

S - CURVATURE OF HOMOGENEOUS FINSLER SPACE WITH SPECIAL (α, β) -METRICS

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ABSTRACT. In this paper, we derive the formula for S -curvature of homogeneous Finsler space with special (α, β) -metric $F = \alpha + \frac{\beta^2}{\alpha}$ and further proved that the almost isotropic S -curvature must vanishes S -curvature. Also we find the explicit formula of the mean Berwald curvature.

1. INTRODUCTION AND PRELIMINARIES

Let (M, F) be a Finsler space, where M be a connected smooth manifold and F be the Finsler metric. A Finsler space (M, F) is homogeneous, if the group of isometries $I(M, F)$ of (M, F) acts transitively on M . As described in [3, 4] for homogeneous Finsler space, M can be written as a coset-space G/H with (α, β) -metric of the form $F = \Phi(s)$, where $s = \frac{\beta}{\alpha}$, with α be a G -invariant Riemannian metric on G/H and β is a G -invariant vector field on G/H . Therefore the Lie algebra \mathfrak{g} of G as a composition of \mathfrak{h} and \mathfrak{m} as;

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}(\text{direct sum of subspaces}). \quad (1.1)$$

Such that $Ad(\mathfrak{h})(\mathfrak{m}) \subset \mathfrak{m}$, $\mathfrak{h} \in \mathfrak{H}$ and we can identify M with the tangent space of (G/H) at the origin $o = H$. Further, the 1-form β corresponds to a vector field X on G/H which is generated by $Ad(H)$ -invariant vector in M with length < 1 . The S -curvature of Finsler space was introduced in Finsler geometry by S. Shen [17], for given composition theorem on Finsler manifold. The authors [1, 2, 4, 9, 12, 16] were investigated and studied S -curvature in Finsler geometry. Thus the non-Riemannian quantity is used for characterization of Finsler metrics among Berwald metric, Riemannian metric and locally Minkowski metric [11, 16]. In this paper we study by curvature properties of homogeneous Finsler space with special (α, β) -metric $F = \alpha + \frac{\beta^2}{\alpha}$ and also finding the formula for S -curvature and mean Berwald curvature.

In this section we recollected some known results and notations. A formal definition of Finsler geometry as follows.

Date: Received: Jun 28, 2018; Accepted: Oct 30, 2018.

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2010 *Mathematics Subject Classification.* Primary 53C25, 53C40; Secondary 53D15.

Key words and phrases. Finsler space, S -curvature, (α, β) -metrics, mean Berwald curvature, homogenous Finsler space.

Definition 1.1. A Minkowski norm on a real vector space V is a continuous function $F : V \rightarrow [0, \infty)$ satisfying the following;

1. F is positive and smooth on $V \setminus \{0\}$;
2. $F(\lambda y) = \lambda F(y)$ for any $\lambda > 0$;
3. With respect to any linear coordinates $y = y^i e_i$, the Hessian matrix

$$g_{ij}(y) = \left(\frac{1}{2} [F^2]_{y^i y^j}(y) \right), \quad (1.2)$$

is positive definite at any $y \neq 0$.

The Hessian matrix $g_{ij}(y)$ and its inverse $g^{ij}(y)$ of a Minkowski norm can be use to rise up or lower down indices of tensors on the vector space.

Given any $y \neq 0$, the Hessian matrix $(g_{ij}(y))$ defines an inner product $\langle \cdot, \cdot \rangle_y$ defines on V be $\langle u, v \rangle_y = g_{ij}(y) u^i v^j$, where, $u = u^i e_i$ and $v = v^i e_i$. This inner product can also be written as

$$\langle u, v \rangle_y = \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]|_{s=t=0} \quad (1.3)$$

which is independent of the choice of the linear coordinates.

A Finsler metric F on a smooth manifold M , $\dim M = n$, is a continuous function $F : TM \rightarrow [0, +\infty)$ such that it is positive and smooth on the slit tangent bundle $TM/0$, its restriction to each tangent space is a Minkowski norm. We generally say that (M, F) is a Finsler manifold or Finsler space.

