Gulf Journal of Mathematics
Vol 7, Issue 3 (2019) 1-6

ON P-2-BÉZOUT RINGS

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ABSTRACT. In this paper, we introduce a generalization of the well -known notion of a P-Bézout rings and a 2-Bézout rings, which we call a P-2-Bézout rings. We establish the transfer of this notion to trivial ring extensions and to pullbacks. There results provide example of non 2-Bézout ring which is P-2-Bézout ring and example of non P-Bézout ring which is P-2-Bézout ring. Our aim is to provide new classe of commutative rings.

1. INTRODUCTION

All rings considered below are commutative with unit and all modules are unital. Recall that a ring R is called P-Bézout if every finitely generated prime ideal P of R is principal [1] and 2-Bézout if every finitely presented ideal of R is principal [2]. We introduce a new concept of a "P-2-Bézout". A ring R is called P-2-Bézout if every finitely presented prime ideal P of R is principal. If R is P-Bézout or 2-Bézout, then R naturally P-2-Bézout.

Let T be a domain and let K be a field which is a retract of T, that is $T = K+M$ where M is a maximal ideal of T. Each subring D of K determines a subring $R = D + M$ of T. This construction arises frequently in algebra, especially in connection with counter examples.

The original of $D + M$ construction involved a valuation domain T with $K = T/M$, where M is the maximal ideal of T and $K \subseteq T$. A thoroughout account of results about $D + M$ construction can be find in [3, 4, 6, 9]

For a nonnegative integer n, an R-module E is n-presented if there is an exact sequence of R-modules:

$$F_n \longrightarrow F_{n-1} \longrightarrow \dots F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-module are respectively, finitely generated and finitely presented R-module.

Date: Received: Sep 9, 2019; Accepted: Nov 2, 2019.
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2010 *Mathematics Subject Classification.* Primary 13A15, 13B10; Secondary 13F05, 13D02.
Key words and phrases. P-2-Bézout rings, pullbacks, trivial rings extension.

Let A be a ring, E be an A-module and $R := A \propto E$ be the set of pairs (a, e) with pairwise addition and multiplication given by: $(a, e)(b, f) = (ab, af + be)$. R is called the trivial ring extension of A by E. Recall that a prime ideal of R always has the form $Q \propto E$, where Q is a prime ideal of A [7, Theorem 25.1 (3)]. Considerable work, part of it summarized in Glaz's book [6] and Huckaba's book [7] where R is called the idealization of E by A, has been concerned with trivial ring extensions. These have proven to be useful in solving many open problems and conjectures for various contexts in (commutative and non-commutative) ring theory. See for instance [6, 7, 11].

Our aim in this paper is to prove that P-2-Bézout rings are not P-Bézout rings and not 2-Bézout rings, in general. Further, we investigate the possible transfer of the P-2-Bézout property to various trivial extension constructions.

2. Main results

In this section, we study the possible transfer of the P-2-Bézout property to various trivial extension contexts. First, we examine the context of trivial ring extensions of a domain A by K-vector space E, where $K := qf(A)$.

Theorem 2.1. *Let A be a domain which is not a field, $K := qf(A)$, E be a K-vector space and $R := A \propto E$ be the trivial extension of A by E.*

(1) *If A is a P-2-Bézout domain, then R is a P-2-Bézout ring.*
(2) *If $E = K$, then A is a P-2-Bézout domain if and only if R is a P-2-Bézout ring.*
(3) *Assume that $Ann_A(e)$ is a finitely generated ideal of A for each $e \neq 0 \in E$. If R is a 2-Bézout ring, then $dim_K E = 1$.*

Before proving Theorem 2.1, we establish the following Lemma.

Lemma 2.2. . *With the notation of Theorem 2.1, let $I := R(a, e)$ be a principal ideal of R, where $a \in A - \{0\}$ and $e \in E$. Then, $I := Aa \propto E = R(a, 0)$.*

Proof. Clearly, $I = R(a, e) = \{(b, f)(a, e)/b \in A, f \in E\} = \{(ba, fa + be)/b \in A, f \in E\}$. But $\{af/f \in E\} = E$, hence $I = Aa \propto E = R(a, 0)$. □

Proof of Theorem 2.1.

(1) Assume that A is a P-2-Bézout domain and let P be a finitely presented prime ideal of R. Then, P has the form $P := Q \propto E \in Spec(R)$ by [7, Theorem 25.1 (3)], where Q is a prime ideal of A. Since $0 \propto E$ is not a finitely generated ideal of R (because A is not a field), then $Q \neq 0$. Moreover, $P := Q \propto E = \sum_{i=1}^{i=n} R(a_i, e_i) = R(a_1, e_1) + \cdots + R(a_n, e_n) = (Aa_1 \propto E) + \cdots + (Aa_n \propto E) = R(a_1, 0) + \cdots + R(a_n, 0) = \sum_{i=1}^{i=n} R(a_i, 0)$. We claim that $Q := \sum_{i=1}^{i=n} Aa_i$ is a finitely presented ideal of A. Indeed, consider the exact sequence of R-modules:

