

SOME RESULTS ON TOTALLY n -*-PARANORMAL OPERATORS

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ABSTRACT. Additional properties for new class called totally n -*-paranormal operators are obtained. Invariant subspaces, reducing subspaces of such operators are discussed. An asymmetric Putnam-Fuglede theorem for totally n -*-paranormal operators are studied.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper $\mathcal{H}$ will denote an infinite dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of all bounded linear operator on $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$, we write $\sigma(T)$ for the spectrum of T , $\sigma_p(T)$ for the eigenvalues of T , $\sigma_w(T)$ for the Weyl's spectrum and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. T satisfies Weyl's theorem if $\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T)$. If K is a subset of $\mathbb{C}$, we write $\text{iso}K$ for the set of isolated points of T . Also, we write $\ker(T)$ and $\mathcal{R}(T)$ for the null space and the range of T . An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and also $T \in \mathcal{B}(\mathcal{H})$ is said to be strictly positive (denoted by $T > 0$) if it is positive and invertible.

Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be hyponormal if $T^*T \geq TT^*$, or equivalently, if $\|Tx\| \geq \|T^*x\|$ for all $x \in \mathcal{H}$. A larger class of operators related to hyponormal operators is the following: $T \in \mathcal{B}(\mathcal{H})$ is called n -*-paranormal ($T \in \mathfrak{C}(n)$) if $\|T^*x\|^n \leq \|T^n x\| \|x\|^{n-1}$ for every $x \in \mathcal{H}$. In particular, in the case $n = 2$, $T \in \mathcal{B}(\mathcal{H})$ is called *-paranormal [17]. This class of operators introduced and studied by K. Tanahashi and A. Uchiyama. The references [12, 13, 14] are among the various extensions of these classes of operators.

In this paper we want to focus on a class of n -*-paranormal which has the translation invariance property.

Definition 1.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be totally n -*-paranormal for a positive integer n such that $n \geq 2$ ($T \in \mathcal{TC}(n)$ for short) if

$$\|(T - \lambda)^*x\|^n \leq \|(T - \lambda)^n x\| \|x\|^{n-1} \quad (1.1)$$

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for all $x \in \mathcal{H}$ and all $\lambda \in \mathbb{C}$. In particular, if $n = 2$, T is called totally $*$ -paranormal.

Let $T \in \mathcal{B}(\mathcal{H})$ and μ_0 be an isolated point of $\sigma(T)$. Then there exists a small enough positive number $r > 0$ such that $\{\mu \in \mathbb{C} : |\mu - \mu_0| \leq r\} \cap \sigma(T) = \{\mu_0\}$. Let

$$E = \frac{1}{2\pi i} \int_{|\mu - \mu_0| = r} (\mu - T)^{-1} d\mu.$$

E is called the Riesz idempotent with respect to μ_0 . Then it is well known that

$$E^2 = E, \quad ET = TE, \quad \sigma(T|_{\mathcal{R}(E)}) = \{\mu_0\} \text{ and } \ker(T - \mu_0) \subset \mathcal{R}(E).$$

In general, it is well known that the Riesz idempotent E is not orthogonal projection and a necessary and sufficient condition for E to be orthogonal is that E is self-adjoint [6]. For hyponormal operator, Stampfli [15] have shown that the Riesz idempotent for an isolated point of the spectrum of T is self-adjoint. That is

$$\mathcal{R}(E) = \ker(E - \mu_0) = \ker((E - \mu_0)^*).$$

The paper is organized as follows. In section one, we give some preliminary facts. In section two, we study some properties of totally n - $*$ -paranormal operators on Hilbert space. In particular, we show that if E is the Riesz idempotent for a non-zero isolated point μ of the spectrum of a totally n - $*$ -paranormal operator T , then E is self-adjoint, and $\mathcal{R}(E) = \ker(T - \mu) = \ker(T - \mu)^*$. Finally, section three is devoted to extend the asymmetric Putnam-Fuglede theorem to the class of totally n - $*$ -paranormal operators.

2. Basic Properties of $TC(n)$

In this section, we study some properties of totally $*$ -paranormal operators. We begin with the following lemma which is characterize the class of totally n - $*$ -paranormal operators.

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is totally n - $*$ -paranormal for positive integer n if and only if*

$$(T - \lambda)^{*(n)}(T - \lambda)^n - n\mu^{n-1}(T - \lambda)(T - \lambda)^* + (n - 1)\mu^n \geq 0 \quad (2.1)$$

for all real number $\mu > 0$ and all $\lambda \in \mathbb{C}$.

Proof. Let $T \in \mathcal{TC}(n)$. By the generalized arithmetic-geometric mean inequality, we have

$$\begin{aligned} \frac{1}{n} \langle \mu^{n-1} |(T - \lambda)^n|^2 x, x \rangle + \frac{n-1}{n} \langle \mu x, x \rangle \\ \geq \langle |(T - \lambda)^n|^2 x, x \rangle^{\frac{1}{n}} \langle x, x \rangle^{\frac{n-1}{n}} \\ \geq \langle |(T - \lambda)^*|^2 x, x \rangle = \langle (T - \lambda)(T - \lambda)^* x, x \rangle. \end{aligned}$$

Conversely, if $x \in \mathcal{H}$ with $\langle |(T - \lambda)^n|^2 x, x \rangle = 0$, multiplying (2.1) by μ^{n-1} and letting $\mu \rightarrow 0$ we have $\langle (T - \lambda)(T - \lambda)^* x, x \rangle = 0$, thus

$$\|(T - \lambda)^n x\| \|x\|^{n-1} \geq \|(T - \lambda)^* x\|^n.$$

If $x \in \mathcal{H}$ with $\langle |(T - \lambda)^n|^2 x, x \rangle \neq 0$. Putting $\mu = \left(\frac{\langle |(T - \lambda)^n|^2 x, x \rangle}{\langle x, x \rangle} \right)^{\frac{1}{n}}$ in (2.1) we have

$$\langle |(T - \lambda)^n|^2 x, x \rangle^{\frac{1}{n}} \langle x, x \rangle^{\frac{n-1}{n}} \geq \langle (T - \lambda)(T - \lambda)^* x, x \rangle$$

for any $x \in \mathcal{H}$. That is, $T \in \mathcal{TC}(n)$. $\square$

Lemma 2.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be totally n -*-paranormal operator. Then $T - \alpha$ and αT are totally n -*-paranormal operators for each $\alpha \in \mathbb{C}$.*

Proof. (i) Suppose that T is totally n -*-paranormal operator. Then for all $x \in \mathcal{H}$, we have

$$\begin{aligned} \|[(T - \alpha) - \lambda]^* x\|^n &= \|(T - (\alpha + \lambda))^* x\|^n \\ &\leq \|(T - (\alpha + \lambda))^n x\| \|x\|^{n-1} = \|[(T - \alpha) - \lambda]^n x\| \|x\|^{n-1}. \end{aligned}$$

Hence $T - \alpha$ is totally n -*-paranormal operator. Now, To prove αT is totally M -*-paranormal operator, we consider two cases:

Case I: If $\alpha = 0$, then $\alpha T = 0$ and so its totally n -*-paranormal operator.

