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INTERPOLATIVE FUZZY Z-CONTRACTION WITH ITS APPLICATION TO FREDHOLM NON-LINEAR INTEGRAL EQUATION

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This paper is dedicated to Professor LAl BAHADUR

Abstract. The aim of this paper is to introduce the notion of interpolative fuzzy Z-contractive condition for a pair of self maps in a fuzzy metric space, which enlarges, unifies and generalizes the existing fuzzy contractions (by Gregori et al. [6]), ψ contraction (by Mihet [11]), fuzzy Z-contractions (by Shukla et al. ([19]) and Tirado contraction (by Tirado [20]), which are for only one self map. Using this, we establish a unique common fixed point theorem for two self maps satisfying condition (S), which was introduced by Shukla et al. in [19] through weak compatibility. As an applications, we discuss fixed point theorem for Fredholm non-linear integral equation. The article also includes an example, which shows the validity of our results.

1. Introduction and preliminaries

In 1965 L. Zadeh [22] introduce the theory of fuzzy sets. Later on in 1978 the concept of fuzzy metric space was introduced by Kramosil and Michalek in [10], which was modified by George and Veeramani [4] in order to obtain a Hausdorff topology for this class of fuzzy metric spaces. Then in year 1988, Grabiec [5] gave a fuzzy version of Banach contraction principle in the setting of fuzzy metric space. In recent past years, various authors tried to generalize the fixed point theorem by modifying and varying the contractive condition (see Gregori and Sapena [6], Mihet [11], Tirado [20] and Wardowski [21]) in the sense of George and Veeramani. In 2019, Shukla et al. [19] introduced a new class of contractive mappings called Z-contractions, and proved some fixed point results for a self map of this new class.
Inspired from the interpolative theory, Karapinar et al. [9] introduce the notion of interpolative Rus-Reich-Ćrić type contraction via simulation function in metric space. Motivated by this paper we introduce interpolative fuzzy Z-contraction for two self maps in the setting of fuzzy metric space, which enlarges, unifies and generalizes the existing contraction. Employing condition (S), we prove

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the existence of unique common fixed point of a pair of interpolative fuzzy Z-contractive self maps in fuzzy metric space through weak compatibility.
The structure of the paper is as follows:
After preliminaries in section 2, we introduce interpolative fuzzy Z-contraction in the setting of fuzzy metric space. Then we study fuzzy contractive mapping due to Gregori and Sapena [6], Mihet [11], Tirado [20] and Wardowski [21] and Shukla et al. [19]. In section 3, we prove the existence of unique common fixed point of an interpolative fuzzy Z- contractive pair of weakly compatible self maps satisfying condition (S). Finally in section 4, we conclude with an application for the solution of Fredholm non-linear integral equation.

1.1. **Preliminaries.** Now, let us recall the definition of fuzzy metric space and other results given in [4].

Definition 1.1. [4] A mapping $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (t-norm for short) if $*$ is continuous and satisfies the following conditions:

(i) $*$ is commutative and associative, i. e. $a*b = b*a$ and $a*(b*c) = (a*b)*c$, for all $a, b, c \in [0,1]$;

(ii) $1 * a = a$, for all $a \in [0,1]$;

(iii) $a * c \leq b * d$, for $a \leq b, c \leq d$ for $a, b, c, d \in [0,1]$.

Note that in view of condition (GV2) we have $M(x,x,t) = 1$, for all $x \in X$ and $t > 0$ and $M(x,y,t) < 1$, for all $x \neq y$ and $t > 0$.

Definition 1.2. [4] A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$;
(GV1) $M(x,y,t) > 0$;
(GV2) $M(x,y,t) = 1$ if and only if $x = y$;
(GV3) $M(x,y,t) = M(y,x,t)$;
(GV4) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$;
(GV5) $M(x,y,) : (0,+\infty) \to (0,1]$ is continuous.

Note that in view of condition $(GV2)$ we have $M(x,x,t) = 1$, for all $x \in X$ and $t > 0$ and $M(x,y,t) < 1$, for all $x \neq y$ and $t > 0$.

Definition 1.3. [4] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be M-Cauchy, or simply Cauchy, if for each $\epsilon \in (0,1)$ and each $t > 0$ there exists an $n_0 \in N$, such that $M(x_n, x_m, t) > 1 - \epsilon$, for all $n, m \geq n_0$. Equivalently, $\{x_n\}$ is Cauchy if $\lim_{m,n \to +\infty} M(x_n, x_m, t) = 1$, for all $t > 0$.

Lemma 1.4. [4] *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x,y,.)$ is non-decreasing for all $x, y \in X$.*

Theorem 1.5. [4] *Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}_{n\in N}$ in X converges to $x \in X$ if and only if* $\lim_{n\to+\infty} M(x_n, x, t) \to 1$.

Definition 1.6. [4] $(X, M, *)$ (or simply X) is called M-complete if every M-Cauchy sequence in X is convergent.