$$0 \longrightarrow Ker(w) \longrightarrow R^n \cong A^n \propto E^n \xrightarrow{w} P \longrightarrow 0$$

where w is defined by $w((\alpha_i,e_i)_{i=1}^{i=n})=\sum_{i=1}^{i=n}(\alpha_i,e_i)(a_i,0)$. But:

$$\begin{aligned} Ker(w) &= \{(\alpha_i,e_i)_{i=1}^{i=n}\in R^n/\sum_{i=1}^{i=n}(\alpha_i,e_i)(a_i,0)=0\}\\ &= \{(\alpha_i,e_i)_{i=1}^{i=n}\in R^n/\sum_{i=1}^{i=n}\alpha_i a_i=0 \ and \sum_{i=1}^{i=n}e_i a_i=0\}\\ &= X\propto F \end{aligned}$$

where $X:=\{(\alpha_i)_{i=1}^{i=n}\in A^n/\sum_{i=1}^{i=n}\alpha_i a_i=0\}$ and $F:=\{(e_i)_{i=1}^{i=n}\in E^n/\sum_{i=1}^{i=n}e_i a_i=0\}$. Therefore, $Ker(w)$ is a finitely generated R-module by [6, Theorem 2.1.2 (3)]. On the other hand, consider the exact sequence of A-modules:

$$0\longrightarrow Ker(u)\longrightarrow A^n\xrightarrow{u} Q\longrightarrow 0 \qquad (*)$$

where u is defined by $u((\beta_i)_{i=1}^{i=n})=\sum_{i=1}^{i=n}\beta_i a_i$. Clearly, $Ker(u)=X$ which is a finitely generated A-module since $ker(w)$ is a finitely generated R-module. Therefore, Q is a finitely presented ideal of A by the exact sequence $(*)$, so $Q=Aa$ for some $a\in A$ since A is a P-2-Bézout ring. Hence, $P=Q\propto E=R(a,0)$ which means that R is a P-2-Bézout ring, as desired.

(2) If A is a P-2-Bézout domain, then so is R by (1). Conversely, assume that R is a P-2-Bézout ring and let $Q:=\sum_{i=1}^{i=n}Ab_i$ be a nonzero finitely presented prime ideal of A, where $b_i\in Q$ for each $i\in\{1,\dots,n\}$. Then, $P:=Q\propto K$ is a finitely presented ideal of R by [8, Lemma 3.3] since $QK=K$. Therefore, $P:=Q\propto K=R(a,e)=Aa\propto E$ for some element (a,e) of R since R is a P-2-Bézout ring. Hence, $Q=Aa$, as desired.

(3) Assume that $\dim_K E\geqslant 2$. Let $e,f\in E$ such that $\{e,f\}$ is a K-linearly independant set. We claim that $R(0,e)+R(0,f)$ is a finitely presented ideal of R. Indeed, consider the exact sequence of R-modules:

$$0\longrightarrow Ker(v)\longrightarrow R\xrightarrow{v} R(0,e)\longrightarrow 0$$

where v is defined by $v(\alpha,k)=(\alpha,k)(0,e)$. Clearly, $Ker(v)=Ann_A(e)\propto E$ which is a finitely generated ideal of R since $Ann_A(e)$ is a finitely generated ideal of A by hypothesis. Hence, $R(0,e)$ is a finitely presented ideal of R by the above exact sequence. On the other hand, assume that $(l,m)(0,e)=(a,k)(0,f)\in R(0,e)\cap R(0,f)$ where $(l,m),(a,k)\in R$. Since $(l,m)(0,e)=(0,le)$ and $(a,k)(0,f)=(0,af)$, then $le=af$, hence $l=a=0$ since $\{e,f\}$ is a K-linearly independant set. So, $R(0,e)\cap R(0,f)=0$. Therefore, $R(0,e)+R(0,f)$ is a finitely presented ideal of R by [6, Corollary 2.1.3]. Our aim is to show that $R(0,e)+R(0,f)$ is not a principal ideal. Deny, there exists $g\in E$ such that $R(0,e)+R(0,f)=R(0,g)=0\propto Ag$. Hence, $Ae+Af=Ag$ moreover $Ke+Kf=Kg$ (since $K:=qf(A)$) and so $\{e,f\}$ is a K-linearly dependent set, a contradiction. Therefore, $R(0,e)+R(0,f)$ is not a principal ideal which means that R is not a 2-Bézout ring and this completes the proof of Theorem 2.1.

Now, we are able to construct a non 2-Bézout ring which is a P-2-Bézout ring.

Example 2.3. Let A be a Bézout and Noetherian domain (for example, a polynomial ring in one indeterminate over any field), $K := qf(A)$, E be a K-vector space such that $\dim_K E \geqslant 2$ and let $R := A \propto E$. Then:

(1) R is a P-2-Bézout ring by Theorem 2.1 (1).
(2) R is not a 2-Bézout ring by Theorem 2.1 (3) since $\dim_K E \geqslant 2$.