Case II: If $\alpha \neq 0$, then for all $x \in \mathcal{H}$

$$\begin{aligned} \|(\alpha T - \lambda)^* x\|^n &= |\alpha|^n \left\| \left(T - \frac{\lambda}{\alpha} \right)^* x \right\|^n \\ &\leq |\alpha|^n \left\| \left(T - \frac{\lambda}{\alpha} \right)^n x \right\| \|x\|^{n-1} \\ &\leq \|(\alpha T - \lambda)^n x\| \|x\|^{n-1}. \end{aligned}$$

Hence αT is totally n -*-paranormal operator. $\square$

Lemma 2.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally n -*-paranormal operator. If $\sigma(T) = \{\lambda\}$, then $T = \lambda$.*

Proof. Suppose that T is a totally n -*-paranormal operator. Then T has invariant translation property. But every quasinilpotent totally n -*-paranormal operator is zero operator, hence $T - \lambda = 0$ and so $T = \lambda$. $\square$

Lemma 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be a totally n -*-paranormal operator. If $\mathfrak{M}$ is a T -invariant subspace, then the restriction $T|_{\mathfrak{M}}$ of T to $\mathfrak{M}$ is also totally n -*-paranormal operator.*

Proof. Let P be the orthogonal projection onto $\mathfrak{M}$. Since $TP = PTP$, we have

$$\begin{aligned} \|(T|_{\mathfrak{M}} - \lambda)^* x\|^n &= \|(T - \lambda)^*|_{\mathfrak{M}} x\|^n = \|P(T - \lambda)^* P x\|^n = \|P(T - \lambda)^* x\|^n \\ &\leq \|(T - \lambda)^n x\| \|x\|^{n-1} = \|((T - \lambda)|_{\mathfrak{M}})^n x\| \|x\|^{n-1} = \|(T|_{\mathfrak{M}} - \lambda)^n x\| \|x\|^{n-1} \end{aligned}$$

for all $x \in \mathfrak{M}$. Thus $T|_{\mathfrak{M}}$ is totally n -*-paranormal operator. $\square$

Theorem 2.5 (Berberian's Extension). *Let $\mathcal{H}$ be a complex Hilbert space. Then there exists a Hilbert space $\mathcal{H}^\circ \supset \mathcal{H}$ and $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}^\circ)$ ($T \mapsto T^\circ$) satisfying: ϕ is an $\star$ -isometric isomorphism preserving the order such that*

- (i) $\phi(T^*) = \phi(T)^*, \phi(I) = \phi(I)^\circ, \phi(\alpha T + \beta S) = \alpha\phi(T) + \beta\phi(S), \phi(TS) = \phi(T)\phi(S), \|\phi(T)\| = \|T\|$, for all $T, S \in \mathcal{B}(\mathcal{H})$ and $\alpha, \beta \in \mathbb{C}$.
- (ii) If $T \leq S$, then $\phi(T) \leq \phi(S)$ for all $T, S \in \mathcal{B}(\mathcal{H})$.
- (iii) $\sigma(T) = \sigma(T^\circ)$ and $\sigma_a(T) = \sigma_a(T^\circ) = \sigma_p(T^\circ)$.

Theorem 2.6. Let $T \in \mathcal{B}(\mathcal{H})$ such that $T \in \mathcal{TC}(n)$ and $\mathfrak{M} \subset \mathcal{H}$ an invariant subspace of T such that $T|_{\mathfrak{M}}$ is normal. Then $\mathfrak{M}$ is reducing for T .

Proof. Let us consider the matrix decomposition

$$T = \begin{pmatrix} N & A \\ 0 & \star \end{pmatrix},$$

where $N = T|_{\mathfrak{M}}$ is a normal operator. If $T \in \mathcal{TC}(n)$, then T has the invariant translation property, so it suffices to assume $\lambda = 0$. Hence

$$\|A^*x\|^2 + \|N^*x\|^2 = \|T^*x\|^2 \leq \|T^n x\|^{\frac{2}{n}} \|x\|^{\frac{2(n-1)}{n}} = \|N^n x\|^{\frac{2}{n}} \|x\|^{\frac{2(n-1)}{n}} \quad (2.2)$$

for all $x \in \mathfrak{M}$.

Let us take the Berberian's extension of the operator T . Then the extension T° has the following matrix decomposition

$$T = \begin{pmatrix} N^\circ & A^\circ \\ 0 & \star \end{pmatrix},$$

where N° and A° are the Berberian's extension of the operators N and A .

Let $y = [x_n]$ denote the equivalence class of the sequence $\{x_n\}_n \subset \mathfrak{M}$. By the inequality (2.2) and Hölder inequality we get

$$\begin{aligned} \|(A^*)^\circ y\|^2 + \|(N^*)^\circ y\|^2 &= \|(T^*)^\circ y\|^2 = \phi(\|T^* x_n\|^2) \leq \phi(\|T^n x\|^{\frac{2}{n}} \|x\|^{\frac{2(n-1)}{n}}) \\ &\leq (\phi(\|T^n x\|))^{\frac{2}{n}} (\phi(\|x\|))^{\frac{2(n-1)}{n}} = \|(T^\circ)^n y\|^{\frac{2}{n}} \|y\|^{\frac{2(n-1)}{n}} \\ &= \|(N^\circ)^n y\|^{\frac{2}{n}} \|y\|^{\frac{2(n-1)}{n}} \end{aligned}$$

By the [2, Theorem 1] we know that the spectrum of normal operator N° is equal to its point spectrum.

If y is an eigenvector of N° , with an eigenvalue α , then we have

$$\|(A^*)^\circ y\|^2 + |\alpha|^2 \|y\|^2 = \|(A^*)^\circ y\|^2 + \|(N^*)^\circ y\|^2 \leq \|(N^\circ)^n y\|^{\frac{2}{n}} \|y\|^{\frac{2(n-1)}{n}} = |\alpha|^2 \|y\|^2.$$

Thus $(A^\circ)^* y = 0$. Now we prove that $A = 0$. Without loss of generality we may assume N is invertible. Put $\lambda = 0$ in (2.1) then we have

$$|N|^{2n} - n\mu^{n-1}|N|^2 + (n-1)\mu^n \geq n\mu^{n-1}AA^*.$$

Put $S = |N|^2$. Then

$$\begin{aligned}
|N|^{2n} - n\mu^{n-1}|N|^2 + (n-1)\mu^n &= S^n - n\mu^{n-1}S + (n-1)\mu^n \\
&= (S^n - \mu^n) - n\mu^{n-1}(S - \mu) \\
&= (S - \mu) \left\{ \sum_{k=0}^{n-1} \mu^k S^{n-1-k} - n\mu^{n-1} \right\} \\
&= (S - \mu)^2 \left\{ \sum_{k=0}^{n-2} (k+1)\mu^k S^{n-2-k} - n\mu^{n-1} \right\} \\
&\geq n\mu^{n-1}AA^*.
\end{aligned}$$

This implies that $A\mathcal{H} \subset \bigcap_{\mu>0} (S - \mu)\mathcal{H} = \bigcap_{\mu \in \mathbb{C}} (S - \mu)\mathcal{H} = \{0\}$ by Douglas's theorem and Putnam's theorem. Hence $A = 0$. $\square$

Lemma 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$. If T is totally $*$ -paranormal or belongs to class $\mathcal{TC}(n)$ for $n \geq 2$ and $\{x_n\}$ is a sequence of unit vectors in $\mathcal{H}$ which satisfies $\lim_{n \rightarrow \infty} \|(T - \lambda)x_n\| = \|T - \lambda\|$, then*

$$\lim_{n \rightarrow \infty} \|(T - \lambda)^k x_n\| = \|T - \lambda\|^k$$

for all $k \in \mathbb{N}$. Hence $\|(T - \lambda)^k\| = \|T - \lambda\|^k$ for all $k \in \mathbb{N}$ and for all $\lambda \in \mathbb{C}$.

