SUMS WHERE FRACTIONAL PARTS APPEAR

RAFAEL JAKIMCZUK

ABSTRACT. In this article we examine sums where fractional parts appear. These sums are a generalization of the well-known De la Vallée Poussin’s formula.

1. Introduction

In a short proof and using very elementary methods we prove the asymptotic formula

\[
\sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\}^2 = \left( (1 - \gamma) - 2 + \log 2 + \log \pi \right) x + O \left( x^{3/2} \right),
\]

(1.1)

where \( \left\{ y \right\} = y - \lfloor y \rfloor \) is the fractional part of \( y \) and the \( m \) are positive integers. We let \( \lfloor y \rfloor \) denote the integer part function. This formula is a generalization of the formula proved by Dirichlet in 1849. Namely

\[
\sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\} = (1 - \gamma)x + O \left( \sqrt{x} \right).
\]

(1.2)

In 1898 de la Vallée Poussin [3] obtained some generalizations of the Dirichlet’s formula doing some restrictions on the divisors \( m \), equation (1.2) is also known as the De la Vallée Poussin’s formula. Pillichshammer [5] obtained another generalization of the Dirichlet’s formula also doing a restriction on the divisors \( m \). The De la Vallée Poussin’s formula is also proved in this article.

There exist studies using more sophisticated methods of the general sum

\[
\sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\}^k
\]

We also obtain asymptotic formulæ for the general sums

\[
\sum_{1 \leq m \leq x} m^\alpha \left\{ \frac{x}{m} \right\},
\]

when \( \alpha > 0 \).

Alabdulmohsin [1] studied the case when \( \alpha \) is a positive integer, however using well-known formulæ we obtain better error terms.

Date: Received: Jan 8, 2019; Accepted: Jul 4, 2019.

* Corresponding author.

2010 Mathematics Subject Classification. Primary 11A99; Secondary 11B99.

Key words and phrases. Fractional parts.
2. Main Results

Lemma 2.1. (Euler’s formula) We have the following asymptotic formula
\[ \sum_{1 \leq n \leq x} \frac{1}{n} = \log x + \gamma + O \left( \frac{1}{x} \right), \]
where \( \gamma = 0.5772156649 \ldots \)

Proof. See [4]. The lemma is proved.

Theorem 2.2. (De la Vallée Poussin’s formula) We have the following asymptotic formula
\[ \sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\} = (1 - \gamma)x + O \left( \sqrt{x} \right). \quad (2.1) \]

Proof. Suppose that \( j \) is a positive integer and suppose that the following inequality holds \( \frac{x}{j+1} < m \leq \frac{x}{j} \), then \( \left\lfloor \frac{x}{m} \right\rfloor = j. \)

Let \( s = \lfloor \sqrt{x} \rfloor \). We have
\[ \sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\} = \sum_{1 \leq m \leq \frac{x}{s}} \left\{ \frac{x}{m} \right\} + \sum_{\frac{x}{s} < m \leq x} \left( \frac{x}{m} - \left\lfloor \frac{x}{m} \right\rfloor \right) \quad (2.2) \]
where by Lemma 2.1
\[ \sum_{\frac{x}{s} < m \leq x} \frac{x}{m} = x \log s + O(s). \quad (2.3) \]

On the other hand, we also have by Lemma 2.1
\[
\sum_{\frac{x}{s} < m \leq x} \left\lfloor \frac{x}{m} \right\rfloor = \sum_{j=1}^{s-1} \sum_{\frac{x}{j+1} < m \leq \frac{x}{j}} j = \sum_{j=1}^{s-1} \sum_{\frac{x}{j} < m \leq \frac{x}{j+1}} 1 = \sum_{j=1}^{s-1} \left( \frac{x}{j} - \frac{x}{s} \right) + O(s)
\]
\[ = -x + x \sum_{j=1}^{s} \frac{1}{j} + O(s) = -x + x \log s + \gamma x + O \left( \frac{x}{s} \right) + O(s). \quad (2.4) \]

Substituting Eqs. (2.3) and (2.4) into Eq. (2.2) we obtain Eq. (2.1), since \( O(s) = O(\frac{x}{s}) = O(\sqrt{x}) \). The theorem is proved.