In the Finsler geometry, important class of Finsler metric is a (α, β) -metrics[8]. A Finsler metric is said to be (α, β) -metric if it can be expressed in the form,

$$F = \phi(s), \quad s = \frac{\beta}{\alpha}, \quad (1.4)$$

where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form. Here $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq 0 < b_0, \quad b = \|\beta(x)\|_\alpha. \quad (1.5)$$

In this article we considered an important examples of such (α, β) -metrics as $F = \alpha + \frac{\beta^2}{\alpha}$ expressed in the following form,

$$F = \alpha(x, y) + \frac{\beta^2(x, y)}{\alpha} \quad (1.6)$$

$$= \phi(s), \quad \phi(s) = 1 + s^2, \quad (1.7)$$

Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}). \quad (1.8)$$

We use the following notations.

$$r_{i0} = r_{ij} y^j, \quad r_{00} = r_{ij} y^i y^j, \quad r_j = b^i r_{ij}, \quad (1.9)$$

$$s_{i0} = s_{ij} y^j, \quad s^j = b^i s_{ij}, \quad r_0 = r_j y^j, \quad s_0 = s_j y^j \quad (1.10)$$

For a Finsler metric $F = F(x, y)$, the geodesics are characterized by system of second order ordinary differential equations,

$$\ddot{x} + 2G^i(x, \dot{x}) = 0, \tag{1.11}$$

where $G^i = G^i(x, y)$ are called spray coefficients and are given by,

$$G^i = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial x^i}, \tag{1.12}$$

in standard local coordinates (x^i, y^i) in TM .

The S -curvature is an important non-Riemannian quantity introduced in [11], defined with respect to a volume form $dv = \sigma(x)dx$ by,

$$S = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} (\ln \sigma_F(x)). \tag{1.13}$$

Here the volume can be the Busemann-Housdorff volume form. $dV_{BH}(x)dx$, where

$$\sigma_{BH}(x) = \frac{vol(B^n)}{vol\{(y^i) \in R^n / F(x, y) < 1\}}, \tag{1.14}$$

or the Holmes-Thompson volume form $dV_{HT} = \sigma_{HT}(x)dx$, where

$$\sigma_{HT}(x) = \frac{1}{vol(B^n)} \int_{F(x,y) < 1} det(g_{ij}(x, y)dy). \tag{1.15}$$

Unless specified, the S -curvature usually defined with respect to the Busemann-Housdorff volume form.

Definition 1.2. A Finsler space (M, F) is said to be have almost isotropic S -curvature if there exists a smooth function on M and closed 1-form η such that:

$$S(x, y) = (n + 1)(c(x)F(y) + \eta(y)), \quad x \in M, \quad y \in T_x M. \tag{1.16}$$

If in the above equation $\eta = 0$, then (M, F) is said to have isotropic S -curvature. If $\eta = 0$ and $c(x)$ is a constant, then (M, F) is said to have constant S -curvature. In [9], author defined the S -curvature of the special (α, β) -metric $F = \alpha\phi(s)$, in local coordinate system by,

$$s = \left\{ 2\psi - \frac{f'(b)}{bf(b)} \right\} (r_0 + s_0) - \frac{\phi}{2\alpha\Delta^2} (r_{00} - 2\alpha Qs_0), \tag{1.17}$$

where

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \Delta = 1 + SQ + (b^2 + s^2)Q', \tag{1.18}$$

$$\phi = -(Q - SQ')(n\Delta + 1 + SQ) - (b^2 - s^2)(1 + SQ)Q'', \quad \psi = \frac{Q'}{2\Delta}, \tag{1.19}$$

and the function $f(b)$ in the formula is as follows,

$$f(b) = \begin{cases} \frac{\int_0^\pi \sin^{n-2} t dt}{\int_0^\pi \frac{\sin^{n-2} t}{\phi(\text{bcost})^n} dt}, & \text{if } dV = dV_{BH}, \\ \frac{\int_0^\pi \sin^{n-2} T(\text{bcost}) dt}{\int_0^\pi \sin^{n-2} t dt}, & \text{if } dV = dV_{HT} \end{cases} \quad (1.20)$$