Proof. If T belongs to class $\mathcal{TC}(n)$, then $T - \lambda$ belongs to class $\mathfrak{C}(n)$ and so the assertions follow from lemma 1 of [17]. $\square$

Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be transloid if $T - \lambda$ is normaloid for all $\lambda \in \mathbb{C}$.

Proposition 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$. If T belongs to class $\mathcal{TC}(n)$ for $n \geq 2$, then T is transloid, i.e., $r(T - \lambda) = \|T - \lambda\|$, where $r(T)$ denotes the spectral radius of T . In particular, T is normaloid.*

Proof. The proof follows immediately from Lemma 2.7. $\square$

Theorem 2.9. *Let T be an invertible totally n -*-paranormal. Then*

$$\|(T - \lambda)^{-1}\| \leq r((T - \lambda)^{-1})^{\frac{n(n-1)}{2}} r(T - \lambda)^{\frac{(n+1)(n-2)}{2}}.$$

Proof. If T belongs to class $\mathcal{TC}(n)$, then $T - \lambda$ belongs to class $\mathfrak{C}(n)$ and so the assertions follow from Theorem 1 of [17]. $\square$

Corollary 2.10. *If $T \in \mathcal{TC}(n)$ and $\sigma(T) \subset S^1 := \{z \in \mathbb{C} : |z| = 1\}$, then T is unitary*

Proof. If T is a totally n -*-paranormal such that $\sigma(T) \subset S^1$, then $r(T) = r(T^{-1}) = 1$. Hence $\|T\| = r(T) = 1$ and $1 = r(T^{-1}) \leq \|T^{-1}\| \leq r(T^{-1})^{\frac{n(n-1)}{2}} r(T)^{\frac{(n+1)(n-2)}{2}} = 1$ implies $\|T^{-1}\| = 1$. It follows that T is invertible and an isometry because

$$\|x\| = \|T^{-1}Tx\| \leq \|Tx\| \leq \|x\|$$

for all $x \in \mathcal{H}$, so T is unitary. $\square$

Lemma 2.11. *If $T \in \mathcal{B}(\mathcal{H})$ belongs to class $\mathcal{TC}(n)$, $\alpha \in \sigma_p(T)$ and a vector $x \in \mathcal{H}$ satisfies $(T - \alpha)x = 0$, then $(T - \alpha)^*x = 0$.*

Proof. If $T \in \mathcal{TC}(n)$, then T has the invariant translation property. So, it suffices to assume $\alpha = 0$. Without loss of generality we may assume that $\|x\| = 1$.

$$\|T^*x\|^n \leq \|T^n x\| \|x\|^{n-1} = \|T^{n-1}\| \|Tx\| = 0$$

implies that $T^*x = 0$. □

Theorem 2.12. *Any isolated point in the spectrum of a totally n -*-paranormal operator is its eigenvalue.*

Proof. Since $T - z$ is totally n -*-paranormal for each complex number z , therefore we can assume the isolated point in the spectrum $\sigma(T)$ to be zero. Choose $R > 0$ such that the only point of $\sigma(T)$ strictly within $\{z : |z| = R\}$ is zero and $\{z : |z| = R\} \cap \sigma(T) = \emptyset$. Set

$$E = \int_{|z|=R} \frac{1}{z - T} dz.$$

Then E is a non-zero projection commuting with T and hence its range span N is invariant under T . This implies that $T|_N$ is totally n -*-paranormal by Lemma 2.4. Also then

$$\sigma(T|_N) = \sigma(T) \cap \{z : |z| < R\} = \{0\}.$$

Thus $T|_N$ is totally n -*-paranormal quasinilpotent operator by Lemma 2.3 is zero. Let $0 \neq x \in N$. Then $Tx = 0$. This proves the theorem. □

Corollary 2.13. *Suppose that T belongs to class $\mathcal{TC}(n)$ and $\alpha, \beta \in \sigma_p(T)$ with $\alpha \neq \beta$. Then $\ker(T - \alpha) \perp \ker(T - \beta)$.*

Proof. Let $x \in \ker(T - \alpha)$ and $y \in \ker(T - \beta)$. Then $Tx = \alpha x$ and $Ty = \beta y$. Therefore

$$\alpha \langle x, y \rangle = \langle \alpha x, y \rangle = \langle Tx, y \rangle = \langle x, T^*y \rangle = \langle x, \bar{\beta}y \rangle = \beta \langle x, y \rangle.$$

Hence $\alpha \langle x, y \rangle = \beta \langle x, y \rangle$ and so $(\alpha - \beta) \langle x, y \rangle = 0$. But $\alpha \neq \beta$, hence $\langle x, y \rangle = 0$. Consequently $\ker(T - \alpha) \perp \ker(T - \beta)$. □

We say that $T \in \mathcal{B}(\mathcal{H})$ has the *single-valued extension property* (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ , the only analytic function $f : U_\lambda \rightarrow \mathcal{H}$ which satisfies the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathcal{B}(\mathcal{H})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathcal{B}(\mathcal{H})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [10, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Corollary 2.14. *If $T \in \mathcal{TC}(n)$, then T has SVEP.*

Proof. Let f be an analytic function such that $(T - \lambda)f(\lambda) = 0$ on an open set U . By assumption, $f(\lambda) \in \ker(T - \lambda)$ for each $\lambda \in D$. Thus $f(\lambda) \perp f(\mu)$ for any two different nonzero numbers λ and μ in D by Corollary 2.13. Therefore, for any sequence $\{\mu_n\}$ of non-zero complex numbers such that $\mu_n \rightarrow \lambda$, thus $\|f(\lambda)\|^2 = \lim_{\mu_n \rightarrow \lambda} \langle f(\lambda), f(\mu_n) \rangle = 0$. That is, T has SVEP. $\square$

In [9], it is shown that totally $*$ -paranormal operators are not hypercyclic. We generalize this result to the class of totally n -*-paranormal operators. Recall that if $T \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$, then $\{T^n x\}$ is called the orbit of x under T , and is denoted by $\text{orb}(T, x)$. If $\text{orb}(T, x)$ is dense in $\mathcal{H}$, then T is called a hypercyclic operator. Recall that an operator T is said to be semi-Fredholm if $\mathcal{R}(T)$ is closed and either $\ker(T)$ or $\mathcal{H}/\mathcal{R}(T)$ is finite dimensional. The semi-Fredholm spectrum $\sigma_{sF}(T)$ of T is defined by

$$\sigma_{sF}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a semi-Fredholm}\}.$$

At this point we cannot prove that every totally n -*-paranormal operator is not hypercyclic.