Before our main theorem we need the following lemma.

Lemma 2.3. The following asymptotic formulae hold
\[ \sum_{m \leq x} \frac{1}{m^2} = -\frac{1}{x} + \zeta(2) + O \left( \frac{1}{x^2} \right), \quad (2.5) \]
\[ \sum_{j=1}^{n} \log j = n \log n - n + \frac{1}{2} \log n + \frac{1}{2} \log(2\pi) + O \left( \frac{1}{n} \right), \quad (2.6) \]
\[ \log x - \log(x - 1) = \frac{1}{x-1} + O \left( \frac{1}{x^2} \right) \quad (x \to \infty). \quad (2.7) \]
Proof. The proof of equation (2.5) is in [2] (page 55). Equation (2.7) is an immediate consequence of the L'Hospital's rule. The proof of equation (2.6) is as follows

\[
\sum_{j=1}^{n} \log j = \int_{1}^{n} \log x + \log n - \frac{1}{2} \log n - \sum_{i=1}^{n-1} c_i = n \log n - n + \frac{1}{2} \log n + 1
\]

\[
- \sum_{i=1}^{\infty} c_i = n \log n - n + \frac{1}{2} \log n + C + O \left( \frac{1}{n} \right),
\]

where \( C = 1 - \sum_{i=1}^{\infty} c_i \) and \( \sum_{i=n}^{\infty} c_i = O \left( \frac{1}{n} \right) \). Note that the area \( \int_{j}^{j+1} \log x \, dx \) is the sum of three areas, the area of the rectangle of base 1 and height \( \log j \), the area of the rectangle triangle of base 1 and height \( \log(j+1) - \log j \) and the area \( c_j \) between the chord and the curve \( \log x \). Note also that the derivative of \( \log x \), namely \( \frac{1}{x} \) is strictly decreasing and consequently the area \( \sum_{i=n}^{\infty} c_i \) is contained in the rectangle triangle of base 1 and height \( 1/n \).

The value of the constant \( C \) is obtained from the Stirling's formula \( n! \sim \sqrt{2\pi n} \frac{n^n}{e^n} \). The lemma is proved. \( \square \)

Theorem 2.4. We have the following asymptotic formula

\[
\sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\}^2 = ((1 - \gamma) - 2 + \log 2 + \log \pi) x + O \left( \frac{x^2}{s} \right). \tag{2.8}
\]

Proof. Let \( s = \lfloor \sqrt{n} \rfloor \). We have

\[
\sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\}^2 = \sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\}^2 + \sum_{\frac{x}{2} < m \leq x} \left( \frac{x}{m} - \left\lfloor \frac{x}{m} \right\rfloor \right)^2
\]

\[
= O \left( \frac{x}{s} \right) + x^2 \sum_{\frac{x}{2} < m \leq x} \frac{1}{m^2} - 2 \sum_{\frac{x}{2} < m \leq x} \frac{x}{m} \left\lfloor \frac{x}{m} \right\rfloor + \sum_{\frac{x}{2} < m \leq x} \left\lfloor \frac{x}{m} \right\rfloor^2, \tag{2.9}
\]

where by (2.5)

\[
x^2 \sum_{\frac{x}{2} < m \leq x} \frac{1}{m^2} = (s - 1)x + O(s^2). \tag{2.10}
\]

On the other hand, we have by Lemma 2.1

\[
\sum_{\frac{x}{2} < m \leq x} \left\lfloor \frac{x}{m} \right\rfloor^2 = \sum_{j=1}^{s-1} \sum_{\frac{x}{2} < m \leq \frac{x}{j}} j^2 = \sum_{j=1}^{s-1} \sum_{\frac{x}{2} < m \leq \frac{x}{j}} (1 + 3 + \cdots + (2j - 1))
\]

