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SUMS WHERE FRACTIONAL PARTS APPEAR

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ABSTRACT. In this article we examine sums where fractional parts appear. These sums are a generalization of the well-known De la Vallée Poussin's formula.

1. INTRODUCTION

In a short proof and using very elementary methods we prove the asymptotic formula

$$\sum_{1\leq m\leq x}\left\{\frac{x}{m}\right\}^2=((1-\gamma)-2+\log 2+\log\pi)x+O\left(x^{\frac{2}{3}}\right), \tag{1.1}$$

where $\{y\}=y-\lfloor y\rfloor$ is the fractional part of y and the m are positive integers. We let $\lfloor y\rfloor$ denote the integer part function. This formula is a generalization of the formula proved by Dirichlet in 1849. Namely

$$\sum_{1\leq m\leq x}\left\{\frac{x}{m}\right\}=(1-\gamma)x+O\left(\sqrt{x}\right). \tag{1.2}$$

In 1898 de la Vallée Poussin [3] obtained some generalizations of the Dirichlet's formula doing some restrictions on the divisors m, equation (1.2) is also known as the De la Vallée Poussin's formula. Pillichshammer [5] obtained another generalization of the Dirichlet's formula also doing a restriction on the divisors m. The De la Vallée Poussin's formula is also proved in this article.

There exist studies using more sophisticated methods of the general sum

$$\sum_{1\leq m\leq x}\left\{\frac{x}{m}\right\}^k$$

We also obtain asymptotic formulae for the general sums

$$\sum_{1\leq m\leq x}m^{\alpha}\left\{\frac{x}{m}\right\},$$

when $\alpha>0$.

Alabdulmohsin [1] studied the case when α is a positive integer, however using well-known formulae we obtain better error terms.

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2. Main Results

Lemma 2.1. *(Euler's formula) We have the following asymptotic formula*

$$\sum_{1\le n\le x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right),$$

where $\gamma=0,5772156649\ldots$.

Proof. See [4]. The lemma is proved. □

Theorem 2.2. *(De la Vallée Poussin's formula) We have the following asymptotic formula*

$$\sum_{1\le m\le x}\left\{\frac{x}{m}\right\}=(1-\gamma)x+O\left(\sqrt{x}\right). \tag{2.1}$$

Proof. Suppose that j is a positive integer and suppose that the following inequality holds $\frac{x}{j+1}<m\le\frac{x}{j}$, then $\left\lfloor\frac{x}{m}\right\rfloor=j$.

Let $s=\lfloor\sqrt{x}\rfloor$. We have

$$\sum_{1\le m\le x}\left\{\frac{x}{m}\right\}=\sum_{1\le m\le\frac{x}{s}}\left\{\frac{x}{m}\right\}+\sum_{\frac{x}{s}<m\le x}\left\{\frac{x}{m}\right\}=O\left(\frac{x}{s}\right)+\sum_{\frac{x}{s}<m\le x}\left(\frac{x}{m}-\left\lfloor\frac{x}{m}\right\rfloor\right) \tag{2.2}$$

where by Lemma 2.1

$$\sum_{\frac{x}{s}<m\le x}\frac{x}{m}=x\log s+O(s). \tag{2.3}$$

On the other hand, we also have by Lemma 2.1

$$\sum_{\frac{x}{s}<m\le x}\left\lfloor\frac{x}{m}\right\rfloor=\sum_{j=1}^{s-1}\sum_{\frac{x}{j+1}<m\le\frac{x}{j}}j=\sum_{j=1}^{s-1}\sum_{\frac{x}{s}<m\le\frac{x}{j}}1=\sum_{j=1}^{s-1}\left(\frac{x}{j}-\frac{x}{s}\right)+O(s)$$

$$=\quad -x+x\sum_{j=1}^{s}\frac{1}{j}+O(s)=-x+x\log s+\gamma x+O\left(\frac{x}{s}\right)+O(s). \tag{2.4}$$

Substituting Eqs. (2.3) and (2.4) into Eq. (2.2) we obtain Eq. (2.1), since $O(s)=O(\frac{x}{s})=O(\sqrt{x})$. The theorem is proved. □

Before our main theorem we need the following lemma.