Proposition 2.15. *If T belongs to class $\mathcal{TC}(n)$ with $\sigma(T) \neq \sigma_{sF}(T)$, then it is not hypercyclic.*

Proof. We may assume that $0 \in \sigma(T) \setminus \sigma_{sF}(T)$. If T is hypercyclic, $\sigma_p(T^*) = \emptyset$ by [8, Corollary 2.4]. By Lemma 2.11, $\sigma_p(T) = \emptyset$. So, we have a contradiction. $\square$

Recall that the joint point spectrum of T is defined by

$$\sigma_{jp}(T) = \{\lambda \in \mathbb{C} : \exists x \neq 0 : Tx = \lambda x \text{ and } T^*x = \bar{\lambda}x\}.$$

Also the joint approximate point spectrum of T is defined by

$$\sigma_{ja}(T) = \{\lambda \in \mathbb{C} : \text{there exists a sequence } \{x_n\} \text{ of unit vectors such that}$$

$$\lim_{n \rightarrow \infty} \|(T - \lambda)x_n\| = \lim_{n \rightarrow \infty} \|(T - \lambda)^*x_n\| = 0\}.$$

Lemma 2.16. *If T belongs to class $\mathcal{TC}(n)$, then $\sigma_{ja}(T) = \sigma_a(T)$. In particular, $\sigma_{jp}(T) = \sigma_p(T)$.*

Proof. It is clear that $\sigma_{ja}(T) \subset \sigma_a(T)$. Let $\lambda \in \sigma_a(T)$. Then there exists a sequence $\{x_n\}$ of unit vectors such that

$$\lim_{n \rightarrow \infty} \|(T - \lambda)x_n\| = 0.$$

Since $\|(T - \lambda)^*x_n\|^n \leq \|(T - \lambda)^n x_n\| \leq \|(T - \lambda)^n\| \|(T - \lambda)x_n\|$, we get

$$\lim_{n \rightarrow \infty} \|(T - \lambda)^*x_n\| = 0.$$

Consequently, $\lambda \in \sigma_{ja}(T)$. $\square$

Proposition 2.17. *If T belongs to class $\mathcal{TC}(n)$, then $\sigma(T) = \sigma_a(T^*)^*$.*

Proof. It is known for any $T \in \mathcal{B}(\mathcal{H})$

$$\sigma(T) = \sigma_a(T) \cup \sigma(T^*)^*$$

by [7, Problem 73]. It follows from Lemma 2.16 that $\sigma_a(T) = \sigma_{ja}(T)$. From the definition of joint approximate point spectrum, it is clear that for any operator $T \in \mathcal{B}(\mathcal{H})$

$$\sigma_{ja}(T) = \sigma_{ja}(T^*)^* \subset \sigma_a(T^*)^*.$$

Therefore,

$$\sigma_a(T) = \sigma_{ja}(T) \subset \sigma_a(T^*)^*.$$

Since $\sigma(T) = \sigma_a(T) \cup \sigma_a(T^*)^*$, $\sigma(T) \subset \sigma_a(T^*)^*$. On the other hand, $\sigma(T) = \sigma_a(T) \cup \sigma_a(T^*)^* \supset \sigma_a(T^*)^*$. So, we conclude that $\sigma(T) = \sigma_a(T^*)^*$. $\square$

Halmos showed in [7] that a partial isometry is subnormal if and only if it is the direct sum of an isometry and zero. We generalize this theorem to the case of a totally n -*-paranormal operator.

Proposition 2.18. *A partial isometry T is quasinormal (i.e., $T^*T^2 = TT^*T$) if and only if T totally n -*-paranormal operator*

Proof. Assume T is partial isometry and totally n -*-paranormal operator. Since $\ker(T)$ is a reducing subspace for T , $T = 0 \oplus B$ where $B = T|_{\ker(T)^\perp}$ is isometry. Hence $T^*T^2 = TT^*T$. $\square$

Theorem 2.19. *Suppose that the subspaces K of $\mathcal{H}$ reduces an operator T on $\mathcal{H}$. Then T is totally M -*-paranormal if and only if T/K and $T/K^\perp$ are totally M -*-paranormal.*

Proof. Denote $T/K = T_1$ and $T/K^\perp = T_2$. If T belongs to class $\mathcal{TE}(n)$, then $\|(T - \lambda)^*x\|^n \leq M \|(T - \lambda)^n x\|$ for all unit vector $x \in \mathcal{H}$ and for every $\lambda \in \mathbb{C}$. Note that $T = T_1$ and $T^* = T_1^*$ on K . For a unit vector x in K we have,

$$\begin{aligned} \|(T_1 - \lambda)^*x\|^n &= \|(T - \lambda)^*x\|^n \leq \|(T - \lambda)^n x\| \\ &= \|(T_1 - \lambda)^n x\|. \end{aligned}$$

This shows that T_1 belongs to class $\mathcal{TE}(n)$. Again for a unit vector $x \in K^\perp$ we have,

$$\begin{aligned} \|(T_2 - \lambda)^*x\|^n &= \|(T - \lambda)^*x\|^n \leq \|(T - \lambda)^n x\| \\ &= \|(T_2 - \lambda)^n x\|. \end{aligned}$$

This shows that T_2 belongs to class $\mathcal{TE}(n)$.

Conversely, assume that T_1 and T_2 are belong to class $\mathcal{TE}(n)$. We know that for every $x \in \mathcal{H}$ can be written as $x = x_1 + x_2$ where $x_1 \in K$ and $x_2 \in K^\perp$. Hence for all $\lambda \in \mathbb{C}$ and all vectors $x \in \mathcal{H}$ we have,

$$\begin{aligned} \|(T - \lambda)^*x\|^n &= \|(T - \lambda)^*x_1 + (T - \lambda)^*x_2\|^n \\ &= \|(T_1 - \lambda)^*x_1 + (T_2 - \lambda)^*x_2\|^n \\ &= \|(T_1 - \lambda)^*x_1\|^n + \|(T_2 - \lambda)^*x_2\|^n \\ &\leq \|(T_1 - \lambda)^n x_1\| \|x_1\|^{n-1} + \|(T_2 - \lambda)^n x_2\| \|x_2\|^{n-1} \\ &= \|(T - \lambda)^n x_1\| \|x_1\|^{n-1} + \|(T - \lambda)^n x_2\| \|x_2\|^{n-1} \\ &\leq \|(T - \lambda)^n x\| \|x\|^{n-1} \end{aligned}$$

by which totally n -*-paranormality of T is obvious. $\square$

Theorem 2.20. *Let T belongs to class $\mathcal{TE}(n)$. Then the span of all eigenvalues of T reduces T .*

Proof. We shall divide the proof in four steps.

Step 1-Firstly,

$$\{x \in \mathcal{H} : Tx = \lambda x\} \subset \{x \in \mathcal{H} : T^*x = \bar{\lambda}x\}$$

for all $\lambda \in \mathbb{C}$ and this fact follows by Lemma 2.11.

Step 2- For each $\lambda \in \mathbb{C}$, the subspace $\{x \in \mathcal{H} : Tx = \lambda x\}$ reduces T . Denote by K the subspace $\{x \in \mathcal{H} : Tx = \lambda x\}$. For $x \in K$, we have $T(Tx) = \lambda(Tx)$ which implies that Tx is in K . Also, $T(T^*x) = \lambda(\bar{\lambda}x) = \lambda(T^*x)$ showing that T^*x belongs to K . Thus K reduces T .

