problem (1.1) depends on the value of the parameter $a \in \mathbb{R}$, analogously, the solution $C(t)\,(t \in \mathbb{R})$ of the Cauchy problem (1.4) depends of an operator A which generates that cosine operator function. Notice that from (1.1), since $a(0) = 1$, it directly follows $y''(0) = a$, and from (1.4) since $C(0) = I$, it follows $C''(0) = A$. Motivated by these mentioned facts, we are interested in determining the cosine operator functions, specially, in real vector space $X = \mathbb{R}^2$, in depending of the given generator A. In that case, because of the isomorphism between the vector space of all linear mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$ and the vector space $\mathcal{M}_2(\mathbb{R})$ (vector space of all real square matrices of the second order), we may identify the generator A and family $C(t)\,(t \in \mathbb{R})$ of cosine operator functions with corresponding matrices from $\mathcal{M}_2(\mathbb{R})$ considering, for example, the standard basis of the space $X = \mathbb{R}^2$. Recently, there have been generalized notions of cosine operator functions, such as, namely, C-cosine function, local C-cosine function and α-times integrated C-cosine function.

Example 1.2. ([10]) Let A be the operator of multiplication by a complex number on the space $\mathbb{R}$. Then A is the generator of the cosine operator function $(C(t)x)(s) = \cos(it\sqrt{A})x(s)$, $t \in \mathbb{R}$.

Proposition 1.3. *([9]) The operators $C(t), C(s), S(t)$ and $S(s)$ commute for any $t, s \in \mathbb{R}$.*

Proposition 1.4. *([10]) The sine operator function $S(\cdot)$ is continuous in the uniform operator topology.*

Proposition 1.5. *([9]) Let $C(t)$ be a cosine operator function with generator A. Then for all $t, s \in \mathbb{R}$, we have the relations*

1. $C(t) = C(-t), \quad S(-t) = -S(t), \quad S(0) = 0$;
2. $S(t+s) + S(t-s) = 2S(t)C(s)$;
3. $S(t+s) = S(t)C(s) + S(s)C(t)$;
4. $C(t+s) - C(t-s) = 2AS(t)S(s)$;
5. $C(2t) = 2C(t)^2 - I, \quad C(t)^2 - AS(t)^2 = I$;
6. $C((n+1)t) = b_0 I + b_1 C(t) + \ldots + b_{n+1} C^{n+1}(t)$, *where* $b_0 + b_1 z + \ldots + b_{n+1} z^{n+1}$ *is the Chebyshev polynomial of the first kind of degree* $n+1$.

The relation between well-posedness for second order abstract Cauchy problems and cosine operator functions is very close to that between first order abstract Cauchy problem and strongly continuous semigroups.

Lemma 1.6. *([1, 6]) Let A be a closed operator on a Banach space X. Then the operator A generates a cosine operator function on X if and only if there exists a Banach space F, with dense imbeddings $[D(A)] \hookrightarrow F \hookrightarrow X$, such that the operator matrix*

$$\mathbf{A} := \begin{bmatrix} 0 & I_F \\ A & 0 \end{bmatrix}, \qquad D(\mathbf{A}) := D(A) \times F,$$

generates a strongly continuous semigroup $\left(e^{t\mathbf{A}}\right)_{t \geq 0}$ in $F \times X$. In this case F is uniquely determined and coincides with the space of strong differentiability of the operator valued mapping $C(\cdot) : \mathbb{R} \to \mathcal{L}(X)$, and there holds

$$e^{t\mathbf{A}} := \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}, \quad t \geq 0.$$

Since the theory of cosine operator functions is close to the operator semigroup theory, it is often developed in parallel to it. At the same time, the theory of cosine operator functions considerably differs from the operator semigroup theory. First of all, these distinctions concern the properties inherent to the corresponding parabolic and hyperbolic partial differential equations ([10]).

2. Results

Let $C(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$ $(t \in \mathbb{R})$ and $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ constant matrix $(\alpha, \beta, \gamma, \delta \in \mathbb{R})$. Then the Cauchy problem (1.4) is

$$\begin{gathered} \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}'' = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \cdot \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}, \\ \begin{bmatrix} a(0) & b(0) \\ c(0) & d(0) \end{bmatrix} = I, \quad \begin{bmatrix} a'(0) & b'(0) \\ c'(0) & d'(0) \end{bmatrix} = 0. \end{gathered} \tag{2.1}$$

So, we ought to find the real functions $a(t), b(t), c(t)$ and $d(t)$ which are the solutions of the system of differential equations

$$\begin{array}{r}a''(t) = \alpha a(t) + \beta c(t)\\ b''(t) = \alpha b(t) + \beta d(t)\\ c''(t) = \gamma a(t) + \delta c(t)\\ d''(t) = \gamma b(t) + \delta d(t)\end{array}. \tag{2.2}$$

and fulfill the initial conditions

$$\begin{array}{c}a(0) = 1, a'(0) = 0, b(0) = 0, b'(0) = 0,\\ c(0) = 0, c'(0) = 0, d(0) = 1, d'(0) = 0.\end{array}. \tag{2.3}$$

These solutions, because of (2.2) and (2.3), satisfy also additional conditions

$$\begin{array}{c}a''(0) = \alpha, a'''(0) = 0, b''(0) = \beta, b'''(0) = 0,\\ c''(0) = \gamma, c'''(0) = 0, d''(0) = \delta, d'''(0) = 0.\end{array}. \tag{2.4}$$

In order to determine unknown functions, for each of unknown functions, beside two initial conditions from (2.3), we can use also two additional conditions from (2.4). From the first and the third equation of the system (2.2) it follows

$$a^{IV}(t) = \alpha a''(t) + \beta(\gamma a(t) + \delta c(t)) = \alpha a''(t) + \beta\gamma a(t) + \delta(a''(t) - \alpha a(t)),$$

$$a^{IV}(t) - (\alpha + \delta)a''(t) + (\alpha\delta - \beta\gamma)a(t) = 0. \tag{2.5}$$