Lemma 1.7. [17] *If $*$ is a continuous t-norm $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences such that $\alpha_n \to \alpha, \beta_n \to \beta$ and $\gamma_n \to \gamma$ as $n \to +\infty$ then*

$$\overline{\lim}_{k\to+\infty}(\alpha_k * \beta_k * \gamma_k) = \alpha * \overline{\lim}_{k\to+\infty}\beta_k * \gamma,$$

and

$$\underline{\lim}_{k\to+\infty}(\alpha_k * \beta_k * \gamma_k) = \alpha * \underline{\lim}_{k\to+\infty}\beta_k * \gamma.$$

Lemma 1.8. [17] *Let $\{f(k, .) : (0, +\infty) \to (0, 1]; k = 0, 1, 2, ..., \}$ be a sequence of functions such that $f(k, .)$ is continuous and monotone increasing for each $k \geq 0$. Then $\overline{\lim}_{k\to+\infty}f(k, t)$ is a left continuous function in t and $\underline{\lim}_{k\to+\infty}f(k, t)$ is a right continuous function in t.*

Definition 1.9. [8] Two self maps S and T in a fuzzy metric space $(X, M, *)$ are said to be weakly compatible if they commute at their coincidence points i.e. for $x \in X, Sx = Tx = y$ implies $Ty = Sy$.

2. Interpolative fuzzy Z-contraction

Let Z denotes the family of functions $\zeta : (0, 1) \times (0, 1) \to \Re^+$, where $\Re^+ = (0, +\infty)$ satisfying the following condition:

$$\zeta(t, s) > s, \text{for } t, s \in (0, 1).$$

Example 2.1. Define $\zeta : (0, 1) \times (0, 1) \to \Re^+$ by

$$\zeta(t, s) = \begin{cases} \frac{s}{t} & \text{if } t \geq s; \\ \frac{t}{s}, & \text{if } s \geq t; \end{cases}$$

Example 2.2. Define $\zeta : (0, 1) \times (0, 1) \to \Re^+$ by

$$\zeta(t, s) = \frac{1}{s + t} + t, \text{ if } s, t \in (0, 1).$$

Definition 2.3. Let $(X, M, *)$ be a fuzzy metric space. A pair (A, B) of self maps in X is said to be interpolative fuzzy Z-contractive if there exists $\zeta \in Z, \beta, \gamma \in [0, 1)$ with $\beta + \gamma < 1$ and for all $x, y \in X$ with $Ax \neq Ay$ and for all $t > 0$,

$$M(Ax, Ay, t) \geq \zeta(M(Ax, Ay, t), R(x, y)), \tag{2.1}$$

where

$$R(x, y) = (M(Ax, Bx, t))^{\beta}(M(Ay, By, t))^{\gamma}(M(Bx, By, t))^{(1-\beta-\gamma)}.$$

Remark 2.4. Taking $B = I$, the identity map in equation (2.1) we obtain

$$M(Ax, Ay, t) \geq \zeta(M(Ax, Ay, t), R(x, y)), \tag{2.2}$$

where

$$R(x, y) = (M(Ax, x, t))^{\beta}(M(Ay, y, t))^{\gamma}(M(x, y, t))^{(1-\beta-\gamma)}.$$

which is interpolative fuzzy Z-contraction, for a self map A.

Remark 2.5. Taking $\beta = 0, \gamma = 0$ in equation (2.2) then $R(x, y) = M(x, y, t)$ and we have
$$M(Ax, Ay, t) \geq \zeta(M(Ax, Ay, t), M(x, y, t)), \tag{2.3}$$
which is precisely the Z-contraction, for a self map A given by Shukla et al.[19].

Gregori and Sapena in [6] defined the fuzzy contractive mappings as follows: Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is called a fuzzy contractive mapping if there exists $k \in (0, 1)$ such that
$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right),$$
for all $x, y \in X$.
Taking $\zeta(t, s) = \frac{s}{k+(1-k)s}$, for all $t, s \in (0, 1)$ and $A = T$ in (2.2), with $\beta, \gamma \in [0, 1)$ we get
$$\frac{1}{M(Tx, Ty, t)} - 1 \leq k \left(\frac{1}{R(x, y)} - 1 \right),$$
for all $x, y \in X$ where
$$R(x, y) = (M(Tx, x, t))^{\beta}(M(Ty, y, t))^{\gamma}(M(x, y, t))^{(1-\beta-\gamma)}.$$

Remark 2.6. Thus the fuzzy contractive mapping defined by Gregori and Sapena in [6] get generalized and we call the mapping T to be interpolative fuzzy contractive mapping. If we take $\beta = 0$ and $\gamma = 0$ then the map T becomes fuzzy contractive(defind in [6]).

In [20], Tirado defined the following contraction:
Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is Tirado contraction if there exists $k \in (0, 1)$ such that
$$1 - M(Tx, Ty, t) \leq k(1 - M(x, y, t)),$$
for all $x, y \in X$.
Taking $\zeta(t, s) = 1 + k(s - 1)$, for all $t, s \in (0, 1)$ and $A = T$ in (2.2), with $\beta, \gamma \in [0, 1)$ we get
$$1 - M(Tx, Ty, t) \leq k(1 - R(x, y)),$$
for all $x, y \in X$ where
$$R(x, y) = (M(Tx, x, t))^{\beta}(M(Ty, y, t))^{\gamma}(M(x, y, t))^{(1-\beta-\gamma)}.$$

Remark 2.7. Thus Tirdo contraction defined by Tirado in [20] get generalized and we call the mapping T to be interpolative Tirado contractive mapping. If we take $\beta = 0$ and $\gamma = 0$ then T becomes Tirado contractive (defind in [20]).