Step 3- If $\lambda \neq \mu$, then $\{x \in \mathcal{H} : Tx = \lambda x\} \perp \{x \in \mathcal{H} : Tx = \mu x\}$. The proof of this fact follows from Corollary 2.13.

Step 4- The span of all eigenvectors of T reduces T and the restriction of T to that span is normal.

The proof follows from steps (1), (2), (3) and using the fact that the restriction of T to any of the eigenspaces of itself is normal; which of course follows from step (2). $\square$

Theorem 2.21. *If $\mathcal{H}$ is finite-dimensional and T belongs to class $\mathcal{TE}(n)$ on $\mathcal{H}$, then T is normal.*

Proof. We shall follow the induction method. For $n = 1$, every operator is normal. Assume that the theorem holds for dimension $< n$. It can be shown that every linear operator in a finite dimensional space has at least one eigenvalue. Let μ be an eigenvalue for T . Let $K = K_T(\mu)$ be the eigenspace of T associated with μ . By Theorem 2.20, K reduces T and T/K is normal. It can be seen that $K^\perp$ also reduces T . This fact along Theorem 2.19 leads us the conclusion that $T/K^\perp$ is totally M -*-paranormal. Hence $T/K^\perp$ is normal by the inductive assumption and we are through. $\square$

Theorem 2.22. *Let T belongs to class $\mathcal{TE}(n)$ and μ be an isolated point of $\sigma(T)$. If E_μ is the Riesz idempotent for μ , then*

$$E(\mathcal{H}) = \ker(T - \mu) = \ker(T - \mu)^*.$$

In particular, E is self-adjoint, i.e., it is an orthogonal projection.

Proof. We first show that $\ker(T - \mu) = \ker(T - \mu)^*$ for each $\mu \in \text{iso}\sigma(T)$. Since a totally n -*-paranormal operator has the invariant translation property, it sufficient to show that $\ker(T) = \ker(T^*)$. Let 0 be an isolated point of $\sigma(T)$. The inclusion $\ker(T) \subset \ker(T^*)$ holds by of Lemma 2.11 and hence $E(\mathcal{H}) = \ker(T)$ reduces T by Theorem 2.6. Put $T = 0 \oplus T_2$ on $\mathcal{H} = E(\mathcal{H}) \oplus E(\mathcal{H})^\perp$. If $0 \in \sigma(T_2)$, then 0 is an isolated point of $\sigma(T_2)$. Since T_2 is also totally n -*-paranormal by Lemma 2.4, $\lambda \in \sigma_p(T_2)$ by Theorem 2.12. Since $\ker(T_2) \subset \ker(T)$, we have

$$\{0\} \neq \ker(T_2) \subset \ker(T) \cap \ker(T)^\perp = \{0\},$$

and it is a contradiction. Hence $0 \notin \sigma(T_2)$ and T_2 is invertible. Thus $\ker(T^*) \subset \ker(T)$ and so $\ker(T) = \ker(T^*)$. Finally, we show that $E = E^*$. Consider the E on $\mathcal{H} = \mathcal{R}(E) \oplus \mathcal{R}(E)^\perp$ in its block operator form $\begin{pmatrix} I & B \\ 0 & 0 \end{pmatrix}$. Observe that E and $T = 0 \oplus T_2$ commute and that T_2 is invertible. This implies that $B = 0$. Hence E is self-adjoint. $\square$

For $T \in \mathcal{B}(\mathcal{H})$, $\lambda \in \sigma(T)$ is said to be a regular point if there exists $S \in \mathcal{B}(\mathcal{H})$ such that $T - \lambda = (T - \lambda)S(T - \lambda)$. T is called reguloid if every isolated point of $\sigma(T)$ is a regular point. It is well-known from [5, Theorem 4.6.4 and 8.4.4] that

$$T - \lambda = (T - \lambda)S(T - \lambda) \text{ for some } S \in \mathcal{B}(\mathcal{H}) \iff T - \lambda \text{ has closed range.}$$

Corollary 2.23. *Let T be a totally n -*-paranormal operator. Then T is reguloid.*

Proof. Let μ_0 be an isolated point of $\sigma(T)$. Using the Riesz idempotent $E = \frac{1}{2\pi i} \int_{\partial D} (\mu - T)^{-1} d\mu$ for μ_0 , we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \text{ where } \sigma(T_1) = \{\mu_0\} \text{ and } \sigma(T_2) = \sigma(T) \setminus \{\mu_0\}.$$

Since T_1 is also totally n -*-paranormal, it follows from Lemma 2.3 that $T_1 = \mu_0$. Therefore by Theorem 2.22,

$$\mathcal{H} = E(\mathcal{H}) \oplus E(\mathcal{H})^\perp = \ker(T - \mu_0) \oplus \ker(T - \mu_0)^\perp. \quad (2.3)$$

Relative to the decomposition (2.3), $T = \mu_0 \oplus T_2$. Therefore

$$T - \mu_0 = 0 \oplus (T_2 - \mu_0)$$

and hence

$$\mathcal{R}(T - \mu_0) = (T - \mu_0)(\mathcal{H}) = 0 \oplus (T_2 - \mu_0)(\ker(T - \mu_0)^\perp).$$

Since $T_2 - \mu_0$ is invertible, $T - \mu_0$ has a closed range. $\square$

Theorem 2.24. *If T belongs to class $\mathcal{TC}(n)$, then*

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T).$$

Proof. It suffices to establish that $0 \in \sigma(T) \setminus \sigma_w(T)$ if and only if $0 \in \pi_{00}(T)$. Since T belongs to class $\mathcal{TC}(n)$, therefore

$$\|T^*x\|^n \leq \|T^n x\|$$

for each unit vector $x \in \mathcal{H}$. By Lemma 2.11, we have

$$\ker(T) \subset \ker(T^*) = \overline{\mathcal{R}(T)}^\perp.$$

Now, let $0 \in \sigma(T) \setminus \sigma_w(T)$. Then T is a Fredholm of index zero. this gives

$$\ker(T) \subset \overline{\mathcal{R}(T)}^\perp = \mathcal{R}(T)^\perp$$

and

$$\dim \ker(T) = \dim \mathcal{R}(T)^\perp < \infty.$$

We obtain therefore $\ker(T) = \mathcal{R}(T)^\perp$. Hence the decomposition $\mathcal{H} = \ker(T) \oplus \ker(T)^\perp$ gives $T = 0 \oplus S$, where S is invertible. This gives

$$\sigma(T) = \{0\} \cup \sigma(S).$$

Since $0 \notin \sigma(S)$, it is isolated in $\sigma(T)$ and is an eigenvalue of finite multiplicity.