2. S -CURVATURE OF HOMOGENEOUS FINSLER SPACE WITH SPECIAL (α, β) -METRICS

Recall that two Finsler spaces (M_1, F_1) and (M_2, F_2) are said to be isometric if there exists a diffeomorphism φ from M_1 onto M_2 , such that $F_1(x, y) = F_2(\varphi(x), \varphi(y))$ for any $x \in M_1$ and $y = T_x M_1$. In 2006[12], S.Deng and Z.Wang proved that the group of isometries of Finsler space is a Lie transformation group on the original manifold which can be used to study homogeneous Finsler spaces. A homogeneous Finsler space M can be written as a coset space G/H with G -invariant (α, β) -metric $F = \alpha + \frac{\beta^2}{\alpha}$, where both Riemannian metric α and the 1-form β are invariant under the action of G . In [7] S.Deng and Z.Hou, specified that, β corresponding to a unique vector u in $T_0(G/H)$ which is fixed under linear isotropy representation of H on $T_0(G/H)$ and $o = H$ is the origin of G/H . And the authors S.Deng and X.Wang gives the following result.

Theorem 2.1. [12] *Let $F = \phi(s)$ be G -invariant (α, β) -metric on the reduction homogeneous manifold G/H with a decomposition of the function*

$$g = h + m. \quad (2.1)$$

Then the S -curvature of F has the form;

$$S(0, y) = -\frac{1}{\alpha(y)} \frac{\Phi}{2\Delta^2} \{-c\langle [u, y]_m, y \rangle - \alpha(y)\langle [u, y]_m, u \rangle\}, \quad y \in m, \quad (2.2)$$

where u is the vector in m corresponding to the 1-form β and we have identified m with tangent space of G/H at origin $o = H$.

In this section, we find the formula of S -curvature for homogeneous Finsler space with special (α, β) -metrics. By (1.1) we can write special (α, β) -metric $F = \alpha + \frac{\beta^2}{\alpha}$ as $F = \phi(s)$, where $\phi(s) = 1 + s^2$. Now, for this special (α, β) -metric by the equations (1.18) and (1.19) and theorem 2.1, we obtained the following quantities:

$$Q = -\frac{2s}{1-s^2}, \quad Q' = \frac{2(s^2+1)}{1+s^4-2s^2} \quad (2.3)$$

$$\Delta = \frac{1-3s^4-2s^2+2b^2(s^2+1)}{(1+m)s^2}, \quad (2.4)$$

$$\Phi = \frac{s^2[(s^6(b^2-1)s^5(n+1) - s^4(3b^2+2)s^3(2b^2n) - s^2(3b^2-1) + b^2)]}{(1-s^2)^4}. \quad (2.5)$$

Since $(G/H, F)$ is homogeneous, it is enough to compute the S -curvature at the origin $o = H$. Let $(U, (x^1, x^2, x^3, \dots, x^n))$ be the local coordinate system with

reference to [12], we have to find the S -curvature in local coordinate system, for that we need to evaluate the following quantities at the origin;

- (1) $r_{ij} = \frac{1}{2} (b_{i;j} + b_{j;i})$ and b_i are defined by $\beta = b_i(x)dx^i$,
- (2) $s_i = b_j s_j^i$ and s_j^i are defined by $s_j^i = \frac{1}{2} \left(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} \right)$,
- (3) $s_0 = s_i y^i$,

and $\rho_0 = \rho_{x^i} y^i$. where, $\rho = \sqrt{1 - \|\beta\|}$ is the length of the 1-form β with respect to α .

Let \langle , \rangle be the corresponding inner product on m . Then we have to consider the Levi-Civita connection of $(G/H, \alpha)$ which will be useful to calculate the S -curvature of homogeneous Finsler space. There are many versions of the formula of the connection for Killing vector fields. But, we are working on the differential of (left) invariant vector fields on G/H and we adopt the formula from [4]. Now for $v \in g$, define a one parameter transformation group $\phi_t, t \in R$ of G/H by,

$$\phi_t(gH) = (\exp(tv)g)H, \quad g \in G. \tag{2.6}$$

Now ϕ_t generates a vector field on G/H which is a Killing vector field (this is called the fundamental vector field generated by v in [12]), and we denote this vector field by \bar{v} . In [16] authors defined the following formula,

$$\langle \nabla_{v_1} v_2 |_o, w \rangle = \frac{1}{2} (\langle -[v_1, v_2]_m, w \rangle + \langle [w, v_2]_m, v_1 \rangle + \langle [w, v_1]_m, v_2 \rangle), \tag{2.7}$$

where, $v_1, v_2, w \in m$, and m is identified with the tangent space, $o = H$ is the origin of the coset space and $[v_1, v_2]_m$ denote the projection of $[v_1, v_2]$ to m corresponding to the decomposition (2.1).