In [11], Mihet defined the class Ψ of mappings as follows:
Let $\psi : (0, 1] \to (0, 1)$ such that ψ is continuous, non decreasing and $\psi(t) > t$, for all $t \in (0, 1)$. Let $\psi \in \Psi$. A mapping $T : X \to X$ is called a fuzzy ψ-contractive mapping if
$$M(x, y, t) > 0 \quad \text{implies} \quad M(Tx, Ty, t) \geq \psi(M(x, y, t)),$$

for all $x, y \in X$ and $t > 0$.
Taking $\zeta(t,s) = \psi(s)$, for all $t, s \in (0,1)$ and $A = T$ in (2.2), with $\beta, \gamma \in [0,1)$ we get

$$M(x,y,t) > 0 \quad \text{implies} \quad M(Tx,Ty,t) \geq \psi(R(x,y)),$$

for all $x, y \in X$ where

$$R(x,y) = (M(Tx,x,t))^{\beta}(M(Ty,y,t))^{\gamma}(M(x,y,t))^{(1-\beta-\gamma)}.$$

Remark 2.8. Thus fuzzy ψ contraction defined by Mihet in [11] get generalized and we call the mapping T to be interpolative fuzzy ψ contractive mapping. If we take $\beta = 0$ and $\gamma = 0$ then T becomes fuzzy ψ contractive (defind in [11]).

In [21], Wardowski defined the following class H of mappings as follows:
Let H be the family of the mappings $\eta : (0,1] \to [0,+\infty)$ satisfying the following conditions:

(H-1) η transforms $(0,1]$ onto $[0,+\infty)$;

(H-2) η is strictly decreasing.

A mapping $T : X \to X$ is called fuzzy H-contractive with respect to $\eta \in H$ if there exists $k \in (0,1)$ satisfying the following condition:

$$\eta(M(Tx,Ty,t)) \leq k\eta(M(x,y,t),$$

for all $x, y \in X$ and $t > 0$.
Taking $\zeta(t,s) = \eta^{-1}(k\eta(s))$, for all $t, s \in (0,1)$ and $A = T$ in (2.2), with $\beta, \gamma \in [0,1)$ we get

$$\eta(M(Tx,Ty,t)) \leq k\eta(R(x,y)),$$

where

$$R(x,y) = (M(Tx,x,t))^{\beta}(M(Ty,y,t))^{\gamma}(M(x,y,t))^{(1-\beta-\gamma)}.$$

Remark 2.9. In view of the remark by Gregori and Minana in [7], every fuzzy H-contractive mapping with respect to $\eta \in H$ is a fuzzy contractive with respect to the function $\zeta \in Z$ get generalized and we call the mapping T to be interpolative fuzzy contractive mapping. If we take $\beta = 0$ and $\gamma = 0$ then T becomes fuzzy H-contractive (defind in [7]).

Now we are extending the definition of property (S) given by Shukla et al. [19] for quadruple to quintuplet (X, M, A, B, ζ) as

Definition 2.10. Let $(X, M, *)$ be a fuzzy metric space and A and B be self-maps in X which are interpolative Z-contractive with respect to $\zeta \in Z$. The Quintuplet (X, M, A, B, ζ) said to satisfy property (S), if for a sequence $\{y_n\}$, with initial point x_0, $Ax_n = Bx_{n+1} = y_n$, for $n = 0, 1, 2, \ldots,$.

$$\inf_{m>n} M(y_n, y_m, t) \leq \inf_{m>n} M(y_{n+1}, y_{m+1}, t),$$

for all $n > m$, implies

$$\lim_{n\to+\infty} \inf_{m>n} \zeta(M(y_{n+1}, y_{m+1}, t), M(y_n, y_m, t)) = 1.$$

Taking $A = T$ and $B = I$, the identity map, and considering the sequence $\{x_n\}$, we get the property (S), for Quadruple (X, M, T, ζ),

$$\inf_{m>n} M(x_n, x_m, t) \leq \inf_{m>n} M(x_{n+1}, x_{m+1}, t),$$

for all $n > m$, implies

$$\lim_{n\to+\infty} \inf_{m>n} \zeta(M(x_{n+1}, x_{m+1}, t), M(x_n, x_m, t)) = 1,$$

which is precisely property (S) given by Shukla et al. [19].

3. Main results

Our first new result is the next:

Theorem 3.1. *Let A and B be self maps in a fuzzy metric space $(X, M, *)$ satisfying the following conditions*

(3.1.1): $A(X) \subseteq B(X)$;
(3.1.2): *The pair (A, B) is interpolative fuzzy Z-contractive;*
(3.1.3): *$B(X)$ is complete;*
(3.1.4): *The pair (A, B) is weakly compatible;*
(3.1.5): *The Quintuple (X, M, A, B, ζ) satisfies property (S).*

Then the map A and the map B have a unique common fixed point in X.