\[
= \sum_{j=1}^{s-1} \sum_{\frac{x}{2} < m \leq \frac{x}{j}} (2j - 1) = O(s^2) - (s - 1)^2 \frac{x}{s} + x \left( 2(s - 1) + \frac{1}{s} - \sum_{k=1}^{s} \frac{1}{k} \right)
\]

\[
= (s - 1)x + x - x \log s - \gamma x + O(s^2) + O \left( \frac{x}{s} \right). \tag{2.11}
\]
Besides, we also have by Lemma 2.1 and Eq. (2.6)

\[
\sum_{\frac{1}{x} < m \leq x} \frac{x}{m} \left\{ \frac{x}{m} \right\} = x \sum_{j=1}^{s-1} \sum_{m \leq \frac{x}{j}} \frac{1}{m} = \sum_{j=1}^{s-1} \sum_{m \leq \frac{x}{j}} \frac{1}{m}
\]

\[
- x(s-1) \sum_{m \leq \frac{x}{s}} \frac{1}{m} = x \sum_{j=1}^{s-1} \left( \log \frac{x}{j} + \gamma + O \left( \frac{j}{x} \right) \right)
\]

\[
- x(s-1) \left( \log \frac{x}{s} + \gamma + O \left( \frac{s}{x} \right) \right) = x \left( (s-1) \log s - \sum_{j=1}^{s-1} \log j \right) + O(s^2)
\]

\[
= x \left( (s-1)(\log s - \log(s-1)) + (s-1) - \frac{1}{2} \log(s-1) \right)
\]

\[
- \frac{1}{2} \log(2\pi) + O \left( \frac{1}{s} \right) + O(s^2).
\]

(2.12)

Substituting Eqs. (2.10), (2.11), and (2.12) into Eq. (2.9) and using Eq. (2.7) we obtain (2.8). The theorem is proved. □

We let \( \sigma_\alpha(m) \) denote the sum of the \( \alpha \)-th powers of the positive divisors of \( m \), where \( \alpha \) is a real number. When \( \alpha = 0 \), \( \sigma_0(m) \) is the number of positive divisors of \( m \), this is often denoted by \( d(n) \). When \( \alpha = 1 \), \( \sigma_1(m) \) is the sum of the positive divisors of \( m \), this is often denoted by \( \sigma(n) \).

The sum of fractional parts

\[
\sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\}
\]

in the de la Valée Poussin’s formula has its origin in the formula

\[
\sum_{1 \leq m \leq x} d(m) = \sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\} = x \sum_{1 \leq m \leq x} \frac{1}{m} - \sum_{1 \leq m \leq x} \left\{ \frac{x}{m} \right\}.
\]

The more general sum of fractional parts

\[
\sum_{1 \leq m \leq N} m^\alpha \left\{ \frac{x}{m} \right\}
\]

has its origin in the formula

\[
\sum_{1 \leq m \leq x} \sigma_\alpha(m) = \sum_{1 \leq m \leq x} m^\alpha \left\{ \frac{x}{m} \right\} = \sum_{1 \leq m \leq x} m^\alpha \left( \frac{x}{m} - \left\{ \frac{x}{m} \right\} \right)
\]

\[
= x \sum_{1 \leq m \leq x} m^{\alpha-1} - \sum_{1 \leq m \leq x} m^\alpha \left\{ \frac{x}{m} \right\}.
\]

(2.14)

We wish to obtain asymptotic formulae for this more general sum of fractional parts (2.13).

**Lemma 2.5.** We have the following formulas

\[
\sum_{m \leq x} \sigma(m) = \frac{1}{2} \zeta(2)x^2 + O(x \log x).
\]

(2.15)
If \( 0 < \beta < 1 \) then
\[
\sum_{m \leq x} \sigma_\beta(m) = \frac{1}{\beta + 1} \zeta(\beta + 1)x^{\beta + 1} + O(x). \tag{2.16}
\]

If \( \beta > 1 \) then
\[
\sum_{m \leq x} \sigma_\beta(m) = \frac{1}{\beta + 1} \zeta(\beta + 1)x^{\beta + 1} + O(x^\beta). \tag{2.17}
\]

If \( \beta > 0 \) then
\[
\sum_{n \leq x} n^\beta = \frac{x^{\beta + 1}}{\beta + 1} + O\left(x^\beta\right). \tag{2.18}
\]

If \( 0 < \beta < 1 \) then
\[
\sum_{n \leq x} \frac{1}{n^\beta} = \frac{x^{-\beta + 1}}{-\beta + 1} + O(1). \tag{2.19}
\]

Proof. See [2] (Chapter 3). The lemma is proved. \( \square \)

In the following Theorem we solve the case \( \alpha > 0 \) for the sum (2.13).