Now by using the formula (2.7), we have to compute r_{00} and s_0 . First consider,

$$b_i = \beta \left(\frac{\partial}{\partial x^i} \right) = \left\langle \bar{u}, \frac{\partial}{\partial x^i} \right\rangle = c \left\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i} \right\rangle, \tag{2.8}$$

by differentiating b_i we get,

$$\frac{\partial b_i}{\partial x^j} = c \frac{\partial}{\partial x^j} \left\langle \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i} \right\rangle \tag{2.9}$$

$$= c \left(\left\langle \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^i} \right\rangle \right) + \left\langle \frac{\partial}{\partial x^n}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} \right\rangle. \tag{2.10}$$

Hence, at the origin we have (here we use the symmetry of the connection:

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0).$$

$$s_{ij}(0) = \frac{1}{2} c \left(\left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \right\rangle + \left\langle \frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right\rangle \right). \tag{2.11}$$

Then by Equation (2.7) we have,

$$s_{ij}(0) = \frac{1}{2} \langle [u_i, u_j]_m, u_n \rangle. \quad (2.12)$$

Since at the origin we have $(a_{ij}) = I_n$, we get;

$$s_j^i(0) = a^{ik}(0) s_{kj}(0) = \sum_{k=1}^n \delta_i^k s_{kj}(0) = s_{ij}(0). \quad (2.13)$$

Hence,

$$s_i(0) = b_l(0) s_i^l(0) = c s_i^n(0) = s_{nl}(0). \quad (2.14)$$

Now for $y = y^i u_i \in m$, we have

$$s_0(y) = y^l s_l(0) = c y^l s_{nl}(0) = \frac{1}{2} y^l \langle [u_n, u_l]_m, u_n \rangle, \quad (2.15)$$

$$= \frac{1}{2} \langle [c u_n, y^l u_l]_m, c u_n \rangle, \quad (2.16)$$

$$= \frac{1}{2} \langle [u, y]_m, u_i \rangle. \quad (2.17)$$

Next we compute r_{ij} . Suppose $i \geq j$. Then we have

$$r_{ij}(0) = \frac{1}{2} (b_{i;j} + b_{j;i})|_0, \quad (2.18)$$

$$= \frac{1}{2} \left(\frac{\partial b_i}{\partial x^j} - b_i \tau_{ji}^l + \frac{\partial b_j}{\partial x^i} - b_l \tau_{ji}^l \right) |_0, \quad (2.19)$$

$$= \frac{1}{2} \left(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i} \right) |_0 - c \tau_{ji}^l(0). \quad (2.20)$$

By using Equation(2.12), we have

$$= \frac{1}{2} \left(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i} \right) |_0 - c \tau_{ij}^l(0) \quad (2.21)$$

$$= \frac{1}{2} \langle [u_i, u_j]_m, u_n \rangle, \quad i \geq j. \quad (2.22)$$

Combining the above equation (2.21) with the connection coefficients τ_{ij}^l defined in[16], we get

$$r_{ij}(0) = -\frac{1}{2} c (\langle [u_n, u_j]_m, u_i \rangle + \langle [u_n, u_i]_m, u_j \rangle), \quad i \geq j. \quad (2.23)$$

Further,

$$r_{00}|_0 = r_{ij}(0) y^i y^j = -c (\langle [u_n, y]_m, y \rangle). \quad (2.24)$$