Proof. Let $x_0 \in X$, be any arbitrary point. Construct sequence $\{y_n\}$, by defining $Ax_n = Bx_{n+1} = y_n$, for $n = 0, 1, 2, \ldots$. First we show that if the two maps A and B have a common fixed point then it is unique. Let u and v be two common fixed points of A and B. Then $u = Au = Bu$ and $v = Av = Bv$. We show that $u = v$. Suppose, on the contrary that $u \neq v$, then $Au \neq Av$. Now

$$\begin{aligned}
R(u,v) &= (M(Au, Bu, t))^{\beta}(M(Av, Bv, t))^{\gamma}(M(Bu, Bv, t))^{(1-\beta-\gamma)},\\
&= (M(u, u, t))^{\beta}(M(v, v, t))^{\gamma}(M(u, v, t))^{(1-\beta-\gamma)},\\
&= (M(u, v, t))^{(1-\beta-\gamma)},
\end{aligned}$$

and

$$\begin{aligned}
M(u, v, t) &= M(Au, Av, t),\\
&\geq \zeta(M(Au, Av, t), R(u, v)), (\text{using } (2.1))\\
&= \zeta\left(M(u, v, t), (M(u, v, t))^{(1-\beta-\gamma)}\right),\\
&> (M(u, v, t))^{1-\beta-\gamma},
\end{aligned}$$

i. e. $M(u, v, t) > (M(u, v, t))^{(1-\beta-\gamma)}$,
implies

$$(M(u, v, t))^{(\beta+\gamma)} > 1,$$

which is a contradiction. So $u = v$. Thus, if the pair (A, B) has a common fixed point then it is unique. Now to see the existence of a common fixed point of self maps A and B, we consider the following cases.

CASE I Suppose $y_n = y_{n+1}$, for some $n \in N$. Now $y_n = Ax_n = Bx_{n+1} = Ax_{n+1} = Bx_{n+2} = y_{n+1}$ we have $Ax_{n+1} = Bx_{n+1}$. Let $Ax_{n+1} = Bx_{n+1} = z$. So x_{n+1} is a point of coincidence of the pair (A, B). As the pair (A, B) is weakly compatible we have $Az = Bz$. Now we show that $Az = z$. Suppose, if possible on the contrary, that $Az \neq z$. Now

$$\begin{aligned} R(x_{n+1}, z) &= (M(Ax_{n+1}, Bx_{n+1}, t))^{\beta}(M(Az, Bz, t)^{\gamma}(M(Bx_{n+1}, Bz, t))^{(1-\beta-\gamma)}, \\ &= (M(z, z, t))^{\beta}(M(Az, Az, t))^{\gamma}(M(z, Az, t))^{(1-\beta-\gamma)}, \\ &= M(z, Az, t)^{(1-\beta-\gamma)}, \end{aligned}$$

$$\begin{aligned} M(z, Az, t) &= M(Ax_{n+1}, Az, t), \\ &\geq \zeta(M(Ax_{n+1}, Az, t), R(x_{n+1}, z)), (\text{using}(2.1)) \\ &= \zeta(M(Ax_{n+1}, Az, t), (M(z, Az, t))^{(1-\beta-\gamma)}), \\ &> (M(z, Az, t))^{(1-\beta-\gamma)}, \end{aligned}$$

i.e.

$$M(z, Az, t) > (M(z, Az, t))^{(1-\beta-\gamma)}, \quad \text{for all } t > 0,$$

implies

$$(M(z, Az, t))^{(\beta+\gamma)} > 1,$$

which is not possible. Hence $Az = z$. Therefore, z is a common fixed point of the pair (A, B) in this case.
So we can assume the consecutive terms of the sequence $\{y_n\}$ are distinct.
Again, to see the existence of common fixed point in other cases, we first show that all the terms of the the sequence $\{y_n\}$ are distinct.
CASE II Suppose $y_n = y_m$, for some $m > n+1$, then as all the consecutive terms of the sequence $\{y_n\}$ are distinct we claim that $y_{n+1} = y_{m+1}$.Suppose if possible on the contrary $y_{n+1} \neq y_{m+1}$ then $y_n \neq y_{n+1} \neq y_{m+1} \neq y_m$ implies $y_n \neq y_m$ which contradicts our assuption.So we have $y_{n+1} = y_{m+1}$. Also

$$\begin{aligned} R(x_{n+1}, x_{n+2}) &= \left\{ \begin{array}{c} (M(Ax_{n+1}, Bx_{n+1}, t))^{\beta}(M(Ax_{n+2}, Bx_{n+2}, t))^{\gamma} \\ (M(Bx_{n+1}, Bx_{n+2}, t))^{(1-\beta-\gamma)} \end{array} \right\} \\ &= \left\{(M(y_{n+1}, y_n, t))^{\beta}(M(y_{n+2}, y_{n+1}, t))^{\gamma}(M(y_n, y_{n+1}, t))^{(1-\beta-\gamma)}\right\} \\ &= (M(y_{n+2}, y_{n+1}, t))^{\gamma}(M(y_n, y_{n+1}, t))^{(1-\gamma)}. \end{aligned}$$

and

$$\begin{aligned} M(y_{n+1}, y_{n+2}, t) &= M(Ax_{n+1}, Ax_{n+2}, t) \\ &\geq \zeta(M(Ax_{n+1}, Ax_{n+2}, t), R(x_{n+1}, x_{n+2})) \\ &= \zeta(M(y_{n+1}, y_{n+2}, t), (M(y_{n+2}, y_{n+1}, t))^{\gamma}(M(y_n, y_{n+1}, t)0^{(1-\gamma)}) \\ &> (M(y_{n+2}, y_{n+1}, t))^{\gamma}(M(y_n, y_{n+1}, t))^{(1-\gamma)}. \end{aligned}$$

i. e.