Conversely, if $0 \in \pi_{00}(T)$, the decomposition $\mathcal{H} = \ker(T) \oplus \ker(T)^\perp$ gives $T = 0 \oplus S$, where S is one-one and totally n -*-paranormal. Once a gain

$$\sigma(T) = \{0\} \cup \sigma(S).$$

If S is not invertible, then 0 being an isolated point of $\sigma(S)$ is an eigenvalue of S by Theorem 2.12. This contradicts that S is one to one. Hence S is invertible and in particular surjective. This would imply that T is a Fredholm of index zero. This completes the proof. $\square$

Theorem 2.25. *If $T \in \mathcal{TC}(n)$ with $\sigma_w(T) = \{0\}$, then T is a compact normal operator.*

Proof. By Theorem 2.24, T satisfy Weyl's theorem and this implies that each element in $\sigma(T) \setminus \sigma_w(T) = \sigma(T) \setminus \{0\}$ is an eigenvalue of T with finite multiplicity, and is isolated in $\sigma(T)$. Hence $\sigma(T) \setminus \{0\}$ is a finite set or a countable set with 0 as its only accumulation point. Put $\sigma(T) \setminus \{\lambda_n\}$, where $\lambda_n \neq \lambda_m$ whenever $n \neq m$ and $\{|\lambda_n|\}$ is a non-increasing sequence. Since T is normaloid, we have $|\lambda_1| = \|T\|$. By Lemma 2.11, we have $(T - \lambda_1)x = 0$ implies $(T - \lambda_1)^*x = 0$. Hence $\ker(T - \lambda_1)$ is a reducing subspace of T . Let E_1 be the orthogonal projection onto $\ker(T - \lambda_1)$. Then $T = \lambda_1 \oplus T_1$ on $\mathcal{H} = E_1\mathcal{H} \oplus (1 - E_1)\mathcal{H}$. Since $T_1 \in \mathcal{TC}(n)$ by Lemma 2.4 and $\sigma_p(T) = \sigma_p(T_1) \cup \{\lambda_1\}$, we have $\lambda_2 \in \sigma_p(T_1)$. By the same argument as above, $\ker(T - \lambda_2) = \ker(T_1 - \lambda_2)$ is a finite dimensional reducing subspace of T which is included in $(1 - E_1)\mathcal{H}$. Put E_2 be the orthogonal projection onto $\ker(T - \lambda_2)$. Then $T = \lambda_1 E_1 \oplus \lambda_2 E_2 \oplus T_2$ on $\mathcal{H} = E_1\mathcal{H} \oplus E_2\mathcal{H} \oplus (1 - E_1 - E_2)\mathcal{H}$. By repeating above

argument, each $\ker(T - \lambda_n)$ is a reducing subspace of T and $\left\| T - \bigoplus_{k=1}^n \lambda_k E_k \right\| = \|T_n\| = |\lambda_{n+1}| \rightarrow 0$ as $n \rightarrow \infty$. Here E_k is the orthogonal projection onto $\ker(T - \lambda_k)$ and $T = \left(\bigoplus_{k=1}^n \lambda_k E_k \right) \oplus T_n$ on $\mathcal{H} = \left(\bigoplus_{k=1}^n E_k \right) \mathcal{H} \oplus \left(1 - \sum_{k=1}^n E_k \right) \mathcal{H}$.

Hence $T = \bigoplus_{k=1}^{\infty} \lambda_k E_k$ is compact and normal because each E_k is a finite rank orthogonal projection which satisfies $E_k E_t = 0$ whenever $k \neq t$ by Corollary 2.13 and $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Corollary 2.26. *Every totally n -*-paranormal T can be written as*

$$T = A \oplus S$$

where A is normal and S is totally n -*-paranormal with $\sigma_w(S) = \sigma(S)$.

Proof. By Theorem 2.24, $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. Let N be the closed linear subspace of $\mathcal{H}$ generated by $\bigcup_{\lambda_j \in \pi_{00}(T)} \ker(T - \lambda_j)$. Then N is reduced by T . The

decomposition $\mathcal{H} = N \oplus N^\perp$ gives $T = A \oplus S$, where A is normal and S is totally n -*-paranormal. One can see that $\sigma(S) = \sigma_w(S)$. $\square$

Theorem 2.27. *If T belongs to class $\mathcal{TE}(n)$ with a single limit point of the spectrum, then T is normal.*

Proof. We can assume the limit point to be zero. By hypothesis, every non-zero point of the spectrum being isolated is an eigenvalue. Totally n -*-paranormality of T implies that each eigenspace of T is reducing and T is normal on that eigenspace. Let M be the closed linear span of $\mathcal{H}$ generated by $\bigcup \ker(T - \lambda_j)$, where λ_j runs over non-zero values in $\sigma(T)$. M is thus a closed linear subspace of $\mathcal{H}$ reducing T and $T|_M$ is normal. But then by the decomposition $\mathcal{H} = M \oplus M^\perp$ we get $T|_{M^\perp}$ to be totally n -*-paranormal quasinilpotent operator and hence is zero. Hence T is normal. $\square$

Theorem 2.28. *If T belongs to class $\mathcal{TE}(n)$ with only a finite number of limits points in its spectrum, then T is normal.*

Proof. Let z_1 be a limit point of $\sigma(T)$ and choose a simple closed curve G which does not intersect $\sigma(T)$ and contains only one limit point z_1 in its interior.

$$E_1 = \int_G \frac{1}{z - T} dz.$$

Then E_1 is a non-zero projection on $\mathcal{H}$ such that $E_1\mathcal{H}$ is invariant under T . Also then

$$\sigma(T|_{E_1\mathcal{H}}) = \sigma(T) \cap G^\circ,$$

where G° denotes the interior of G . Hence $T|_{E_1\mathcal{H}}$ can have only one limit point and therefore is normal by Theorem 2.27. Hence T is reduced by $E_1\mathcal{H}$ by Theorem 2.6. Now considering T on $(E_1\mathcal{H})^\perp$ and continuing the same process we conclude that T being a direct sum of normal operators is normal. $\square$

Theorem 2.29. *If T and S are belong to class $\mathcal{TE}(n)$, then*

$$S, T : \text{Weyl} \iff ST : \text{Weyl}.$$

Proof. If S and T are Weyl, then S and T are Fredholm and $\text{ind}(S) = \text{ind}(T) = 0$. By [4], ST is Weyl and by the index product theorem, $\text{ind}(ST) = \text{ind}(S) + \text{ind}(T) = 0$. Hence ST is Weyl.

Conversely, if ST is Weyl, then ST is Fredholm and $\text{ind}(ST) = 0$. Since S and T are totally n -*-paranormal, $\ker(T) \subset \ker(T^*)$ and $\ker(S) \subset \ker(S^*)$. Since $\ker(S^*) \subset \ker(ST)^*$, $\dim \ker(S) \leq \dim \ker(S^*) \leq \dim \ker(ST)^* < \infty$. Thus $\ker(S)$ and $\ker(S^*)$ are finite dimensional. By Schechter [16, Chapter 5, Theorem 3.5], S and T are Fredholm. Since $0 = \text{ind}(ST) = \text{ind}(S) + \text{ind}(T)$, by the index product theorem, $\text{ind}(S) = \text{ind}(T) = 0$. Hence S and T are Weyl. $\square$

Theorem 2.30. *If T belongs to class $\mathcal{TE}(n)$ and f is analytic on a neighborhood of $\sigma(T)$, then $\sigma_w(f(T)) = f(\sigma_w(T))$.*