Now by substituting the above quantities and the values of (2.3),(2.4) and (2.5) into the equation (2.2) we get the formula for S -curvature of homogeneous Finsler space with the special (α, β) -metric $F = \alpha + \frac{\beta^2}{\alpha}$ at the origin as follows,

$$s(0, y) = -\frac{1}{\alpha(y)} \frac{A}{2[1 - 3s^4 - 2s^2 + 2b^2(s^2 + 1)]^2} c \langle [u, y]_m, y \rangle + \frac{2s\alpha(y)}{1 - s^2} \langle [u, y]_m, u \rangle \quad (2.25)$$

where,

$$A = s^2[s^6(b^2 - 1) - s^5(n + 1) - s^4(3b^2 + 2) + s^3(2b^2n) - s^2(3b^2 - 1) + s(2nb^2 + n + 1) - b^2] \tag{2.26}$$

By observing above finally we state,

Theorem 2.2. *Let $F = \alpha + \frac{\beta^2}{\alpha}$ be a G -invariant (α, β) -metric on the reductive homogeneous manifold G/H , with the decomposition of the Lie algebra.*

$$g = h + m \tag{2.27}$$

The S -curvature of the Finsler special (α, β) - metric $F = \alpha + \frac{\beta^2}{\alpha}$ has the form,

$$s(0, y) = -\frac{1}{\alpha(y)} \frac{A}{2[1 - 3s^4 - 2s^2 + 2b^2(s^2 + 1)]^2} \left\{ c\langle [u, y]_m, y \rangle + \frac{2s\alpha(y)}{1 - s^2} \langle [u, y]_m, u \rangle \right\} \tag{2.28}$$

where,

$$A = s^2[s^6(b^2 - 1) - s^5(n + 1) - s^4(3b^2 + 2) + s^3(2b^2n) - s^2(3b^2 - 1) + s(2nb^2 + n + 1) - b^2]$$

and u is the vector in m corresponding to the 1-form β and we identified m with the tangent space of G/H at the origin $o = H$.

As a direct application of the above formula we get,

Theorem 2.3. *Let $(G/H, F)$ be as in the above theorem 2.2. Then homogeneous Finsler space with special (α, β) -metric F has isotropic S -curvature if and only if F has vanishing S -curvature.*

Proof. By using the equation (2.28) it is enough to show the direct part. So, suppose F has isotropic S -curvature;

$$S(x, y) = (n + 1)c(x)F(y), \quad x \in G/H, \quad y \in T_x(G/H). \tag{2.29}$$

Letting $x = 0$ and $y = u$ then (2.28) we get $c(0) = 0$. Hence $S(0, y) = 0$ for all $y \in T_0(G/H)$. Since F is a homogeneous metric, we must have $S = 0$ everywhere. \square

3. Mean Berwald Curvature of Homogeneous Finsler space with special (α, β) -metrics.

Mean Berwald curvature is an one more important properties in non-Riemannian Finsler geometry. In this section we find the formula of mean Berwald curvature of homogeneous Finsler space with special (α, β) -metric $F = \alpha + \frac{\beta^2}{\alpha}$. In [11] X.Cheng and Z.Shen, gives the mean Berwald curvature for Finsler geometry as follows.

$$E_{ij} = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{\partial G^m}{\partial y^m} \right), \tag{3.1}$$

where, $G^m = G^m(x, y)$ are the spray coefficients. Then by S -curvature can be defined as,

$$S = \frac{\partial G^m}{\partial y^m} - (\ln \sigma(x))_{x^k} y^k. \quad (3.2)$$

Here, $\sigma(x)$ is the function of x , the second term in the above equation $(\ln \sigma(x))$ is also function of x only.

Therefore,

$$\frac{\partial^2}{\partial y^i \partial y^j} [(\ln \sigma(x))_{x^k} y^k] = 0. \quad (3.3)$$

Hence, the formula for mean Berwald curvature obtained by direct computations,

$$\frac{\partial^2 S}{\partial y^i \partial y^j} = \frac{\partial^2}{\partial y^i \partial y^j} \left[\frac{\partial G^m}{\partial y^m} - (\ln \sigma(x))_{x^k} y^k \right], \quad (3.4)$$

$$= \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{\partial G^m}{\partial y^m} \right) = 2E_{ij}, \quad (3.5)$$

So, that clearly by partially differentiating the expression of S -curvature derived in section 2. We determine the formula for mean Berwald curvature of the homogeneous Finsler space.

i.e.,

$$2E_{ij} = \frac{\partial^2 S}{\partial y^i \partial y^j}. \quad (3.6)$$

Before the computations we consider the identities,

$$\frac{\partial s}{\partial y^m} = \frac{1}{\alpha} \left(b_m - s \frac{y^m}{\alpha} \right), \quad \frac{\partial \alpha}{\partial y^m} = \frac{y^m}{\alpha}, \quad (3.7)$$

Now, we prove the following theorem.