Next, we explore a different context; namely the trivial ring extension of a local domain (A, M) by an A-module E such that $ME = 0$.

Proposition 2.4. *Let A be a local domain with maximal ideal M, $E \neq 0$ an A-module such that $ME = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E. Assume that every nonzero prime ideal of A is maximal, M is not a finitely generated ideal of A and E is an (A/M)-vector space of finite rank. Then, R is a P-2-Bézout ring.*

Proof. Assume that every nonzero prime ideal of A is maximal, M is not a finitely generated ideal of A and E is an (A/M)-vector space of finite rank. Our aim is to show that R is a P-2-Bézout ring. We claim that $M \propto E$ and $0 \propto E$ are not finitely presented prime ideals of R. Indeed:

• $M \propto E$ is not finitely presented (since M is not a finitely generated ideal of A).

• $0 \propto E$ is not finitely presented. Deny, let $(e_i)_{i=1}^{i=n}$ be a basis of the (A/M)-vector space E. Consider the exact sequence of R-modules:

$$0 \longrightarrow Ker(u) \longrightarrow R^n \xrightarrow{u} 0 \propto E \longrightarrow 0$$

where $u((c_i, g_i)_{i=1}^{i=n}) = \sum_{i=1}^{i=n}(c_i, g_i)(0, e_i) = (0, \sum_{i=1}^{i=n} c_i e_i)$. Hence, $Ker(u) = M^n \propto E^n$ since $(e_i)_{i=1}^{i=n}$ is a basis of E, moreover is a finitely generated R-module (since $0 \propto E$ is a finitely presented ideal of R) and so M is a finitely generated ideal of A, a contradiction.Therefore, there is no finitely presented prime ideal of R and so R is a P-2-Bézout ring. □

Now, we are able to construct a non P-Bézout ring which is a P-2-Bézout ring.

Example 2.5. Let (A, M) be a nondiscrete valuation domain, E be an (A/M)-vector space of finite rank $\geqslant 2$ and let $R := A \propto E$. Then:

(1) R is a P-2-Bézout ring.
(2) R is not a P-Bézout ring.

Proof. (1) By Theorem 2.4 (1).

(2) Let $(e_i)_{i=1}^{i=n}$ be a basis of the (A/M)-vector space E. Then, $0 \propto E$ is a finitely generated prime ideal of R which is not principal. Deny, $0 \propto E = R(0, e)$ for some $(0, e) \in 0 \propto E$, a contradiction since $\dim_{A/M} E \geqslant 2$.

□

Next, we studying the possible transfer of P-2-Bézout property to pullbacks.

Proposition 2.6. *Let $T := K + M$ be a valuation domain of dimension one, where K is a field and M is the unique prime ideal of T and $R := D + M$*

where D is a subring of K such that either D is not a field or D is a field and $[K : D] = \infty$. Then R is a P-2-Bézout domain if and only if so is D.

We need the following Lemma before proving Proposition 2.6.

Lemma 2.7. *Let $T := K + M$ be a domain, where K is a field and M is a maximal ideal of T and $R := D + M$ where D is a subring of K such that $qf(D) = K$. If R is P-2-Bézout, then so is D.*

Proof. Let Q be a nonzero finitely presented prime ideal of D. Then, $P := Q \otimes_D R = QR = Q(D + M) = QD + QM = Q + M$ is a finitely presented prime ideal of R since R is faithfully flat over D and $QM = QKM = KM = M$. Hence, $P := R(a + m) = Da + M$ for some $a \in D$ and $m \in M$ since R is a P-2-Bézout. Therefore, $Q = Da$ is a principal ideal of D, as desired. □

Proof of Proposition 2.6. If R is P-2-Bézout, then D is P-2-Bézout by Lemma 2.7. Conversely, assume that D is P-2-Bézout and let P be a finitely presented prime ideal of R. Since M is not a finitely generated ideal of R by [6, Lemma 5.2.1] (since D is not a field or D is a field and $[K : D] = \infty$), P is of the form $P = Q + M$ by [10, Lemma 3], where Q is a nonzero prime ideal of D. Hence, Q is a finitely presented prime ideal of D since R is faithfully flat over D and $P = Q + M = Q \otimes_D R$ and so Q is a principal prime ideal of D since D is P-2-Bézout. Therefore, $P = Q \otimes_D R$ is a principal ideal of R since R is faithfully flat over D and so R is P-2-Bézout, thus completing the proof.

Example 2.8. Let D be a Bézout domain which is not a field, let $T = K + M$ be a one dimensional valuation, where K is a field such that $qf(D)$ is a proper subfield of K and M is the unique prime ideal of T. Then:

(1) $D + M$ is a P-2-Bézout domain by Proposition 2.6.
(2) $D+M$ is not a Bézout domain since not a coherent domain by [5, Theorem 5.2.10].

Open Problem. Is the property P-2-Bézout stable by homomorphic images and finite direct products?

Acknowledgement. The authors would like to thank the referee for his careful reading of the manuscript.

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