**Theorem 2.6.** If \( \alpha = 1 \) we have the following asymptotic formula
\[
\sum_{1 \leq m \leq x} m \{ \frac{x}{m} \} = \left( 1 - \frac{1}{2} \zeta(2) \right)x^2 + O(x \log x).
\]

If \( 0 < \alpha < 1 \) we have the following asymptotic formula
\[
\sum_{1 \leq m \leq x} m^\alpha \left\{ \frac{x}{m} \right\} = \left( \frac{1}{\alpha} - \frac{1}{\alpha + 1} \zeta(\alpha + 1) \right)x^{\alpha + 1} + O(x).
\]

If \( \alpha > 1 \) we have the following asymptotic formula
\[
\sum_{1 \leq m \leq x} m^\alpha \left\{ \frac{x}{m} \right\} = \left( \frac{1}{\alpha} - \frac{1}{\alpha + 1} \zeta(\alpha + 1) \right)x^{\alpha + 1} + O(x^\alpha).
\]

Proof. We have by (2.14)
\[
\sum_{m \leq x} m^\alpha \left\{ \frac{x}{m} \right\} = x \sum_{m \leq x} m^{\alpha - 1} - \sum_{m \leq x} \sigma_\alpha(m). \tag{2.20}
\]

If \( 0 < \alpha < 1 \) then equations (2.20), (2.16) and (2.19) give
\[
\sum_{m \leq x} m^\alpha \left\{ \frac{x}{m} \right\} = x \sum_{m \leq x} m^{\alpha - 1} - \sum_{m \leq x} \sigma_\alpha(m) = x \left( \frac{x^{-(1-\alpha)+1}}{-\alpha + 1} + O(1) \right)
\]
\[
- \frac{1}{\alpha + 1} \zeta(\alpha + 1)x^{\alpha + 1} + O(x)
\]
\[
= \left( \frac{1}{\alpha} - \frac{1}{\alpha + 1} \zeta(\alpha + 1) \right)x^{\alpha + 1} + O(x).
\]
If $\alpha = 1$ then Eqs. (2.20) and (2.15) give
\[
\sum_{m \leq x} m \left\{ \frac{x}{m} \right\} = x \sum_{m \leq x} 1 - \sum_{m \leq x} \sigma_1(m) = x^2 - \frac{1}{2} \zeta(2)x^2 + O(x \log x).
\]

If $\alpha > 1$ then equations (2.20), (2.17) and (2.18) give
\[
\sum_{m \leq x} m^\alpha \left\{ \frac{x}{m} \right\} = x \sum_{m \leq x} m^{\alpha-1} - \sum_{m \leq x} \sigma_\alpha(m) = x \left( \frac{x^\alpha}{\alpha} + O(x^{\alpha-1}) \right)
- \frac{1}{\alpha + 1} \zeta(\alpha + 1)x^{\alpha+1} + O(x^\alpha) = \frac{x^{\alpha+1}}{\alpha} - \frac{1}{\alpha + 1} \zeta(\alpha + 1)x^{\alpha+1} + O(x^\alpha)
= \left( \frac{1}{\alpha} - \frac{1}{\alpha + 1} \zeta(\alpha + 1) \right) x^{\alpha+1} + O(x^\alpha).
\]

The theorem is proved. \qed

**Acknowledgement.** The author is very grateful to Universidad Nacional de Luján.

**References**


1 **División Matemática, Universidad Nacional de Luján, Buenos Aires, Argentina.**

*Email address: jakimczu@mail.unlu.edu.ar*