$$M(y_{n+1}, y_{n+2}, t) > (M(y_{n+2}, y_{n+1}t))^{\gamma}(M(y_n, y_{n+1}, t))^{(1-\gamma)},$$

implies

$$(M(y_{n+2}, y_{n+1}, t))^{1-\gamma} > (M(y_n, y_{n+1}, t))^{1-\gamma}.$$

Thus

$$M(y_n, y_{n+1}, t) < M(y_{n+1}, y_{n+2}, t). \tag{3.1}$$

So
$M(y_n, y_{n+1}, t) < M(y_{n+1}, y_{n+2}, t) < M(y_{n+2}, y_{n+3}, t) < ... < M(y_m, y_{m+1}, t)$. i. e. $M(y_n, y_{n+1}, t) < M(y_n, y_{n+1}, t)$, which is not possible. So this case does not arise. Thus, we conclude that $y_n \neq y_m$ for distinct $n, m \in N$. Thus the elements of the sequence $\{y_n\}$ are distinct. Now we show that the sequence $\{y_n\}$ is M-Cauchy.

STEP 1 In this step we prove that $\lim_{n\to+\infty} M(y_n, y_{n+1}, t) = 1$, for all $t > 0$. From equation (3.1) we have

$$M(y_n, y_{n+1}, t) < M(y_{n+1}, y_{n+2}, t),$$

for all $t > 0$. Thus, $\{M(y_n, y_{n+1}, t)\}$, for each $t > 0$, is an strictly increasing sequence, which is bounded above by 1. Let $\lim_{n\to+\infty} M(y_n, y_{n+1}, t) = a(t)$, for $t > 0$. We claim that $a(t) = 1$. Suppose if possible on the contrary that for some $s, a(s) < 1$. Then

$$\begin{aligned} R(x_n, x_{n+1}) &= \left\{ \begin{array}{c} (M(Ax_n, Bx_n, t))^{\beta}(M(Ax_{n+1}, Bx_{n+1}, t))^{\gamma} \\ (M(Bx_n, Bx_{n+1}, t)^{(1-\beta-\gamma)} \end{array} \right\} \\ &= \left\{ \begin{array}{c} (M(y_n, y_{n-1}, t))^{\beta}(M(y_{n+1}, y_n, t))^{\gamma} \\ (M(y_{n-1}, y_n, t)^{(1-\beta-\gamma)} \end{array} \right\} \\ &= (M(y_{n+1}, y_n, t))^{\gamma}(M(y_{n-1}, y_n, t))^{(1-\gamma)}. \end{aligned}$$

Therefore,

$$\lim_{n\to+\infty} R(x_n, x_{n+1}) = (a(s))^{\gamma}(a(s))^{(1-\gamma)} = a(s).$$

Also

$$\begin{aligned} M(y_{n+1}, y_n, t) &= M(Ax_{n+1}, Ax_n, t) \\ &\geq \zeta(M(Ax_{n+1}, Ax_n, t), R(x_n, x_{n+1})) \\ &> R(x_n, x_{n+1}) \end{aligned}$$

i.e.

$$M(y_{n+1}, y_n, t) > R(x_n, x_{n+1}).$$

Taking infimum we have

$$\inf M(y_{n+1}, y_n, t) \geq \inf R(x_n, x_{n+1}).$$

As the quintuple (X, M, A, B, ζ) satisfies property (S) we have

$$\lim_{n\to+\infty} \inf \zeta(M(y_{n+1}, y_n, t), R(x_n, x_{n+1})) = 1. \tag{3.2}$$

Also,

$$\begin{aligned} M(y_{n+1}, y_n, t) &\geq \zeta(M(Ax_{n+1}, Ax_n, t), R(x_n, x_{n+1})) \\ &> R(x_n, x_{n+1}). \end{aligned}$$

Taking infimum we get

$$\begin{aligned} \inf M(y_{n+1}, y_n, t) &\geq \inf \zeta(M(Ax_{n+1}, Ax_n, t), R(x_n, x_{n+1})) \\ &\geq \inf R(x_n, x_{n+1}). \end{aligned}$$

i.e.

$$\begin{aligned} M(y_{n+1}, y_n, t) &\geq \inf \zeta(M(Ax_{n+1}, Ax_n, t), R(x_n, x_{n+1})) \\ &\geq M(y_{n-1}, y_n, t). \end{aligned}$$

Therefore, for all $n \to +\infty$ and using property (S) of the quintuple (X, M, A, B, ζ), we have

$$\lim_{n\to+\infty} \inf M(y_n, y_{n+1}, s) = a(s) = 1$$

which contradicts our hypothesis. Hence the claim. Therefore,for all $t > 0$

$$\lim_{n\to+\infty} M(y_n, y_{n+1}, t) = 1. \tag{3.3}$$

Now we prove that the sequence $\{y_n\}$ is M-Cauchy. Suppose if possible on the contrary that it is not true then there exist $\eta \in (0, 1), t_0 > 0$ and sequences $\{p(n)\}, \{q(n)\}$ ($p(n$ being the smallest one of index)

$$n < p(n) < q(n), M(y_{p(n)}, y_{q(n)}, t_0) \leq 1 - \eta, M(y_{p(n)-1}, y_{q(n)}, t_0) > 1 - \eta. \tag{3.4}$$

STEP 2 In this step we show that $\lim_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0) = 1 - \eta$.
Now for all $n \geq 1, 0 < \lambda < t_0/2$, we obtain,

$$\begin{aligned} 1 - \eta &\geq M(y_{p(n)}, y_{q(n)}, t_0), \qquad (\text{using}(3.4)) \\ &\geq M(y_{p(n)}, y_{p(n)-1}, \lambda) * M(y_{p(n)-1}, y_{q(n)-1}, t_0 - 2\lambda) * M(y_{q(n)-1}, y_{q(n)}, \lambda). \end{aligned} \tag{3.5}$$