The equation (2.5) is a differential equation with constant coefficients. Its characteristic equation, for particular integrals $a(t) = e^{rt}$, is

$$r^4 - (\alpha + \delta)r^2 + (\alpha\delta - \beta\gamma) = 0 \tag{2.6}$$

and from this it follows

$$r^2 = \frac{\alpha + \delta \pm \sqrt{(\alpha + \delta)^2 - 4(\alpha\delta - \beta\gamma)}}{2} = \frac{\alpha + \delta \pm \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{2}. \tag{2.7}$$

We are going to find the solutions of the system (2.2) which satisfy the conditions (2.3), depending on the sign of the expression $(\alpha - \delta)^2 + 4\beta\gamma$ and yet some additional conditions on the elements of the matrix A. After that, we will consider also the solutions in some special cases of the matrix A.

2.1. **The solutions of the Cauchy problem, depending on the sign of the expression $(\alpha-\delta)^2+4\beta\gamma$, with some additional conditions.**

I) Case: $(\alpha - \delta)^2 + 4\beta\gamma > 0$, with $\beta \neq 0, \gamma \neq 0$ and $\alpha\delta - \beta\gamma \neq 0$.
Denote $A = \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}$. Then from (2.7) we have

$$r = \pm\sqrt{\frac{\alpha + \delta \pm A}{2}} = \pm\frac{\sqrt{2}}{2}\sqrt{\alpha + \delta \pm A}. \tag{2.8}$$

Thus, the general solution of the differential equation (2.5) is

$$a(t) = C_1 e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} + C_2 e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} + C_3 e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t} + C_4 e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t}.$$

From the conditions $a(0) = 1, a'(0) = 0, a''(0) = \alpha, a'''(0) = 0$ we get
$a(0) = C_1 + C_2 + C_3 + C_4 = 1$
$a'(0) = \frac{\sqrt{2}}{2}\sqrt{\alpha + \delta - A}(C_1 - C_2) + \frac{\sqrt{2}}{2}\sqrt{\alpha + \delta + A}(C_3 - C_4) = 0$

$a''(0) = \frac{1}{2}(\alpha+\delta-A)(C_1+C_2) + \frac{1}{2}(\alpha+\delta+A)(C_3+C_4) = \alpha$
$a'''(0) = \left(\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\right)^3 (C_1-C_2) + \left(\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\right)^3 (C_3-C_4) = 0.$

By solving this system of equations, since $A > 0$, we find

$$C_1 = C_2 = \frac{\delta+A-\alpha}{4A} \quad \text{and} \quad C_3 = C_4 = \frac{A+\alpha-\delta}{4A},$$

so for the particular solution of the equation (2.5) we get

$$\begin{aligned} a(t) &= \frac{\delta+A-\alpha}{4A}\left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t}\right) \\ &+ \frac{A+\alpha-\delta}{4A}\left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t}\right). \end{aligned} \tag{2.9}$$

From the first equation of the system (2.2) for $\beta \neq 0$ we have

$$\begin{aligned} c(t) &= \frac{1}{\beta}(a''(t) - \alpha a(t)) = \frac{(\delta-\alpha-A)(\delta-\alpha+A)}{8\beta A} \\ &\cdot \left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} - e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t} - e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t}\right). \end{aligned} \tag{2.10}$$

We see that $c(0) = 0$, and it is easy to see that $c'(0) = 0$. From the second and fourth equation of the system (2.2) on the same way we get general solution

$$b(t) = C_1 e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} + C_2 e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} + C_3 e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t} + C_4 e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t},$$

and from $b(0) = 0$, $b'(0) = 0$, $b''(0) = \beta$, $b'''(0) = 0$ we get the system of equations
$b(0) = C_1 + C_2 + C_3 + C_4 = 0$
$b'(0) = \frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}(C_1-C_2) + \frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}(C_3-C_4) = 0$
$b''(0) = \frac{1}{2}(\alpha+\delta-A)(C_1+C_2) + \frac{1}{2}(\alpha+\delta+A)(C_3+C_4) = \beta$
$b'''(0) = \left(\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\right)^3 (C_1-C_2) + \left(\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\right)^3 (C_3-C_4) = 0.$
By solving this system we come up to

$$C_1 = C_2 = \frac{-\beta}{2A} \quad \text{and} \quad C_3 = C_4 = \frac{\beta}{2A},$$

so we have

$$b(t) = \frac{-\beta}{2A}\left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} - e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t} - e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t}\right). \tag{2.11}$$

From the second equation of the system (2.2) we have

$$\begin{aligned} d(t) &= \frac{1}{\beta}(b''(t) - \alpha b(t)) = \frac{-\delta+A+\alpha}{4A}\left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta-A}\,t}\right) \\ &+ \frac{\delta+A-\alpha}{4A}\left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta+A}\,t}\right). \end{aligned} \tag{2.12}$$

Obviously $d(0) = 1$, and it is not difficult to check out that the condition $d'(0) = 0$ is satisfied.

