CHARACTERIZATION OF TOTAL LINE-CUT GRAPHS

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ABSTRACT. The total line-cut graph of a graph \( G = (V, E) \), denoted by \( TLc(G) \), is the graph with point set \( E(G) \cup W(G) \), where \( W(G) \) is the set of cutpoints of \( G \), in which two points are adjacent if and only if they correspond to adjacent lines of \( G \) or correspond to adjacent or coadjacent cutpoints of \( G \) or one point corresponds to a line \( e \) of \( G \) and the other corresponds to a cutpoint \( c \) of \( G \) such that \( e \) is incident with \( c \). In this paper, we offer a structural characterization of total line-cut graphs.

1. Introduction

By a graph \( G=(V, E) \), we mean a finite, undirected graphs without loops or multiple lines. For any graph \( G \), let \( V(G), E(G), W(G) \) and \( U(G) \) denote the point set, line set, cutpoint set and block set of \( G \), respectively. The lines and cutpoints of a graph are called its members. A pendant point is a point of degree one and a line incident (nonincident) with a pendant point is called pendant (nonpendant) line. The neighborhood of a point \( u \) in \( V \) is the set \( N(u) \) consisting of all points \( v \) which are adjacent with \( u \). A cutpoint of a connected graph \( G \) is the one whose removal increases the number of components. A nonseparable graph is connected, nontrivial and has no cutpoints. A block of a graph \( G \) is a maximal nonseparable subgraph. A block is called pendantblock of a graph if it contains exactly one cutpoint of \( G \). The line graph \( L(G) \) of \( G \) is the graph whose point set is \( E(G) \) in which two points are adjacent if and only if they are adjacent in \( G \). If \( B = \{u_1, u_2, ..., u_n; n \geq 2\} \) is a block of \( G \), then we say that point \( u_1 \) and block \( B \) are incident with each other, as are \( u_2 \) and \( B \) and so on. If a block is incident with cutpoints \( c_1, c_2, ..., c_r, r \geq 2 \), we say that \( c_i \) and \( c_j \) are coadjacent where \( i \neq j \) and \( 1 \leq i, j \leq r \). The cutpoint graph \( C(G) \) of a graph \( G \) is the graph whose point set corresponds to the cutpoints of \( G \) and in which two points of \( C(G) \) are adjacent if the cutpoints of \( G \) to which they correspond lie on a common block [3]. For graph theoretic terminology, we refer to [3, 5].

Kulli and Muddebihal [4] introduced the idea of a lict graph and litact graph. In [1], M. Acharya et al. called lict graph as a line-cut graph and gave the characterization of line-cut graph. In [2], we called litact graph as a total line-cut graph.

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Definition 1.1. The total line-cut graph (also known as litact graph) of a graph \( G = (V, E) \), denoted by \( T_{Lc}(G) \), is the graph with point set \( E(G) \cup W(G) \), where \( W(G) \) is the set of cutpoints of \( G \), in which two points are adjacent if and only if they correspond to adjacent lines of \( G \) or correspond to adjacent or coadjacent cutpoints of \( G \) or one point corresponds to a line \( e \) of \( G \) and the other corresponds to a cutpoint \( c \) of \( G \) such that \( e \) is incident with \( c \).

Figure 1. illustrates a graph and its total line-cut graph.

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\begin{align*}
\text{Figure 1. A graph } G \text{ and its total line-cut graph } T_{Lc}(G). \\
\text{The point } c_i (e_i) \text{ of total line-cut graph } T_{Lc}(G) \text{ corresponding to a cutpoint } c_i \text{ (line } e_i) \text{ of } G \text{ and is referred to as cutpoint (line) vertex.}
\end{align*}
\]

2. Main result

A graph \( G \) is a total line-cut graph if and only if it is isomorphic to the total line-cut graph \( T_{Lc}(H) \) of some graph \( H \). Let \( G=(V,E) \) be a graph and let \( V' \subseteq V(G) \). The induced subgraph \( <V'> \) of \( G \) is called a clique of \( G \) if \( <V'> \) is isomorphic with a complete graph of order \( |V'| \). A clique is maximal if it is not a subgraph of a clique of larger order.

The following theorem gives the characterization of total line-cut graph.

Theorem 2.1. The following statements are equivalent:

(I) \( G = (V, E) \) is a total line-cut graph, i.e \( G \cong T_{Lc}(H) \) of some graph \( H \).

(II) The lines of \( G \) can be partitioned among the three types of maximal cliques, namely: maximal cliques induced by the line vertices of \( G \), maximal cliques induced by the cutpoint vertices of \( G \) and maximal cliques induced by the line vertices and cutpoint vertices of \( G \), satisfying following conditions;

(1) In the maximal cliques \( G_i \) which are induced by the line vertices of \( G \), no point lies in more than two maximal cliques and for each clique \( G_i \) in the partition
    (a) if each point of \( G_i \) lies in two cliques of the partition, then \( G - E(G_i) \) is connected and
(b) if all but one point, \( v \), of \( G_i \) lies in two cliques of the partition, then \( G - E(G_i) - v \) disconnected. (that is, \( G \) does not contain a pendant point.)

(2) In the maximal cliques \( G_i \) which are induced by the cutpoint vertices or line vertices and cutpoint vertices of \( G \)

(a) no line vertex of \( G \) lies in more than two maximal cliques

(b) if cutpoint vertex \( c_i \) lies in one or two (more than two) cliques, then corresponding cutpoint \( c_i \) lies on pendantblock (nonpendantblock) of \( H \).