Theorem 3.1. *Let $(G/H, F)$ be a homogeneous Finsler space with special (α, β) -metric of the form $F = \alpha + \frac{\beta^2}{\alpha}$, where α be a G -invariant vector on G/H . Then the mean Berwald curvature of homogeneous Finsler space is of the form,*

$$E_{ij} = \frac{1}{\alpha(y)} \{ \langle [u, u_j]_m, u_j \rangle + \langle [u, u_j]_m, u_i \rangle + \langle [u, y]_m, y \rangle k_1 \} \\ + \langle [u, y]_m, u \rangle \frac{k_2}{(\alpha^3 + \alpha^3 s^4 - 2\alpha^3 s^2)^2} \quad (3.8)$$

where, k_1, k_2 in (3.17) and (3.21) respectively.

Proof. Let us consider special Finsler (α, β) -metric $F = \phi(s)$, where, $\phi(s) = 1 + s^2$. Put $\psi = \phi - s\phi$, then we have

$$Q = \frac{\phi'}{\psi} = \frac{2s}{1 - s^2}, \quad Q' = \frac{2(s^2 + 1)}{1 + s^4 - 2s^2}, \quad \Delta = \frac{1 - 3s^4 - 2s^2 + 2b^2(s^2 + 1)}{(1 - s^2)^2}, \quad (3.9)$$

$$\begin{aligned} \phi = & \frac{4s^2[s^6(b^2 - 1) - s^5(5n + 1) - s^4(3b^2 + 2)]}{(1 - s^2)^4} \\ & + \frac{2b^n s^3 - 4s^2(3b^2 + 1) + s(2nb^2 + n + 1) - 4b^2}{(1 - s^2)^4}. \end{aligned} \quad (3.10)$$

Letting , $c = 1$ in the formula of S -curvature defined in equation (2.26)

$$\begin{aligned} s(0, y) = & -\frac{1}{\alpha(y)} \frac{A}{2[1 - 3s^4 - 2s^2 + 2b^2(s^2 + 1)]^2} \\ & - \left\{ \langle [u, y]_m, y \rangle + \frac{2s\alpha(y)}{1 - s^2} \langle [u, y]_m, u \rangle \right\}. \end{aligned} \quad (3.11)$$

Now using (3.6) we obtain,

$$\begin{aligned} 2E_{ij} = & \frac{\partial^2 S(0, y)}{\partial y^i \partial y^j} \\ = & \frac{A}{2[1 - 3s^4 - 2s^2 + 2b^2(s^2 + 1)]^2} \\ & \times \frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{1}{\alpha(y) \langle [u, y]_m, y \rangle} + \frac{2s}{1 - s^2} \langle [u, y]_m, u \rangle \right\}. \end{aligned} \quad (3.12)$$

$$\begin{aligned} = & \frac{A}{2[1 - 3s^4 - 2s^2 + 2b^2(s^2 + 1)]^2} \left\{ \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{1}{\alpha(y) \langle [u, y]_m, y \rangle} \right) \right\} \\ & + \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{2s}{1 - s^2} \right) \langle [u, y]_m, u \rangle. \end{aligned} \quad (3.13)$$

where, $A = s^2[4s^6(b^2 - 1) - 4s^5(s + 3n + 1) - 4s^4(3b^2 + 2) + 8b^2ns - 4s^2(3b^2 + 1)4s(n + 1) + 4b^2(2sn - 1)]$.