Example 2.1. For $\alpha = 4$, $\beta = 2$, $\gamma = -1$, $\delta = 1$ we have

$$A = \sqrt{(\alpha-\delta)^2 + 4\beta\gamma} = 1,$$

$a(t) = \frac{-1}{2}\left(e^{\sqrt{2}\,t} + e^{-\sqrt{2}\,t}\right) + e^{\sqrt{3}\,t} + e^{-\sqrt{3}\,t} = -\cosh(\sqrt{2}\,t) + 2\cosh(\sqrt{3}\,t),$
$b(t) = -\left(e^{\sqrt{2}\,t} + e^{-\sqrt{2}\,t} - e^{\sqrt{3}\,t} - e^{-\sqrt{3}\,t}\right) = 2\cosh(\sqrt{3}\,t) - 2\cosh(\sqrt{2}\,t),$
$c(t) = \frac{1}{2}\left(e^{\sqrt{2}\,t} + e^{-\sqrt{2}\,t} - e^{\sqrt{3}\,t} - e^{-\sqrt{3}\,t}\right) = \cosh(\sqrt{2}\,t) - \cosh(\sqrt{3}\,t),$
$d(t) = e^{\sqrt{2}\,t} + e^{-\sqrt{2}\,t} - \frac{1}{2}\left(e^{3t} + e^{-3t}\right) = 2\cosh(\sqrt{2}\,t) - \cosh(\sqrt{3}\,t),$
hence matrix $A = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ generates cosine operator function

$$C(t) = \begin{bmatrix} -\cosh(\sqrt{2}\,t) + 2\cosh(\sqrt{3}\,t) & 2\cosh(\sqrt{3}\,t) - 2\cosh(\sqrt{2}\,t) \\ \cosh(\sqrt{2}\,t) - \cosh(\sqrt{3}\,t) & 2\cosh(\sqrt{2}\,t) - \cosh(\sqrt{3}\,t) \end{bmatrix}$$

Remark 2.2. Notice that the functions given by relations (2.9)-(2.12) are expressed by the cosine functions or hyperbolic cosine functions or with the combination of these two types of functions, depending on the sign of the numbers $\alpha+\delta\pm A$ in (2.8). The exceptions are the cases when $\alpha+\delta-A = 0$ or $\alpha+\delta+A = 0$. It is possible only if $\alpha\delta = \beta\gamma$. Really, from $\alpha+\delta = \pm A$ we have

$$A^2 = (\alpha-\delta)^2 + 4\beta\gamma = (\alpha+\delta)^2$$

and so it follows $\alpha\delta = \beta\gamma$. That case we shall generally study later in subsection 2.2 (case VI), with some additional conditions.

II) Case: $(\alpha-\delta)^2 + 4\beta\gamma = 0$, with $\beta \neq 0$, $\gamma \neq 0$ and $\alpha+\delta \neq 0$.
Then from (2.7) we have that $r = \pm\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}$. The general solution of the differential equation (2.5) is

$$a(t) = (C_1 + C_2 t)e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\,t} + (C_3 + C_4 t)e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\,t}$$

Using the conditions $a(0) = 1, a'(0) = 0, a''(0) = \alpha, a'''(0) = 0$ gives us the following system of equations
$a(0) = C_1 + C_3 = 1$
$a'(0) = \frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}(C_1 - C_3) + C_2 + C_4 = 0$
$a''(0) = \frac{\alpha+\delta}{2}(C_1 + C_3) + \sqrt{2(\alpha+\delta)}(C_2 - C_4) = \alpha$
$a'''(0) = \left(\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\right)^3 (C_1 - C_3) + \frac{\alpha+\delta}{2}(3C_2 + 3C_4) = 0.$
The solution of this system is

$$C_1 = C_3 = \frac{1}{2},\ C_2 = \frac{\alpha-\delta}{4\sqrt{2(\alpha+\delta)}} \quad \text{and} \quad C_4 = -C_2 = \frac{\delta-\alpha}{4\sqrt{2(\alpha+\delta)}}.$$

Hence

$$a(t) = \frac{\left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\,t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\,t}\right)}{2} + \frac{(\alpha-\delta)t\left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\,t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\,t}\right)}{4\sqrt{2(\alpha+\delta)}}. \quad (2.13)$$

For $\beta \neq 0$, from the first equation of the system (2.2) we get

$$\begin{aligned} c(t) &= \frac{1}{\beta}(a''(t) - \alpha a(t)) \\ &= \frac{1}{\beta} t \frac{\alpha - \delta}{4\sqrt{2(\alpha+\delta)}} \left(\frac{\alpha+\delta}{2} - \alpha \right) \left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} \right). \end{aligned}$$

From $(\alpha-\delta)^2 + 4\beta\gamma = 0$, follows $\frac{-(\alpha-\delta)^2}{8\beta} = \frac{\gamma}{2}$, so

$$c(t) = \frac{\gamma t}{2\sqrt{2(\alpha+\delta)}} \left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} - e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} \right). \tag{2.14}$$

This function fulfills the initial conditions $c(0) = 0$, $c'(0) = 0$.
Analogously, from the second and fourth equation of the system (2.2) we get the general solution

$$b(t) = (C_1 + C_2 t) e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} + (C_3 + C_4 t) e^{\frac{-\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t}$$

and from $b(0) = 0$, $b'(0) = 0$ and $b''(0) = \beta$, $b'''(0) = 0$ we get the system of equations
$b(0) = C_1 + C_3 = 0$
$b'(0) = \frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}(C_1 - C_3) + C_2 + C_4 = 0$
$b''(0) = \frac{\alpha+\delta}{2}(C_1 + C_3) + \sqrt{2(\alpha+\delta)}(C_2 - C_4) = \beta$
$b'''(0) = \left(\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\right)^3 (C_1 - C_3) + \frac{\alpha+\delta}{2}(3C_2 + 3C_4) = 0$
which solution is

$$C_1 = C_3 = 0,\ C_2 = \frac{\beta}{2\sqrt{2(\alpha+\delta)}} \quad \text{and} \quad C_4 = -C_2 = \frac{-\beta}{2\sqrt{2(\alpha+\delta)}}.$$

Hence,

$$b(t) = \frac{\beta t}{2\sqrt{2(\alpha+\delta)}} \left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} - e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} \right). \tag{2.15}$$

From the second equation of the system (2.2) we have

$$\begin{aligned} d(t) = \frac{1}{\beta}(b''(t) - \alpha b(t)) &= \frac{1}{2} \left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} + e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} \right) \\ &+ \frac{(\delta-\alpha)t}{4\sqrt{2(\alpha+\delta)}} \left(e^{\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} - e^{-\frac{\sqrt{2}}{2}\sqrt{\alpha+\delta}\, t} \right). \end{aligned} \tag{2.16}$$

Obviously $d(0) = 1$, and it is not difficult to check out that the condition $d'(0) = 0$ is satisfied.