Proof. \((I) \implies (II)\)

Let \( G \) ba a total line-cut graph. Therefore \( G \cong TL_c(H) \) for some graph \( H \). We assume that \( G \) has no isolated points. By definition of total line-cut graph, the lines incident on a point \( v \) of \( H \) with degree \( \deg(v) = p \), that is not a cutpoint, induces a maximal clique of \( G \) with order \( p \). The lines incident on a cutpoint \( c \) of \( H \) with \( \deg(v) = p \) induces a maximal clique of \( G \) with order \( p + 1 \) having \( c \) as one of its points. Let \( U(G) = \{B_1, B_2, ..., B_n\} \), \( n \geq 2 \) be the block set of \( G \) and \( C(B_i) \) be the number of cutpoints of a connected graph \( G \) which are the points of the block \( B_i \). Then cutpoint vertices induces a maximal clique of order \( 1 + \sum_{i=1}^{n} (C(B_i) - 1) \). Because every line of \( G \) either results from two adjacent lines of \( H \) or from a cutpoint of \( H \) and a line of \( H \) that is incident with that cutpoint or adjacent or coadjacent cutpoints of \( H \), then every line of \( G \) is contained in precisely one such clique. This is illustrated in Figure 2.

Figure 2. A graph \( G \) and a graph \( H \) such that \( G \cong TL_c(H) \).

Note that \( V(G) = E(H) \cup W(H) \), where \( W(H) \) is the set of cutpoints of \( H \). Clearly \( e_i \) is a pendant line of \( H \) then the corresponding line vertex in \( G \) is contained in only one maximal clique. If \( e_i \) is a nonpendant line of \( H \), then the corresponding line vertex in \( G \) is contained in precisely two maximal cliques. Therefore no line vertex of \( G \) lies in more than two maximal cliques. Also if cutpoint of \( H \) lies on pendantblock, then the corresponding cutpoint vertex in
$G$ is contained in one or two maximal cliques. If cutpoint of $H$ lies on nonpendant block, then the corresponding cutpoint vertex in $G$ is contained in more than two maximal cliques. Thus the lines of $G$ can be partitioned among the maximal cliques of $G$ in such a way that no line vertex of $G$ lies in more than two maximal cliques and each cutpoint vertex lies in more than one maximal cliques.

In the maximal clique $G_i$ induced by the line vertices of $G$, if each point of $G_i$ is contained in two maximal cliques, then $G_i$ is induced by the lines incident with a noncutpoint, $v$, of $H$. Suppose a point $v_j$ of $G_j$ is also contained in maximal clique $G_j$. Then the maximal clique $G_j$ must result from points of $H$ belongs to $N(v)$, the neighborhood of $v$ in $H$. Because $H - v$ is connected, then $G - E(G_i) - v$ is connected, a contradiction. Therefore Condition 1(a) is satisfied.

Figure 3. illustrates a graph $G$ that does not satisfy Condition 1(a) and is not a total line-cut graph.

Figure 3. A graph $G$ that is not a total line-cut graph.

In the maximal clique $G_i$ induced by the line vertices of $G$, if all but one point, $v$, of $G_i$ is not lies in two cliques of the partition, then all points of $G_i$ lies in only one maximal cliques. Because $H - v$ is connected, then $G - E(G_i) - v$ is connected, a contradiction. Therefore Condition 1(b) is satisfied.

Figure 4. illustrates graphs that do not satisfy Condition 1(b) and are not total line-cut graphs.

Figure 4. Graphs that are not total line-cut graphs.

Since if $e_i$ is a pendant line of $H$, then the corresponding line vertex in $G$ is contained in only one maximal clique. If $e_i$ is a nonpendant line of $H$, then the corresponding line vertex in $G$ is contained in precisely two maximal cliques. Therefore Condition 2(a) is satisfied.
Figure 5. illustrates a graph $G$ that does not satisfy Condition 2(a) and is not a total line-cut graph.

If $H$ has only one cutpoint, then corresponding cutpoint vertex lies on only one maximal clique. Suppose $H$ has more than one cutpoint. If cutpoint lies on pendantblock of $H$, then corresponding cutpoint vertex lies in maximal clique induced by the line vertices and cutpoint vertices and the maximal clique induced by the cutpoint vertices of $G$. Therefore cutpoint vertices lies in one or two maximal cliques. And if cutpoint lies on nonpendantblock of $H$, then the corresponding cutpoint vertex contained in at least three maximal cliques of $G$. Therefore Condition 2(b) is satisfied.

Figure 6. illustrates a graph $G$ that does not satisfy Condition 2(b) and is not a total line-cut graph.

$(II) \implies (I)$

Let $P(G) = \{G_1, G_2, ..., G_n\}$ be set of cliques of $G$ induced by the line vertices of $G$, $C(G) = \{c_1, c_2, ..., c_n\}$ be the set of cutpoint vertices of $G$. We provide the construction of a graph $H$ whose total line-cut graph is $G$. The point set of $H$ is the union of sets $P(G)$, $C(G)$ and the set $U$ of line vertices of $G$ that belongs to only one of the cliques $G_i$. Thus, $V(H) = P(G) \cup C(G) \cup U$ and two of these points are adjacent whenever the point belongs to set $U$ and the point belongs to set $C(G)$ are adjacent or the points of the cliques $G_i$ belongs to set $P(G)$ and the point belongs to set $C(G)$ are adjacent or two cliques of the set $P(G)$ are incident with a common point or two cutpoints are adjacent if the cliques induced by the corresponding cutpoint vertices and line vertices have a point in common. For this graph $H$, $G \cong TL_c(H)$. Hence, $G$ is a total line-cut graph.
Figures 7 and 8 illustrate construction of a graph $H$ such that $G \cong TL_c(H)$ for a graph $G$ that satisfies Condition (II).

Figure 7. A graph $G$ and a graph $H$ such that $G \cong TL_c(H)$.

Figure 8. A graph $G$ and a graph $H$ such that $G \cong TL_c(H)$.


References


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