Let

$$h_1(t) = \overline{\lim}_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t), t > 0.$$

Taking limit supremum on both sides of equation (3.5), and using the properties of M and $*$, and by Lemma 1.8, we obtain

$$\begin{aligned} 1 - \eta &\geq 1 * \overline{\lim}_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0 - 2\lambda) * 1 \qquad (\text{using}(3.3)) \\ &= h_1(t_0 - 2\lambda). \end{aligned}$$

Since M is bounded with range in $(0, 1]$ continuous and non-decreasing in the third variable t, it follows from Lemma 1.8, that h_1 is continuous from the left. Therefore, for $\lambda \to 0$, we obtain

$$h_1(t_0) = \overline{\lim}_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0) \leq 1 - \eta. \tag{3.6}$$

Let

$$h_2(t) = \underline{\lim}_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t), t > 0.$$

Again, for all $n \geq 1, t_0 > 0$

$$\begin{aligned} M(y_{p(n)-1}, y_{q(n)-1}, \lambda + t_0) &\geq M(y_{p(n)-1}, y_{q(n)}, \lambda) * M(y_{q(n)}, y_{q(n)-1}, t_0) \\ &> (1 - \eta) * M(y_{q(n)}, y_{q(n)-1}, \lambda) \quad (\text{using}(3.4)) \end{aligned}$$

Taking limit infimum as $n \to +\infty$ in above inequality we obtain

$$\begin{aligned} h_2(\lambda + t_0) &= \underline{\lim}_{n\to+\infty} M(y_{q(n)-1}, y_{q(n)-1}, \lambda + t_0), \\ &\geq (1 - \eta) * \underline{\lim}_{n\to+\infty} M(y_{q(n)}, y_{q(n)-1}, \lambda), \qquad (\text{using}(3.4)) \\ &= (1 - \eta) * 1 \\ &= 1 - \eta. \end{aligned}$$

Since M is bounded with range in $(0,1]$ continuous and non-decreasing in the third variable t, it follows from Lemma 1.8, that h_2 is continuous from the right. Therefore, for $\lambda \to 0$, we obtain

$$\underline{\lim}_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0) \geq 1-\eta. \tag{3.7}$$

Combining the inequalities (3.6) and (3.7)we get

$$\lim_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0) = 1-\eta. \tag{3.8}$$

STEP 3 In this step we show that $\lim_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0) = 1-\eta$.
From equation (3.4) we have

$$\overline{\lim}_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0) \leq 1-\eta. \tag{3.9}$$

Also for all $n \geq 1$ and $t_0 > 0$ we have

$$M(y_{p(n)}, y_{q(n)}, t_0+2\lambda) \geq M(y_{p(n)}, y_{p(n)-1}, \lambda)*M(y_{p(n)-1}, y_{q(n)-1}, t_0)*M(y_{q(n)-1}, y_{q(n)}, \lambda)$$

Taking limit infmum as $n \to +\infty$ in the above inequality, using (3.3), (3.8) and the properties of M and $*$ and by Lemma 1.7, we obtain

$$\begin{aligned} \underline{\lim}_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0+2\lambda) &\geq 1 * \underline{\lim}_{n\to+\infty} M(y_{p(n)-1}, y_{q(n)-1}, t_0) * 1 \\ &= 1-\eta, \quad (\text{using}(3.8)). \end{aligned} \tag{3.10}$$

Since M is bounded with range in $(0,1]$ continuous and, by Lemma 1.8, non-decreasing in the third variable t, it follows by Lemma 1.8 that $\underline{\lim}_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0)$ is continuous function of t from the right. Therefore, for $\lambda \to 0$, we obtain

$$\underline{\lim}_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0) \geq 1-\eta. \tag{3.11}$$

Combining inequalities (3.9) and (3.11) we get

$$\lim_{n\to+\infty} M(y_{p(n)}, y_{q(n)}, t_0) = 1-\eta. \tag{3.12}$$

STEP 4 In this step we show that the sequence $\{y_n\}$ is an M-Cauchy sequence.
Now

$$\begin{aligned} R(x_{p(n)}, x_{q(n)}) &= \left\{ \begin{array}{c} (M(Ax_{p(n)}, Bx_{p(n)}, t))^{\beta} (M(Ax_{q(n)}, Bx_{q(n)}, t))^{\gamma} \\ ((M(Bx_{p(n)}, Bx_{q(n)}, t))^{1-\beta-\gamma} \end{array} \right\}; \\ &= \left\{ \begin{array}{c} (M(y_{p(n)}, y_{p(n)-1}, t))^{\beta} (M(y_{q(n)}, y_{q(n)-1}, t))^{\gamma} \\ ((M(y_{p(n)-1}, y_{q(n)-1}, t))^{1-\beta-\gamma} \end{array} \right\} \end{aligned}$$

Therefore

$$\lim_{n\to+\infty} \inf_{q(n)>p(n)} R(x_{p(n)}, x_{q(n)}) = (1-\eta)^{(1-\beta-\gamma)}. \tag{3.13}$$