Remark 2.3. The functions given by relations (2.13)-(2.16) are, clearly, ordinary or hyperbolic cosine and sine functions, depending on the sign of the number $\alpha + \delta$.

Example 2.4. For $\alpha = -3$, $\beta = -2$, $\gamma = 2$, $\delta = 1$, we have

$$A = \sqrt{(\alpha-\delta)^2 + 4\beta\gamma} = 0, \quad \frac{\sqrt{2}}{2}\sqrt{\alpha+\delta} = i,$$

$$\begin{aligned}
a(t) &= \tfrac{1}{2}(e^{it}+e^{-it}) + \tfrac{-4t}{4\sqrt{2(-2)}}(e^{it}-e^{-it}) = \cos t - t\cdot\sin t,\\
b(t) &= \tfrac{-2t}{2\sqrt{2(-2)}}(e^{it}-e^{-it}) = -t\cdot\sin t,\\
c(t) &= \tfrac{2t}{2\sqrt{2(-2)}}(e^{it}-e^{-it}) = t\cdot\sin t,\\
d(t) &= \tfrac{1}{2}(e^{it}+e^{-it}) - \tfrac{-4t}{4\sqrt{2(-2)}}(e^{it}-e^{-it}) = \cos t + t\cdot\sin t.
\end{aligned}$$

Thus the matrix A generates cosine operator function

$$C(t) = \begin{bmatrix} \cos t - t\sin t & -t\sin t \\ t\sin t & \cos t + t\sin t \end{bmatrix}.$$

III) Case: $D = (\alpha+\delta)^2 - 4(\alpha\delta - \beta\gamma) < 0$, with $\alpha,\beta,\gamma,\delta \neq 0$. Then from (2.7) we have

$$\sqrt{\frac{\alpha+\delta\pm\sqrt{D}}{2}} = r = x+iy \in \mathbb{C}$$

$$\frac{\alpha+\delta\pm\sqrt{D}}{2} = (x^2-y^2) + i\cdot 2xy$$

$$x^2 - y^2 = \frac{\alpha+\delta}{2} \quad \text{and} \quad \pm\sqrt{-D} = 4xy, \quad \text{i.e}$$

$$x^2 = y^2 + \frac{\alpha+\delta}{2} \quad \text{and} \quad 16x^2y^2 = -D$$

$$16y^2\left(y^2 + \frac{\alpha+\delta}{2}\right) + D = 0$$

$$16y^4 + 8(\alpha+\delta)y^2 + (\alpha+\delta)^2 - 4(\alpha\delta-\beta\gamma) = 0$$

$$y^2 = t \geq 0$$

$$\begin{aligned}
t_{1,2} &= \frac{-8(\alpha+\delta) \pm \sqrt{64(\alpha+\delta)^2 - 64[(\alpha+\delta)^2 - 4(\alpha\delta-\beta\gamma)]}}{32}\\
&= \frac{-1}{4}(\alpha+\delta) \pm \frac{1}{2}\sqrt{\alpha\delta-\beta\gamma}.
\end{aligned}$$

Since $D < 0$ we get

$$\alpha\delta - \beta\gamma > \frac{1}{4}(\alpha+\delta)^2$$

$$\sqrt{\alpha\delta-\beta\gamma} > \frac{1}{2}|\alpha+\delta|. \tag{2.17}$$

i) If $\alpha+\delta > 0$ then from (2.17) we get $\sqrt{\alpha\delta-\beta\gamma} > \frac{1}{2}(\alpha+\delta)$, i.e.

$$-\frac{1}{2}\sqrt{\alpha\delta-\beta\gamma} < -\frac{1}{4}(\alpha+\delta).$$

Then

$$t_1 = -\frac{1}{4}(\alpha+\delta) - \frac{1}{2}\sqrt{\alpha\delta-\beta\gamma} < -\frac{1}{4}(\alpha+\delta) - \frac{1}{4}(\alpha+\delta) < 0,$$

so the equation $y^2 = t_1$ has no real solutions y.

$$t_2 = -\frac{1}{4}(\alpha+\delta) + \frac{1}{2}\sqrt{\alpha\delta-\beta\gamma} > -\frac{1}{4}(\alpha+\delta) + \frac{1}{4}(\alpha+\delta) = 0.$$

$$y^2 = t_2 > 0 \Rightarrow y = \pm\sqrt{t_2}$$

ii) If $\alpha+\delta<0$ then from (2.17) we get $\sqrt{\alpha\delta-\beta\gamma}>-\frac{1}{2}(\alpha+\delta)$,

$$t_1=-\frac{1}{4}(\alpha+\delta)-\frac{1}{2}\sqrt{\alpha\delta-\beta\gamma}<-\frac{1}{4}(\alpha+\delta)+\frac{1}{4}(\alpha+\delta)=0,$$

so the equation $y^2=t_1$ has no real solutions y.

$$t_2=-\frac{1}{4}(\alpha+\delta)+\frac{1}{2}\sqrt{\alpha\delta-\beta\gamma}>-\frac{1}{4}(\alpha+\delta)-\frac{1}{4}(\alpha+\delta)>0,$$

so the equation $y^2=t_2$ has real solutions y.
Hence, in both cases i) and ii) we have

$$y=\pm\sqrt{t_2}=\pm\frac{1}{2}\sqrt{2\sqrt{\alpha\delta-\beta\gamma}-\alpha-\delta}.$$