Lemma 2.3. *The following asymptotic formulae hold*

$$\sum_{m\le x}\frac{1}{m^2}=-\frac{1}{x}+\zeta(2)+O\left(\frac{1}{x^2}\right), \tag{2.5}$$

$$\sum_{j=1}^{n}\log j=n\log n-n+\frac{1}{2}\log n+\frac{1}{2}\log(2\pi)+O\left(\frac{1}{n}\right), \tag{2.6}$$

$$\log x-\log(x-1)=\frac{1}{x-1}+O\left(\frac{1}{x^2}\right)\qquad(x\to\infty). \tag{2.7}$$

Proof. The proof of equation (2.5) is in [2] (page 55). Equation (2.7) is an immediate consequence of the L'Hospital's rule. The proof of equation (2.6) is as follows

$$\sum_{j=1}^{n}\log j=\int_1^n \log x+\log n-\frac{1}{2}\log n-\sum_{i=1}^{n-1}c_i=n\log n-n+\frac{1}{2}\log n+1$$
$$-\sum_{i=1}^{\infty}c_i+\sum_{i=n}^{\infty}c_i=n\log n-n+\frac{1}{2}\log n+C+O\left(\frac{1}{n}\right),$$

where $C=1-\sum_{i=1}^{\infty}c_i$ and $\sum_{i=n}^{\infty}c_i=O\left(\frac{1}{n}\right)$. Note that the area $\int_j^{j+1}\log x dx$ is the sum of three areas, the area of the rectangle of base 1 and height $\log j$, the area of the rectangle triangle of base 1 and height $\log(j+1)-\log j$ and the area c_j between the chord and the curve $\log x$. Note also that the derivative of $\log x$, namely $1/x$ is strictly decreasing and consequently the area $\sum_{i=n}^{\infty}c_i$ is contained in the rectangle triangle of base 1 and height $1/n$.

The value of the constant C is obtained from the Stirling's formula $n!\sim\sqrt{2\pi}\frac{n^n\sqrt{n}}{e^n}$. The lemma is proved. □

Theorem 2.4. *We have the following asymptotic formula*

$$\sum_{1\le m\le x}\left\{\frac{x}{m}\right\}^2=((1-\gamma)-2+\log 2+\log\pi)x+O\left(x^{\frac{2}{3}}\right). \tag{2.8}$$

Proof. Let $s=\lfloor\sqrt[3]{n}\rfloor$. We have

$$\sum_{1\le m\le x}\left\{\frac{x}{m}\right\}^2=\sum_{1\le m\le\frac{x}{s}}\left\{\frac{x}{m}\right\}^2+\sum_{\frac{x}{s}<m\le x}\left(\frac{x}{m}-\left\lfloor\frac{x}{m}\right\rfloor\right)^2$$
$$=O\left(\frac{x}{s}\right)+x^2\sum_{\frac{x}{s}<m\le x}\frac{1}{m^2}-2\sum_{\frac{x}{s}<m\le x}\frac{x}{m}\left\lfloor\frac{x}{m}\right\rfloor+\sum_{\frac{x}{s}<m\le x}\left\lfloor\frac{x}{m}\right\rfloor^2, \tag{2.9}$$

where by (2.5)

$$x^2\sum_{\frac{x}{s}<m\le x}\frac{1}{m^2}=(s-1)x+O(s^2). \tag{2.10}$$

On the other hand, we have by Lemma 2.1

$$\sum_{\frac{x}{s}<m\le x}\left\lfloor\frac{x}{m}\right\rfloor^2=\sum_{j=1}^{s-1}\sum_{\frac{x}{j+1}<m\le\frac{x}{j}}j^2=\sum_{j=1}^{s-1}\sum_{\frac{x}{j+1}<m\le\frac{x}{j}}(1+3+\cdots+(2j-1))$$
$$=\sum_{j=1}^{s-1}\sum_{\frac{x}{s}<m\le\frac{x}{j}}(2j-1)=O(s^2)-(s-1)^2\frac{x}{s}+x\left(2(s-1)+\frac{1}{s}-\sum_{k=1}^{s}\frac{1}{k}\right)$$
$$=(s-1)x+x-x\log s-\gamma x+O(s^2)+O\left(\frac{x}{s}\right). \tag{2.11}$$