And

$$\begin{aligned} M(y_{p(n)}, y_{q(n)}, t) &= M(Ax_{p(n)}, Ax_{q(n)}, t) \\ &\geq \zeta(M(Ax_{p(n)}, Ax_{q(n)}, t), R(x_{p(n)}, x_{q(n)}) \\ &> R(x_{p(n)}, x_{q(n)}). \end{aligned} \tag{3.14}$$

i. e. $R(x_{p(n)}, x_{q(n)}) < M(y_{p(n)}, y_{q(n)}, t)$. Taking infimum we get

$$\inf_{q(n)>p(n)} R(x_{p(n)}, x_{q(n)}) \leq \inf_{q(n)>p(n)} M(y_{p(n)}, y_{q(n)}, t).$$

So, by property (S) of the Quintuple (X, M, A, B, ζ) we have

$$\lim_{n\to+\infty} \inf_{q(n)>p(n)} \zeta\left(M(y_{p(n)}, y_{q(n)}, t), R(x_{p(n)}, x_{q(n)})\right) = 1. \tag{3.15}$$

Taking infimum for $q(n) > p(n)$ in inequality (3.14) we obtain

$$\begin{aligned} \inf_{q(n)>p(n)} M(y_{p(n)}, y_{q(n)}, t) &\geq \inf_{q(n)>p(n)} \zeta\left(M(Ax_{p(n)}, Ax_{q(n)}, t), R(x_{p(n)}, x_{q(n)})\right) \\ &> \inf_{q(n)>p(n)} R(x_{p(n)}, x_{q(n)}). \end{aligned}$$

Therefore, for all $n \to +\infty$ we have

$$(1-\eta) \geq \lim_{n\to+\infty} \inf_{q(n)>p(n)} \zeta\left(M(y_{p(n)}, y_{q(n)}, t), R(x_{p(n)}, x_{q(n)})\right) \geq (1-\eta)^(1-\beta-\gamma),$$

which is not possible inview of equation (3.15). So,$\{y_n\}$ is an M-Cauchy sequence in $B(X)$ which is M-complete. Therefore there exists $u \in B(X)$ such that

$$\{y_n\} \to u. \tag{3.16}$$

i. e.

$$\{Ax_n\} \to u \quad \text{and} \quad \{Bx_{n+1}\} \to u. \tag{3.17}$$

As $u \in B(X)$ there exists $v \in X$ such that

$$u = Bv. \tag{3.18}$$

STEP 5 Now we show that $Bv = Av$. Suppose, on the contrary, that $Av \neq Bv(= u)$. Then exists a positive integer n_0 such that $Bv \neq Bx_n$, for all $n \geq n_0$.

$$\begin{aligned} R(x_n, v) &= (M(Ax_n, Bx_n, t))^\beta (M(Av, Bv, t))^\gamma (M(Bx_n, Bv, t)^{(1-\beta-\gamma)} \\ &= (M(y_n, y_{n-1}, t))^\beta (M(Av, u, t))^\gamma (M(Bx_n, u, t)^{(1-\beta-\gamma)}. \end{aligned}$$

Taking $x = x_n$ and $y = v$ in equation (2.1) we get

$$\begin{aligned} M(Ax_n, Av, t) &\geq \zeta(M(Ax_n, Av, t), R(x_n, v)) \\ &> R(x_n, v) \\ &= (M(y_n, y_{n-1}, t))^\beta (M(Av, u, t))^\gamma (M(Bx_n, u, t)^{(1-\beta-\gamma)}. \end{aligned}$$

Taking limt as $n \to +\infty$ and using equations(3.3),(3.17) and (3.18) we get $M(u, Av, t) \geq (M(Av, u, t))^\gamma$ i. e. $(M(Av, u, t))^{1-\gamma} > 1$, which is not possible if $Av \neq u$. Hence $Av = u$ and we have

$$Av = Bv = u. \tag{3.19}$$

As the pair of self maps (A, B) is weakly compatible we have

$$Au = Bu. \tag{3.20}$$

STEP 6 Now we show that $Au = u$. Suppose, on the contrary that $Au \neq u$. Then $Bu \neq u$.

$$\begin{aligned} R(u, v) &= (M(Av, Bv, t))^\beta (M(Au, Bu, t))^\gamma (M(Bv, Bu, t))^{(1-\beta-\gamma)} \qquad (\text{using}(3.19, 3.20)) \\ &= (M(u, Au, t))^{(1-\beta-\gamma)} \end{aligned}$$

and

$$\begin{aligned} M(u,Au,t) &= M(Av,Au,t) \qquad \text{(using(3.19))} \\ &\geq \zeta(M(Av,Au,t),R(v,u)) \\ &> R(v,u), \\ &= (M(u,Au,t))^{(1-\beta-\gamma)} \end{aligned}$$

i. e. $(M(u,Au,t))^{(\beta+\gamma)} > 1$, which is not possible as left hand side is less than 1.Thus, $Au = Bu = u$.

□

Taking $B = I$ in Theorem 3.1, then the sequence $\{x_n\} = \{A^n x_0\}_{n=0,1,2,\cdots}$ becomes a Picard sequence and we have

Corollary 3.2. *Let A be a self map on a M-complete fuzzy metric space $(X,M,*)$ with the following conditions:*

(3.2.1) *The map A is interpolative fuzzy Z-contractive,*

(3.2.2) *The quadruple (X,M,A,ζ) has property (S).*

Then the map A has a unique fixed point in X.