Then from $x=\frac{\pm\sqrt{-D}}{4y}$ it follows

$$x=\pm\frac{1}{2}\sqrt{2\sqrt{\alpha\delta-\beta\gamma}+\alpha+\delta}.$$

Now we have

$$r=x+i\cdot y=\pm\frac{1}{2}\left[\sqrt{2\sqrt{\alpha\delta-\beta\gamma}+\alpha+\delta}+i\cdot\sqrt{2\sqrt{\alpha\delta-\beta\gamma}-\alpha-\delta}\right].$$

Denote:

$$A=\frac{1}{2}\sqrt{2\sqrt{\alpha\delta-\beta\gamma}+\alpha+\delta},\quad B=\frac{1}{2}\sqrt{2\sqrt{\alpha\delta-\beta\gamma}-\alpha-\delta}.$$

Then the general solution of the differential equation(2.5) is:

$$a(t)=C_1e^{At}\cos(Bt)+C_2e^{At}\sin(Bt)+C_3e^{-At}\cos(Bt)+C_4e^{-At}\sin(Bt).$$

By calculating the derivatives of the function $a(t)$ and using the conditions: $a(0)=1, a'(0)=0, a''(0)=\alpha, a'''(0)=0$, we obtain the system of equations:
$a(0)=C_1+C_3=1$
$a'(0)=(C_1-C_3)A+(C_2+C_4)B=0$
$a''(0)=(C_1+C_3)(A^2-B^2)+2(C_2-C_4)AB=\alpha$
$a'''(0)=(C_1-C_3)A^3-(C_2+C_4)B^3+3(C_3-C_1)AB^2+3(C_2+C_4)A^2B=0.$
After solving this system we get
$C_1=C_3=\frac{1}{2},\quad C_2=\frac{\alpha-\delta}{2\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}$ and $C_4=-C_2=\frac{-(\alpha-\delta)}{2\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}$.
So, considering particular solution is

$$a(t)=\frac{1}{2}(e^{At}+e^{-At})\cos(Bt)+\frac{\alpha-\delta}{2\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}(e^{At}-e^{-At})\sin(Bt),$$

$$a(t)=\cosh(At)\cos(Bt)+\frac{\alpha-\delta}{\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}\sinh(At)\sin(Bt).\qquad(2.18)$$

Since

$$a''(t)=\left(A^2-B^2+\frac{2AB(\alpha-\delta)}{\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}\right)\cosh(At)\cos(Bt)+$$

$$+\left(-2AB+\frac{(A^2-B^2)(\alpha-\delta)}{\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}\right)\sinh(At)\sin(Bt),$$

and $\beta \neq 0$, from the first equation of system (2.2) it follows

$$c(t)=\frac{1}{\beta}(a''(t)-\alpha\cdot a(t))=\frac{1}{\beta}\cdot\frac{2\beta\gamma}{\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}\cdot\sinh(At)\cdot\sin(Bt).$$

Thus,

$$c(t)=\frac{2\gamma}{\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}\cdot\sinh(At)\cdot\sin(Bt). \tag{2.19}$$

On the same way, from the second and fourth equation of the system (2.2), we get

$$b(t)=C_1e^{At}\cos(Bt)+C_2e^{At}\sin(Bt)+C_3e^{-At}\cos(Bt)+C_4e^{-At}\sin(Bt),$$

and from the conditions: $b(0)=0, b'(0)=0, b''(0)=\beta, b'''(0)=0$, we have:
$b(0)=C_1+C_3=0$
$b'(0)=(C_1-C_3)A+(C_2+C_4)B=0$
$b''(0)=(C_1+C_3)(A^2-B^2)+2(C_2-C_4)AB=\beta$
$b'''(0)=(C_1-C_3)A^3-(C_2+C_4)B^3+3(C_3-C_1)AB^2+3(C_2+C_4)A^2B=0.$
After solving this system we get $C_1=C_3=0,\quad C_2=\frac{\beta}{\sqrt{-D}}$ and $C_4=-C_2$ and so

$$b(t)=\frac{2\beta}{\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}\cdot\sinh(At)\cdot\sin(Bt). \tag{2.20}$$

Since $\beta \neq 0$, from the second equation of the system (2.2) we get

$$d(t)=\frac{1}{\beta}(b''(t)-\alpha\cdot b(t))$$

$$d(t)=\cosh(At)\cos(Bt)+\frac{\delta-\alpha}{\sqrt{4(\alpha\delta-\beta\gamma)-(\alpha+\delta)^2}}\cdot\sinh(At)\cdot\sin(Bt). \tag{2.21}$$

Example 2.5. If we take $\alpha=-2,\ \beta=-1,\ \gamma=7,\ \delta=3$ then

$$D=(\alpha+\delta)^2-4(\alpha\delta-\beta\gamma)=1-4=-3<0,\quad A=\frac{\sqrt{3}}{2},\quad B=\frac{1}{2},$$

so the solution of the system (2.2) which satisfies the initial conditions (2.3) is

$$a(t)=\cosh\left(\frac{\sqrt{3}}{2}t\right)\cos\left(\frac{t}{2}\right)-\frac{5}{\sqrt{3}}\sinh\left(\frac{\sqrt{3}}{2}t\right)\sin\left(\frac{t}{2}\right),$$

$$b(t)=-\frac{2}{\sqrt{3}}\sinh\left(\frac{\sqrt{3}}{2}t\right)\sin\left(\frac{t}{2}\right),$$

$$c(t)=\frac{14}{\sqrt{3}}\sinh\left(\frac{\sqrt{3}}{2}t\right)\sin\left(\frac{t}{2}\right),$$

$$d(t)=\cosh\left(\frac{\sqrt{3}}{2}t\right)\cos\left(\frac{t}{2}\right)+\frac{5}{\sqrt{3}}\sinh\left(\frac{\sqrt{3}}{2}t\right)\sin\left(\frac{t}{2}\right).$$

2.2. **Some special cases of the matrix A.** Here we analyze some special cases that have been omitted from the subsection 2.1.