Now, consider the first term,

$$\begin{aligned} \frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{1}{\alpha(y) \langle [u, y]_m, y \rangle} \right\} = & \frac{1}{\alpha(y)} \frac{\partial^2}{\partial y^i \partial y^j} \langle [u, y]_m, y \rangle \\ & + \langle [u, y]_m, y \rangle \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{1}{\alpha(y)} \right), \end{aligned} \quad (3.14)$$

but we get,

$$\frac{\partial \langle [u, y]_m, y \rangle}{\partial y^i} = \langle [u, u_i]_m, y \rangle + \langle [u, y]_m, u_i \rangle, \quad (3.15)$$

$$\frac{\partial \langle [u, y]_m, y \rangle}{\partial y^i \partial y^j} = \langle [u, u_i]_m, u_j \rangle + \langle [u, u_j]_m, u_i \rangle, \quad (3.16)$$

where,

$$k_1 = \alpha^3 \delta_i^j + 3\alpha y_i y_j. \quad (3.17)$$

Therefore we have,

$$\frac{\partial^2}{\partial y^i \partial y^j} \left\{ \frac{1}{\alpha(y) \langle [u, y]_m, y \rangle} \right\}, \quad (3.18)$$

where, k_1 defined in the above equation (3.17). Similarly, by computation the value of second term of the equation (3.13) becomes,

$$\frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{2s}{1-s^2} \langle [u, y]_m, u \rangle \right) = \langle [u, y]_m, u \rangle \frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{2s}{1-s^2} \right), \quad (3.19)$$

$$\frac{\partial^2}{\partial y^i \partial y^j} \left(\frac{2s}{1-s^2} \langle [u, y]_m, u \rangle \right) = \langle [u, y]_m, u \rangle \frac{k_2}{(\alpha^3 + \alpha^3 s^4 - 2\alpha^3 s^2)^2}, \quad (3.20)$$

where,

$$\begin{aligned} k_2 = & -2\alpha^3 s^4 b_j y^i - 4\alpha^2 s^3 b_j y^i - 10\alpha^2 b_j y^i - [8s b_j (\alpha^3 s^2 - 1) \\ & + 4\alpha s^2 y^j (-\alpha s^4 - 3s^3 - 3\alpha) + s^2 - 3\alpha + 4\alpha^3 s^3 y^j (3s^4 - 6s^2 y^j + 4s + 3) \\ & - 16s^5 y^j + 2\alpha^3 s^2 b_j] (b_i - s y^i / \alpha) - 4s^2 \delta_i^j. \end{aligned} \quad (3.21)$$

Hence substituting (3.17),(3.20) into (3.13) we get,

$$\begin{aligned} E_{ij} = & \frac{A}{2[1-3s^4-2s^2+2b^2(s^2+1)]^2} \left\{ \frac{1}{\alpha(y) \langle [u, u_j]_m, u_j \rangle} + \langle [u, u_j]_m, u_i \rangle \right. \\ & \left. + \langle [u, y]_m, y \rangle k_1 \right\} + \langle [u, y_m, u] \rangle \frac{k_2}{(\alpha^3 + \alpha^3 s^4 - 2\alpha^3 s^2)^2}, \end{aligned} \quad (3.22)$$

where,

$$k_1 = \alpha^3 \delta_i^j + 3\alpha y_i y_j, \quad (3.23)$$

$$\begin{aligned} k_2 = & -2\alpha^3 s^4 b_j y^i - 4\alpha^2 s^3 b_j y^i - 10\alpha^2 b_j y^i - [8s b_j (\alpha^3 s^2 - 1) \\ & + 4\alpha s^2 y^j (-\alpha s^4 - 3s^3 - 3\alpha s^2 - 3\alpha) + 4\alpha^3 s^3 y^j (3s^4 - 6s^2 y^j + 4s + 3) \\ & - 16s^5 y^j + 2\alpha^3 s^2 b_j] (b_i - s y^i / \alpha) - 4s^2 \delta_i^j. \end{aligned} \quad (3.24)$$

Therefore the equation (3.22) is the mean Berwald curvature of homogeneous special (α, β) -metrics. \square

4. CONCLUSION

S -curvature is an important quantities in Finsler geometry it has some well-known interrelations with the other quantities such as flag curvature, Ricci curvature scalar curvature etc.

In this paper, we investigated the curvature properties of homogeneous Finsler spaces with special (α, β) -metric and using the Busemann-Hausdorff volume form to find the formula for S -Curvature and also proved that a homogeneous Finsler space has almost isotropic S -curvature if and only if Finsler space has vanishing S -curvature and further we derived the formula for mean Berwald curvature E_{ij} of homogeneous Finsler space.

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