Proof. Suppose that p is any polynomial. Let

$$p(T) - \lambda = a_0(T - \mu_1) \cdots (T - \mu_n).$$

Since T belongs to class $\mathcal{TE}(n)$, $T - \mu_j$ are belong to class $\mathcal{TE}(n)$ for each $j = 1, 2, \dots, n$. It thus follows from Theorem 2.29 that

$$\begin{aligned} \lambda \notin \sigma_w(p(T)) &\iff p(T) - \lambda \text{ is Weyl} \\ &\iff a_0(T - \mu_1) \cdots (T - \mu_n) \text{ is Weyl} \\ &\iff T - \mu_j \text{ is Weyl for each } j = 1, 2, \dots, n \\ &\iff \lambda \notin p(\sigma_w(T)) \end{aligned}$$

which says that $\sigma_w(p(T)) = p(\sigma_w(T))$. If f is analytic on a neighborhood of $\sigma(T)$, then there is a sequence $\{p_n\}$ such that $p_n \rightarrow f$ uniformly on $\sigma(T)$. Since $p_n(T)$ commutes with $f(T)$, we have

$$f(\sigma_w(T)) = \lim_{n \rightarrow \infty} p_n(\sigma_w(T)) = \lim_{n \rightarrow \infty} \sigma_w(p_n(T)) = \sigma_w(f(T)).$$

□

Corollary 2.31. *If T belongs to class $\mathcal{TE}(n)$ and f is analytic on a neighborhood of $\sigma(T)$, then Weyl's theorem holds for $f(T)$.*

Proof. Since T belongs to class $\mathcal{TE}(n)$, it is isoloid by Theorem 2.12. Also Weyl's theorem holds for T by Theorem 2.24, it follows from [1] that

$$f(\sigma_w(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T)).$$

Since $f(\sigma_w(T)) = \sigma_w(f(T))$ by Theorem 2.30,

$$\sigma_w(f(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T)).$$

Therefore, $f(T)$ satisfies Weyl's theorem. □

Recall that an operator $B \in \mathcal{B}(\mathcal{H})$ is said to be simply polaroid if the isolated points of the spectrum of the operator are simple poles (i.e., order one poles) of the resolvent of the operator. The following notation and terminology will be required.

The quasi-nilpotent part of an operator $T \in \mathcal{B}(\mathcal{H})$ is the set

$$H_0(T) = \{x \in \mathcal{H} : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

If $T \in \mathcal{B}(\mathcal{H})$, the analytic core $K(T)$ is the set of all $x \in \mathcal{H}$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in \mathcal{H}$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$.

$H_0(T)$ and $K(T)$ are generally non-closed hyperinvariant subspaces of T such that $\ker(T^n) \subseteq H_0(T)$ for all $n \in \mathbb{N} \cup \{0\}$ and $TK(T) = K(T)$; also if $\lambda \in \text{iso}\sigma(T)$, then $\mathcal{H} = H_0(T - \lambda) \oplus K(T - \lambda)$, where $H_0(T - \lambda)$ and $K(T - \lambda)$ are closed [1].

Theorem 2.32. *If T belongs to class $\mathcal{TE}(n)$, then T is simply polaroid.*

Proof. Let $\lambda \in \text{iso}\sigma(T)$, where $T \in \mathcal{TE}(n)$. Then

$$\mathcal{H} = H_0(T - \lambda) \oplus K(T - \lambda),$$

where $H_0(T - \lambda)$ and $K(T - \lambda)$ are closed, $\sigma(A) = \sigma(T|_{H_0(T - \lambda)}) = \{\lambda\}$ and $\sigma(T|_{K(T - \lambda)}) = \sigma(T) \setminus \{\lambda\}$. If $\lambda = 0$, then, A being normal, $A = 0$ and $H_0(T) = \ker(T)$. If instead $\lambda \neq 0$, then (recall, operators in $\mathcal{TE}(n)$ have the invariant

translation property and closed under scalar multiplication) we may assume that $\lambda = 1$. Applying Corollary 2.10 it follows that A is unitary. By Theorem 1.5.14 of [11] that $A = I|_{H_0(T-\lambda)}$ which implies that $H_0(T-1) = \ker(T-1)$. Thus, in either case, we have that $H_0(T-\lambda) = \ker(T-\lambda)$. The proof now follows from the implications

$$\begin{aligned}\mathcal{H} &= \ker(T-\lambda) \oplus K(T-\lambda) \\ \implies (T-\lambda)\mathcal{H} &= 0 \oplus (T-\lambda)K(T-\lambda) = K(T-\lambda) \\ \implies \mathcal{H} &= \ker(T-\lambda) \oplus (T-\lambda)\mathcal{H}.\end{aligned}$$

□

3. An asymmetric Putnam-Fuglede theorem

The classical Putnam-Fuglede theorem asserts that if $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ are normal operators and $TX = XS$ for some $X \in \mathcal{B}(\mathcal{H})$, then $T^*X = XS^*$. Let us overwrite the Putnam-Fuglede theorem in an asymmetric form: if $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ are normal operators and $TX = XS^*$ for some $X \in \mathcal{B}(\mathcal{H})$, then $T^*X = XS$. In this section, we mainly extend the asymmetric Putnam-Fuglede theorem to the class of totally n -*-paranormal operators.

Definition 3.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces with Schauder bases $\{x_n\}$ and $\{y_n\}$ respectively. An operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ with $Tx_n = \lambda_n y_n$, where $\{\lambda_n\}$ is some fixed scalar sequence, is called a diagonal operator (with respect to $\{x_n\}$ and $\{y_n\}$).

Proposition 3.2. Let $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{TC}(n)$, then the residual spectrum of T^* is empty. In particular, we have $\sigma_a(T^*) = \sigma(T^*)$.

Proof. It follows from Lemma 2.4 and Lemma 2.11 that each $T \in \mathcal{TC}(n)$ is a direct sum of diagonal operator and totally n -*-paranormal without point spectrum. Let us assume that T has no eigenvalues. Then since $\ker(T-\lambda) = \{0\}$, we have $\mathcal{R}(T-\lambda)^* = \mathcal{H}$ for all $\lambda \in \mathbb{C}$. Hence $\sigma_r(T^*) = \emptyset$. Moreover,

$$\sigma(T^*) = \sigma_p(T^*) \cup \sigma_c(T^*) \subset \sigma_a(T^*),$$

where $\sigma_c(T)$ is the continuous spectrum of T . This completes the proof. □

Theorem 3.3. Let $T, S \in \mathcal{B}(\mathcal{H})$. If $T, S \in \mathcal{TC}(n)$ are such that $TX = XS^*$ for some $X \in \mathcal{B}(\mathcal{H})$, then $T^*X = XS$.