IV) Case $\beta = 0$, $\gamma \neq 0$, $\alpha \neq \delta$.

Then the system (2.2) becomes

$$\begin{aligned} a''(t) &= \alpha a(t) \\ b''(t) &= \alpha b(t) \\ c''(t) &= \gamma a(t) + \delta c(t) \\ d''(t) &= \gamma b(t) + \delta d(t) \end{aligned} \tag{2.22}$$

From the third and first equation of the system (2.22) we get

$$c^{IV}(t) = \alpha\gamma a(t) + \delta c''(t) = \alpha(c''(t) - \delta c(t)) + \delta c''(t), \text{ i.e}$$

$$c^{IV}(t) - (\alpha+\delta)c''(t) + \alpha\delta c(t) = 0 \tag{2.23}$$

The general solution of the equation (2.23) is

$$c(t) = C_1 e^{\sqrt{\alpha}\,t} + C_2 e^{-\sqrt{\alpha}\,t} + C_3 e^{\sqrt{\delta}\,t} + C_4 e^{-\sqrt{\delta}\,t},$$

so from the conditions $c(0) = 0, c'(0) = 0, c''(0) = \gamma, c'''(0) = 0$ we get the system of the equations

$$\begin{aligned} & c(0) = C_1 + C_2 + C_3 + C_4 = 0 \\ & c'(0) = \sqrt{\alpha}(C_1 - C_2) + \sqrt{\delta}(C_3 - C_4) = 0 \\ & c''(0) = \alpha(C_1 + C_2) + \delta(C_3 + C_4) = \gamma \\ & c'''(0) = (\sqrt{\alpha})^3 (C_1 - C_2) + \left(\sqrt{\delta}\right)^3 (C_3 - C_4) = 0. \end{aligned}$$

Because $\alpha \neq \delta$, the solution of this system of the equations is

$$C_1 = C_2 = \frac{\gamma}{2(\alpha-\delta)} \quad \text{and} \quad C_3 = C_4 = -C_2 = \frac{-\gamma}{2(\alpha-\delta)},$$

hence the particular solution of the differential equation (2.23) is

$$c(t) = \frac{\gamma}{2(\alpha-\delta)}\left(e^{\sqrt{\alpha}\,t} + e^{-\sqrt{\alpha}\,t}\right) + \frac{-\gamma}{2(\alpha-\delta)}\left(e^{\sqrt{\delta}\,t} + e^{-\sqrt{\delta}\,t}\right). \tag{2.24}$$

Now from the third equation of the system (2.22), since $\gamma \neq 0$, follows

$$a(t) = \frac{1}{2}\left(e^{\sqrt{\alpha}\,t} + e^{-\sqrt{\alpha}\,t}\right). \tag{2.25}$$

On the similar way, by using the fourth and second equation of the system (2.22) and from the conditions (2.3) and (2.4) for the function $d(t)$, we get

$$d(t) = \frac{1}{2}\left(e^{\sqrt{\delta}\,t} + e^{-\sqrt{\delta}\,t}\right). \tag{2.26}$$

Then from the second equation of the system (2.22) we get

$$b(t) = 0. \tag{2.27}$$

V) Case $\gamma = 0$, $\beta \neq 0$, $\alpha \neq \delta$. Then, the system (2.2) becomes

$$\begin{aligned} a''(t) &= \alpha a(t) + \beta c(t) \\ b''(t) &= \alpha b(t) + \beta d(t) \\ c''(t) &= \delta c(t) \\ d''(t) &= \delta d(t). \end{aligned} \tag{2.28}$$

From the first and third equation of the system (2.28) we have

$$a^{IV}(t) = \alpha a''(t) + \beta\delta c(t) = \alpha a''(t) + \delta(a''(t) - \alpha a(t)),$$

$$a^{IV}(t) - (\alpha + \delta)a''(t) + \alpha\delta a(t) = 0 \tag{2.29}$$

As the general solution of the equation (2.29) equals

$$a(t) = C_1 e^{\sqrt{\alpha}\, t} + C_2 e^{-\sqrt{\alpha}\, t} + C_3 e^{\sqrt{\delta}\, t} + C_4 e^{-\sqrt{\delta}\, t}$$

so from the conditions $a(0) = 1, a'(0) = 0, a''(0) = \alpha, a'''(0) = 0$ we get the system of the equations

$$\begin{aligned} a(0) &= C_1 + C_2 + C_3 + C_4 = 1 \\ a'(0) &= \sqrt{\alpha}(C_1 - C_2) + \sqrt{\delta}(C_3 - C_4) = 0 \\ a''(0) &= \alpha(C_1 + C_2) + \delta(C_3 + C_4) = \alpha \\ a'''(0) &= (\sqrt{\alpha})^3 (C_1 - C_2) + \left(\sqrt{\delta}\right)^3 (C_3 - C_4) = 0. \end{aligned}$$

The solution of this system of equations is:

$$C_1 = C_2 = \frac{1}{2} \quad \text{and} \quad C_3 = C_4 = 0,$$

hence the particular solution of the differential equation (2.29) equals

$$a(t) = \frac{1}{2}\left(e^{\sqrt{\alpha}\, t} + e^{-\sqrt{\alpha}\, t}\right). \tag{2.30}$$