Remark 3.3. If we take $\beta = 0$ and $\gamma = 0$ in then above corollary, we obtain Theorem 3.13 of Shukla et al. [19].

Example 3.4. (of Theorem 3.1) Consider the sequence $\{x_n\}$ of real numbers with $x_n = \frac{n}{n+1}$, for $n \in N$. Then the sequence $\{x_n\}$ is strictly increasing sequence and $0 < x_n < 1$, for all $n \in N$. Also $\lim_{n\to+\infty} x_n = 1$. Taking $X = \{x_n : n \in N\} \cup \{1\}$ and define a fuzzy set M on $X \times X \times (0,+\infty)$ by:

$$M(x,y,t) = \begin{cases} 1; & \text{if } x = y, \\ \min\{x,y\}, & \text{if } x \neq y. \end{cases}$$

for all $x, y \in X, t \in (0,+\infty)$. Then $(X,M,*_m)$ is an M-complete fuzzy metric space. Define a function $\zeta : (0,1] \times (0,1] \to \Re$ by

$$\zeta(t,s) = \begin{cases} t; & \text{if } t > s, \\ \sqrt{s}, & \text{otherwise.} \end{cases}$$

for all $s, t \in (0,1]$ and $\beta = \frac{1}{2}$ and $\gamma = \frac{1}{3}$ then $\zeta \in Z$.
Define self map A on X by $A(x_n) = x_{n+1}$, for all $n \in N$ and $A(1) = 1$, and $B = I$, the identity map on X.Then the Quadruple (X,M,A,ζ) satisfies property (S). Also the mapping A is a interpolative fuzzy Z-contractive with respect to the function ζ. Thus, all the conditions of Theorem 3.1 are satisfied and $x = 1$ is the unique fixed point of map A.

4. **Application**

In this section by applying our theorem 3.1, we prove the existence of unique solution of Fredholm non-linear equation. To apply our result we need certain

specific conditions for existence of unique solution.
We consider the following non-linear integral equation :

$$z(t) = \int_a^b K(t,s)h(z(s))ds + g(t), \tag{4.1}$$

for all $t \in \Omega = [a,b], a, b \in R, K \in C(\Omega \times \Omega, R), g, h \in C(\Omega, R)$.
We denote λ_1 to be

$$\lambda \sup_{t\in\Omega} \int_a^b \mid K(t,s) \mid ds = \lambda_1.$$

Theorem 4.1. *: Consider the equation (4.1) with $K \in C(\Omega \times \Omega, R)$ and $g \in C(\Omega, R)$. Suppose*
(4.1.1) *that for a positive number λ*

$$h(z) - h(y) \leq \lambda(z - y),$$

(4.1.2) *and $\lambda_1 \in [0,1) = \frac{1}{3}$.*
Then the equation (4.1) has a unique solution in $C(\Omega, R)$.

Proof. Taking $X = C(\Omega, R)$. Then X is a complete metric space with sup- metric

$$d(z,y) = \sup_{t\in\Omega} |z(t) - y(t)|$$

and the space $(X, M, *)$ is a complete metric space if we take $M(z,y,t) = \frac{t}{t+d(z,y)}$,for all $z, y \in X$ and $t > 0$ and taking $p * q = pq$, for all $p, q \in [0,1]$.
Define $S : X \to X$, for all $t \in \Omega$, by

$$(Az)(t) = \int_a^b K(t,s)h(z(s))ds + g(t), \tag{4.2}$$

Now

$$\begin{aligned}
(Az)(t) - (Ay)(t) &= \int_a^b K(t,s)\left(h(z(s)) - h(y(s))\right)ds, \\
&\leq \lambda \int_a^b K(t,s)(z(s) - y(s))ds, \\
&\leq \lambda \int_a^b K(t,s) \sup_{t\in\Omega} |z(s) - y(s)| ds, \ \text{(using(4.1.1))} \\
&= \lambda \int_a^b K(t,s) d(z,y) ds.
\end{aligned}$$

i. e.

$$(Az)(t) - (Ay)(t) \leq \lambda \int_a^b K(t,s)d(z,y)ds.$$

Taking supremum over $t \in \Omega$ and using condition(4.1.2), we get

$$\begin{aligned}
d(Sz, Sy) &\leq \lambda d(z,y) \int_a^b |K(t,s)| ds \\
&= \lambda_1 d(z,y), \ (\text{as } \lambda \sup_{t\in\Omega} \textstyle\int_a^b \mid K(t,s) \mid ds = \lambda_1).
\end{aligned}$$

Also, for all $z, y \in X, t > 0$ we have

$$\begin{aligned}\frac{1}{M(Az, Ay, t)} - 1 &= \frac{d(Az, Ay)}{t}\\ &\leq \lambda_1 \frac{d(z, y)}{t}\\ &= \frac{d(z, y)}{3t}\\ &= \frac{1}{3}\{\frac{1}{M(z, y, t)} - 1\}.\end{aligned}$$

Taking

$$\zeta(t, s) = \frac{3s}{1 + 2s},$$

for all $t, s \in (0, 1]$. Therefore the contractive condition of Theorem 3.1 is satisfied with $B = I$ and $\beta = \gamma = 0$. Hence by Theorem 3.1, the equation (4.2) has a unique solution in $C(\Omega, R)$. □

5. Conflicts of Interest

The autors declare that they have no conflict of interest regarding the publication of the paper.

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