Proof. Suppose that $T \in \mathcal{TC}(n)$ and hence has the invariant translation property, so we may assume $\lambda = 0$. Let $X = U|X|$ be a polar decomposition of X , with $U : \overline{|X|}\mathcal{H} \rightarrow \overline{X}\mathcal{H}$ unitary operator. Then the operator equation $TX = XS^*$ is equivalent to $\tilde{T}|X| = |X|S^*$, where $\tilde{T} := U^{-1}TU \oplus 0_{\ker(|X|)}$. The operator $\tilde{T}$ belongs to class $\mathcal{TC}(n)$. Indeed we have

$$\begin{aligned}\|(\tilde{T})^*(x+y)\|^n &= \|(U^{-1}T^*U \oplus 0)(x+y)\|^n = \|T^*Ux\|^n \\ &\leq \|T^n Ux\| \|Ux\|^{n-1} = \|(U^{-1}T^n U \oplus 0)(x+y)\| \|x\|^{n-1} \\ &\leq \|(\tilde{T})^n(x+y)\| \|x+y\|^{n-1},\end{aligned}$$

for each $x \in \overline{\mathcal{R}(|X|)}$ and $y \in \ker(|X|)$. Thus it is enough to show that for two totally n -*-paranormal operators T, S and positive operator X such that $TX = XS^*$, the equality $T^*X = XS$ holds true.

Let us fix totally n -*-paranormal operators T, S and positive operator X such that $TX = XS^*$. Hence the subspace $\overline{X\mathcal{H}}$ is invariant for T . Since the subspace $\ker(X)$ is invariant for S^* , then $\overline{X\mathcal{H}}$ is also invariant for S . As a consequence we have the following matrices representations with respect to the decomposition $\mathcal{H} = \overline{X\mathcal{H}} \oplus \ker(X)$.

$$X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ and } S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}.$$

It follows from Lemma 2.4 that T_1, S_1 are belong to $\mathcal{TC}(n)$. The equation $TX = XS^*$ implies that $T_1A = AS_1^*$. By Lemma 3.2 we have $\sigma(S_1^*) = \sigma_a(S_1^*)$.

The Berberian's extensions $T_1^\circ, S_1^\circ, A^\circ$ of the operators T_1, S_1, A satisfy the equation

$$T_1^\circ A^\circ = A^\circ (S_1^*)^\circ \quad (3.1)$$

and $\sigma(S_1^*)^\circ = \sigma(S_1^*) = \sigma_a(S_1^*) = \sigma_p((S_1^*)^\circ)$. The equation (3.1) is equivalent to

$$(\alpha - T_1^\circ)A^\circ = A^\circ(\alpha - (S_1^*)^\circ),$$

for $\alpha \in \mathbb{C}$. Thus if $\alpha \in \sigma_r(T_1^\circ)$, then $\alpha \in \sigma_r((S_1^*)^\circ)$, where $\sigma_r(T)$ is the residual spectrum of T . But by Lemma 3.2 we get $\sigma((S_1^*)^\circ) = \emptyset$. As a consequence $\sigma_r(T_1^\circ) = \emptyset$. So $\sigma(T_1^\circ) = \sigma_a(T_1^\circ) = \sigma_p(T_1^\circ)$. Moreover, the operator T_1° belongs to class $\mathcal{TC}(n)$ (see the proof of Lemma 2.6). Thus by Lemma 2.11 the operator T_1° is diagonal, so it is normal. Normality of T_1° shows that T_1 is normal. Hence by Lemma 2.11 we get $T_2 = 0$.

The equation (3.1) is equivalent to $A^\circ(T_1^*)^\circ = S_1^\circ A^\circ$. Thus we can repeat the above argument and show that the operator S_1 is normal and $S_2 = 0$.

Finally, to show that $T^*X = XS$ it is enough to show that $T_1^*A = AS_1$, but it is consequence of the classical Putnam-Fuglede theorem. $\square$

Definition 3.4. We say that the operator $T \in \mathcal{B}(\mathcal{H})$ satisfies the Putnam-Fuglede theorem if and only if for all operators $X, N \in \mathcal{B}(\mathcal{H})$ such that N is normal and $TX = XN$, it holds that $T^*X = XN^*$.

Proposition 3.5. [3] *The operator $T \in \mathcal{B}(\mathcal{H})$ satisfies the Putnam-Fuglede theorem if and only if each invariant subspace $\mathfrak{M} \subset \mathcal{H}$ of T such that $T|_{\mathfrak{M}}$ is normal, is reducing for T .*

Theorem 3.6. *Let $T \in \mathcal{B}(\mathcal{H})$. If T belongs to $\mathcal{TC}(n)$ and N normal and $TX = XN$, then $T^*X = XN^*$.*

Proof. The proof follows immediately from Lemma 2.6 and Proposition 3.5. $\square$

Lemma 3.7. *Let $T \in \mathcal{B}(\mathcal{H})$. If T and T^* are belong to class $\mathcal{TC}(n)$, then T is normal.*

Proof. If T belongs to class $\mathcal{TC}(n)$, then

$$\|(T - \lambda)^*x\|^n \leq \|(T - \lambda)^n x\| \leq \|(T - \lambda)^{n-1}\| \|(T - \lambda)x\| \quad (3.2)$$

for all $\lambda \in \mathbb{C}$ and unit vector $x \in \mathcal{H}$. Similarly, if T^* belongs to class $\mathcal{TC}(n)$, then

$$\|(T - \lambda)x\|^n \leq \|(T - \lambda)^*x\|^n \leq \|(T - \lambda)^{n-1}\| \|(T - \lambda)^*x\| \quad (3.3)$$

for all $\lambda \in \mathbb{C}$ and unit vector $x \in \mathcal{H}$. Hence from relations (3.2) and (3.3), we have

$$\|(T - \lambda)^*x\| \leq \|(T - \lambda)^{n-1}\|^{\frac{n^2}{(n+1)(n^2-1)}} \text{ and } \|(T - \lambda)x\| \leq \|(T - \lambda)^{n-1}\|^{\frac{n^2}{(n+1)(n^2-1)}}, \quad (3.4)$$

Thus

$$\|(T - \lambda)x\| = \|(T - \lambda)^*x\| \text{ for all } x \in \mathcal{H} \text{ and } \lambda \in \mathbb{C}.$$

Therefore, $T - \lambda$ is normal and consequently T is normal. $\square$

Theorem 3.8. *Let T and S belong to class $\mathcal{TC}(n)$ and $TX = XS^*$. Then*

- (i) $\overline{\mathcal{R}(X)}$ reduces T and $\ker(X)$ reduces S .
- (ii) $T|_{\overline{\mathcal{R}(X)}}$ and $S^*|_{\ker(S)^\perp}$ are unitarily equivalent normal operators.

Proof. (i) By Theorem 3.3, $TXX^* = XS^*X^* = XX^*T$. Thus T commutes with XX^* and so $\overline{\mathcal{R}(X)} = \overline{\mathcal{R}(XX^*)}$ reduces T . Similarly S commutes with X^*X and $\ker(X) = \ker(X^*X)$ reduces S .

(ii) Let $X = UP$ be the polar decomposition of X . Since S commutes with P as above, we have

$$(TU - US^*)P = 0.$$

Let $\mathcal{H}_1 = \ker(X)^\perp$ and let $\mathcal{H}_2 = \overline{\mathcal{R}(X)}$; let $T_2 = T|_{\mathcal{H}_2}$ and $S_1 = S|_{\mathcal{H}_1}$. Let $V : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be defined by $Vx = Ux$ for all $x \in \mathcal{H}_1$. The equation above then becomes

$$T_2V = VS_1^*.$$

Since V is an invertible isometry we have that T_2 is unitarily equivalent to S_1^* and since T_2 and S_1 are both totally M -*-paranormal and hence they are normal by Lemma 3.7. $\square$

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