From the first equation of the system (2.28), $\beta \neq 0$, we obtain

$$c(t) = \frac{1}{\beta}\left(a''(t) - \alpha a(t)\right) = 0. \tag{2.31}$$

From the second and fourth equation of the system (2.28) we get

$$b^{IV}(t) = \alpha b''(t) + \beta\delta d(t) = \alpha b''(t) + \delta(b''(t) - \alpha a(t)),$$

$$b^{IV}(t) - (\alpha + \delta)b''(t) + \alpha\delta b(t) = 0. \tag{2.32}$$

The general solution of the equation (2.32) is

$$b(t) = C_1 e^{\sqrt{\alpha}\, t} + C_2 e^{-\sqrt{\alpha}\, t} + C_3 e^{\sqrt{\delta}\, t} + C_4 e^{-\sqrt{\delta}\, t},$$

so from the conditions $b(0) = 0, b'(0) = 0, b''(0) = \beta, b'''(0) = 0$, we get the system of the equations

$$\begin{aligned} b(0) &= C_1 + C_2 + C_3 + C_4 = 0 \\ b'(0) &= \sqrt{\alpha}(C_1 - C_2) + \sqrt{\delta}(C_3 - C_4) = 0 \\ b''(0) &= \alpha(C_1 + C_2) + \delta(C_3 + C_4) = \beta \\ b'''(0) &= (\sqrt{\alpha})^3 (C_1 - C_2) + \left(\sqrt{\delta}\right)^3 (C_3 - C_4) = 0. \end{aligned}$$

As $\alpha \neq \delta$, the solution of this system is

$$C_1 = C_2 = \frac{\beta}{2(\alpha-\delta)} \quad \text{and} \quad C_3 = C_4 = \frac{-\beta}{2(\alpha-\delta)},$$

so the particular solution of the differential equation (2.32) is

$$b(t) = \frac{\beta}{2(\alpha-\delta)}\left(e^{\sqrt{\alpha}\,t} + e^{-\sqrt{\alpha}\,t}\right) + \frac{-\beta}{2(\alpha-\delta)}\left(e^{\sqrt{\delta}\,t} + e^{-\sqrt{\delta}\,t}\right). \tag{2.33}$$

From the second equation of the system (2.28) and $\beta \neq 0$, we get

$$d(t) = \frac{1}{\beta}\left(b''(t) - \alpha b(t)\right) = \frac{1}{2}\left(e^{\sqrt{\delta}\,t} + e^{-\sqrt{\delta}\,t}\right). \tag{2.34}$$

VI) Case $\alpha\delta - \beta\gamma = 0$, with $\alpha + \delta \neq 0$ and $\alpha, \beta, \gamma, \delta \neq 0$. In this case, from (2.6) we have $r^2 = 0$ or $r^2 = \alpha + \delta$, so the general solution of the differential equation (2.5) is

$$a(t) = C_1 + C_2 t + C_3 e^{\sqrt{\alpha+\delta}\,t} + C_4 e^{-\sqrt{\alpha+\delta}\,t}.$$

From the conditions $a(0) = 1, a'(0) = 0, a''(0) = \alpha, a'''(0) = 0$, we get the system of the equations

$$\begin{aligned}
&a(0) = C_1 + C_3 + C_4 = 1\\
&a'(0) = C_2 + \sqrt{\alpha+\delta}(C_3 - C_4) = 0\\
&a''(0) = (\alpha+\delta)(C_3 + C_4) = \alpha\\
&a'''(0) = \left(\sqrt{\alpha+\delta}\right)^3 (C_3 - C_4) = 0.
\end{aligned}$$

The solution of this system is $C_1 = \frac{\delta}{\alpha+\delta}$, $C_2 = 0$, $C_3 = C_4 = \frac{\alpha}{2(\alpha+\delta)}$, so the corresponding particular solution is

$$a(t) = \frac{\delta}{\alpha+\delta} + \frac{\alpha}{2(\alpha+\delta)}\left(e^{\sqrt{\alpha+\delta}\,t} + e^{-\sqrt{\alpha+\delta}\,t}\right). \tag{2.35}$$

From the first equation of the system (2.2) and since $\beta \neq 0$, we get

$$c(t) = \frac{1}{\beta}\left(\frac{-\alpha\delta}{\alpha+\delta} + \frac{\alpha\delta}{2(\alpha+\delta)}\left(e^{\sqrt{\alpha+\delta}\,t} + e^{-\sqrt{\alpha+\delta}\,t}\right)\right).$$

Since $\frac{\alpha\delta}{\beta} = \gamma$, we may write

$$c(t) = \frac{-\gamma}{\alpha+\delta} + \frac{\gamma}{2(\alpha+\delta)}\left(e^{\sqrt{\alpha+\delta}\,t} + e^{-\sqrt{\alpha+\delta}\,t}\right). \tag{2.36}$$

Analogously from the second and fourth equation of the system (2.2) and conditions $b(0) = 0, b'(0) = 0, b''(0) = \beta, b'''(0) = 0$, we get the general solution

$$b(t) = C_1 + C_2 t + C_3 e^{\sqrt{\alpha+\delta}\,t} + C_4 e^{-\sqrt{\alpha+\delta}\,t},$$

where

$$\begin{aligned}
&b(0) = C_1 + C_3 + C_4 = 0\\
&b'(0) = C_2 + \sqrt{\alpha+\delta}(C_3 - C_4) = 0\\
&b''(0) = (\alpha+\delta)(C_3 + C_4) = \beta\\
&b'''(0) = \left(\sqrt{\alpha+\delta}\right)^3 (C_3 - C_4) = 0.
\end{aligned}$$

The solution of this system is $C_1 = \frac{-\beta}{\alpha+\delta}, C_2 = 0, C_3 = C_4 = \frac{\beta}{2(\alpha+\delta)}$, hence the corresponding particular solution is

$$b(t) = \frac{-\beta}{\alpha+\delta} + \frac{\beta}{2(\alpha+\delta)} \left(e^{\sqrt{\alpha+\delta}\,t} + e^{-\sqrt{\alpha+\delta}\,t} \right). \tag{2.37}$$