Besides, we also have by Lemma 2.1 and Eq. (2.6)

$$\begin{aligned}
\sum_{\frac{x}{s}<m\le x}\frac{x}{m}\left\lfloor\frac{x}{m}\right\rfloor &= x\sum_{j=1}^{s-1}\sum_{\frac{x}{j+1}<m\le\frac{x}{j}}\frac{j}{m} = x\sum_{j=1}^{s-1}\sum_{\frac{x}{s}<m\le\frac{x}{j}}\frac{1}{m} = x\sum_{j=1}^{s-1}\sum_{m\le\frac{x}{j}}\frac{1}{m}\\
&- x(s-1)\sum_{m\le\frac{x}{s}}\frac{1}{m} = x\sum_{j=1}^{s-1}\left(\log\frac{x}{j}+\gamma+O\left(\frac{j}{x}\right)\right)\\
&- x(s-1)\left(\log\frac{x}{s}+\gamma+O\left(\frac{s}{x}\right)\right) = x\left((s-1)\log s-\sum_{j=1}^{s-1}\log j\right)+O(s^2)\\
&= x\Bigg((s-1)(\log s-\log(s-1))+(s-1)-\frac{1}{2}\log(s-1)\\
&- \frac{1}{2}\log(2\pi)+O\left(\frac{1}{s}\right)\Bigg)+O(s^2). \qquad (2.12)
\end{aligned}$$

Substituting Eqs. (2.10), (2.11), and (2.12) into Eq. (2.9) and using Eq. (2.7) we obtain (2.8). The theorem is proved. □

We let $\sigma_\alpha(m)$ denote the sum of the α-th powers of the positive divisors of m, where α is a real number. When $\alpha = 0$, $\sigma_0(m)$ is the number of positive divisors of m, this is often denoted by $d(n)$. When $\alpha = 1$, $\sigma_1(m)$ is the sum of the positive divisors of m, this is often denoted by $\sigma(n)$.

The sum of fractional parts

$$\sum_{1\le m\le x}\left\{\frac{x}{m}\right\}$$

in the de la Valée Poussin's formula has its origin in the formula

$$\sum_{1\le m\le x} d(m) = \sum_{1\le m\le x}\left\lfloor\frac{x}{m}\right\rfloor = x\sum_{1\le m\le x}\frac{1}{m} - \sum_{1\le m\le x}\left\{\frac{x}{m}\right\}.$$

The more general sum of fractional parts

$$\sum_{1\le m\le N} m^\alpha\left\{\frac{x}{m}\right\} \qquad (2.13)$$

has its origin in the formula

$$\begin{aligned}
\sum_{1\le m\le x}\sigma_\alpha(m) &= \sum_{1\le m\le x} m^\alpha\left\lfloor\frac{x}{m}\right\rfloor = \sum_{1\le m\le x} m^\alpha\left(\frac{x}{m}-\left\{\frac{x}{m}\right\}\right)\\
&= x\sum_{1\le m\le x} m^{\alpha-1} - \sum_{1\le m\le x} m^\alpha\left\{\frac{x}{m}\right\}. \qquad (2.14)
\end{aligned}$$

We wish to obtain asymptotic formulae for this more general sum of fractional parts (2.13).