Then, from the fourth equation of the system (2.2) it follows

$$d(t) = \frac{\alpha}{\alpha+\delta} + \frac{\delta}{2(\alpha+\delta)} \left(e^{\sqrt{\alpha+\delta}\,t} + e^{-\sqrt{\alpha+\delta}\,t} \right). \tag{2.38}$$

Specially, for $\alpha = \beta = \gamma = \delta \neq 0$, from the relations (2.35)-(2.38) follows

$$a(t) = d(t) = \frac{1}{2} + \frac{1}{4}\left(e^{\sqrt{2\alpha}\,t} + e^{-\sqrt{2\alpha}\,t} \right), \; b(t) = c(t) = \frac{-1}{2} + \frac{1}{4}\left(e^{\sqrt{2\alpha}\,t} + e^{-\sqrt{2\alpha}\,t} \right).$$

For example, if $\alpha = -8$, then

$$a(t) = d(t) = \frac{1}{2} + \frac{1}{2}\cos(4t), \quad b(t) = c(t) = \frac{-1}{2} + \frac{1}{2}\cos(4t).$$

If $\delta = \beta$, $\gamma = \alpha$, according to the relations (2.35)-(2.38), it is, specially,

$$\begin{aligned} a(t) &= \tfrac{\beta}{\alpha+\beta} + \tfrac{\alpha}{2(\alpha+\beta)} \left(e^{\sqrt{\alpha+\beta}\,t} + e^{-\sqrt{\alpha+\beta}\,t} \right), \\ b(t) &= \tfrac{-\beta}{\alpha+\beta} + \tfrac{\beta}{2(\alpha+\beta)} \left(e^{\sqrt{\alpha+\beta}\,t} + e^{-\sqrt{\alpha+\beta}\,t} \right), \\ c(t) &= \tfrac{-\alpha}{\alpha+\beta} + \tfrac{\alpha}{2(\alpha+\beta)} \left(e^{\sqrt{\alpha+\beta}\,t} + e^{-\sqrt{\alpha+\beta}\,t} \right), \\ d(t) &= \tfrac{\alpha}{\alpha+\beta} + \tfrac{\beta}{2(\alpha+\beta)} \left(e^{\sqrt{\alpha+\beta}\,t} + e^{-\sqrt{\alpha+\beta}\,t} \right). \end{aligned}$$

VII) Case

It is not hard to get also the following results:

a) for $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C(t) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

b) for $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$, $C(t) = \begin{bmatrix} 1 & \frac{\alpha}{2}t^2 \\ 0 & 1 \end{bmatrix}$,

c) for $A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$, $C(t) = \begin{bmatrix} \frac{1}{2}\left(e^{\sqrt{\alpha}\,t} + e^{-\sqrt{\alpha}\,t} \right) & 0 \\ 0 & \frac{1}{2}\left(e^{\sqrt{\alpha}\,t} + e^{-\sqrt{\alpha}\,t} \right) \end{bmatrix}$.

All obtained results we may join in the following theorem.

Theorem 2.6. *The elements of cosine operator function* $C(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$ ($t \in \mathbb{R}$), *generated by real matrix* $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, *are given on the following ways:*

I) in case $(\alpha - \delta)^2 + 4\beta\gamma > 0$, *with* $\beta \neq 0$, $\gamma \neq 0$ *and* $\alpha\delta - \beta\gamma \neq 0$, *it values the relations* (2.9)-(2.12)

II) in case $(\alpha - \delta)^2 + 4\beta\gamma = 0$, *with* $\beta \neq 0$, $\gamma \neq 0$ *and* $\alpha + \delta \neq 0$, *it values the relations* (2.13)-(2.16)

III) in case $(\alpha - \delta)^2 + 4\beta\gamma < 0$, *with* $\alpha, \beta, \gamma, \delta \neq 0$, *it values the relations* (2.18)-(2.21)

IV) in case $\beta = 0$, $\gamma \neq 0$, $\alpha \neq \delta$, *it values the relations* (2.24)-(2.27)

V) in case $\gamma = 0$, $\beta \neq 0$, $\alpha \neq \delta$, it values the relations (2.30)*,* (2.31)*,* (2.33) *and* (2.34)
VI) in case $\alpha\delta - \beta\gamma = 0$, $\gamma \neq 0$, with $\alpha + \delta \neq 0$ and $\alpha, \beta, \gamma, \delta \neq 0$, it values the relations (2.35)-(2.38)
VII) in cases of matrices $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$ it values the results from Case VII a), b) and c).

Remark 2.7. On the same way, by solving corresponding Cauchy problem (2.1) (i.e. (2.2)-(2.3)) we may determine and discuss the cosine operator functions as family of linear operators from real vector space $\mathbb{R}^2$ to $\mathbb{R}^2$ and in other special cases of given generator A.

Remark 2.8. We may notice that founded cosine operator functions, if we consider them as matrices, satisfy the condition $C(0) = I$ and the D'Alambert functional equation in a vector space of matrices $\mathcal{M}_2(\mathbb{R})$, hence these results have certain significance in linear algebra too. The methods presented in this paper may be extended to a higher dimensional systems. We are planning to do the similar research in a real vector space $\mathbb{R}^3$, with some types of matrix A as a generator of cosine operator functions.

3. Conclusion

The benefit of the results in this paper is twofold, since besides determining the cosine functions depending on real parameters α, β, γ and δ, it also gives explicit formulas for the solutions $a(t), b(t), c(t)$ and $d(t)$ of the considering Cauchy problems for the systems of differential equations of the second order. The sine operator functions with the same generator A are associated to these cosine operator functions by the formula (1.3). The theory of cosine operator functions and the results presented here may be applied in solving various second order differential equations with some boundary conditions in Banach spaces, especially in $\mathbb{R}^2$ (well-posed Cauchy problems), such as, for example, wave equations, evolution equations and so on. It has also applications to the controllability of some linear systems of second order (see [4]).

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