Lemma 2.5. *We have the following formulas*

$$\sum_{m\le x}\sigma(m) = \frac{1}{2}\zeta(2)x^2 + O(x\log x). \qquad (2.15)$$

If $0 < \beta < 1$ *then*

$$\sum_{m \leq x} \sigma_\beta(m) = \frac{1}{\beta+1}\zeta(\beta+1)x^{\beta+1} + O(x). \tag{2.16}$$

If $\beta > 1$ *then*

$$\sum_{m \leq x} \sigma_\beta(m) = \frac{1}{\beta+1}\zeta(\beta+1)x^{\beta+1} + O(x^\beta). \tag{2.17}$$

If $\beta > 0$ *then*

$$\sum_{n \leq x} n^\beta = \frac{x^{\beta+1}}{\beta+1} + O\left(x^\beta\right). \tag{2.18}$$

If $0 < \beta < 1$ *then*

$$\sum_{n \leq x} \frac{1}{n^\beta} = \frac{x^{-\beta+1}}{-\beta+1} + O\left(1\right). \tag{2.19}$$

Proof. See [2] (Chapter 3). The lemma is proved. □

In the following Theorem we solve the case $\alpha > 0$ for the sum (2.13).

Theorem 2.6. *If* $\alpha = 1$ *we have the following asymptotic formula*

$$\sum_{1 \leq m \leq x} m \left\{\frac{x}{m}\right\} = \left(1 - \frac{1}{2}\zeta(2)\right) x^2 + O\left(x \log x\right).$$

If $0 < \alpha < 1$ *we have the following asymptotic formula*

$$\sum_{1 \leq m \leq x} m^\alpha \left\{\frac{x}{m}\right\} = \left(\frac{1}{\alpha} - \frac{1}{\alpha+1}\zeta(\alpha+1)\right) x^{\alpha+1} + O\left(x\right).$$

If $\alpha > 1$ *we have the following asymptotic formula*

$$\sum_{1 \leq m \leq x} m^\alpha \left\{\frac{x}{m}\right\} = \left(\frac{1}{\alpha} - \frac{1}{\alpha+1}\zeta(\alpha+1)\right) x^{\alpha+1} + O\left(x^\alpha\right).$$

Proof. We have by (2.14)

$$\sum_{m \leq x} m^\alpha \left\{\frac{x}{m}\right\} = x \sum_{m \leq x} m^{\alpha-1} - \sum_{m \leq x} \sigma_\alpha(m). \tag{2.20}$$

If $0 < \alpha < 1$ then equations (2.20), (2.16) and (2.19) give

$$\begin{aligned} \sum_{m \leq x} m^\alpha \left\{\frac{x}{m}\right\} &= x \sum_{m \leq x} m^{\alpha-1} - \sum_{m \leq x} \sigma_\alpha(m) = x \left(\frac{x^{-(1-\alpha)+1}}{-(1-\alpha)+1} + O\left(1\right)\right) \\ &- \frac{1}{\alpha+1}\zeta(\alpha+1)x^{\alpha+1} + O\left(x\right) \\ &= \left(\frac{1}{\alpha} - \frac{1}{\alpha+1}\zeta(\alpha+1)\right) x^{\alpha+1} + O\left(x\right). \end{aligned}$$

If $\alpha = 1$ then Eqs. (2.20) and (2.15) give

$$\sum_{m\leq x} m\left\{\frac{x}{m}\right\} = x\sum_{m\leq x} 1 - \sum_{m\leq x}\sigma_1(m) = x^2 - \frac{1}{2}\zeta(2)x^2 + O\left(x\log x\right).$$

If $\alpha > 1$ then equations (2.20), (2.17) and (2.18) give

$$\begin{aligned}\sum_{m\leq x} m^{\alpha}\left\{\frac{x}{m}\right\} &= x\sum_{m\leq x} m^{\alpha-1} - \sum_{m\leq x}\sigma_{\alpha}(m) = x\left(\frac{x^{\alpha}}{\alpha} + O\left(x^{\alpha-1}\right)\right)\\ &- \frac{1}{\alpha+1}\zeta(\alpha+1)x^{\alpha+1} + O\left(x^{\alpha}\right) = \frac{x^{\alpha+1}}{\alpha} - \frac{1}{\alpha+1}\zeta(\alpha+1)x^{\alpha+1} + O\left(x^{\alpha}\right)\\ &= \left(\frac{1}{\alpha} - \frac{1}{\alpha+1}\zeta(\alpha+1)\right)x^{\alpha+1} + O\left(x^{\alpha}\right).\end{aligned}$$

The theorem is proved. □

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