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CONTROLLABILITY FOR A SECOND-ORDER EVOLUTION INCLUSION

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Abstract. By using a suitable fixed point theorem a sufficient condition for controllability is obtained for a class of second-order evolution inclusions.

1. Introduction

In this note we study second-order differential inclusions of the form

$$x''(t) \in A(t)x(t) + F(t, x(t)) + (Bu)(t), \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1.1}$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $x_0, x_1 \in X$, $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{\mathcal{U}(t,s)\}_{t,s\in[0,T]}$, B is a linear operator from the Banach space U to X and the the control $u(.)$ belongs to $L^2([0,T], U)$.

The general framework of evolution operators $\{A(t)\}_{t\geq 0}$ that define problem (1.1) has been developed by Kozak in [6].

The present paper is motivated by several very recent papers ([1, 2, 4]) where existence results and qualitative properties of solutions for problem (1.1) with $B \equiv 0$ have been obtained by using fixed point techniques. The aim of this note is to obtain a sufficient condition for controllability of problem (1.1) when the set-valued map $F(.,.)$ has convex values and is upper semicontinuous. The proof of our result relies on a fixed point theorem due to Martelli ([8]) and follows the general ideas in [3]. In fact, our result may be considered as an extension to problem (1.1) of the results in [3] obtained for other classes of differential inclusions.

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main results.

2. Preliminaries

Let denote by I the interval $[0,T]$, $T > 0$ and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$. As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to$

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X endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|\mathrm{d}t$. With $B(X)$ we denote the Banach space of linear nounded operators on X.

We recall that a set-valued map $G : I \to \mathcal{P}(X)$ with nonempty bounded closed convex values is called strong measurable if for any $x \in X$, the function $t \to d(x, G(t)) = \inf\{|x - z|; z \in G(t)\}$ is measurable.

Let Y, Z two Hausdorff topological spaces and $G : Y \to \mathcal{P}(Z)$ a set-valued map. $G(.)$ it is called upper semicontinuous in $x \in \mathrm{Dom}(G)$ if for any open set U containing $G(x)$ there exists $\eta > 0$ such that $\forall\, x' \in B(x, \eta)$, $G(x') \subset U$.

$G(.)$ it is called totally bounded if $F(D)$ is relatively compact for any bounded set D of Y. $G(.)$ it is called completely upper semicontinuous if $G(.)$ is totally bounded and upper semicontinuous.

If $G(.)$ is totally bounded with nonempty compact values then $G(.)$ is upper semicontinuous if and only if it has a closed graph (e.g., [3]).

The set-valued map $G(.) : Y \to \mathcal{P}(Y)$ has a fixed point if there exists $x \in Y$ such that $x \in G(x)$.

In what follows $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{\mathcal{U}(t,s)\}_{t,s\in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t. The next definition is taken from [6].

Definition 2.1. A family of bounded linear operators $\mathcal{U}(t,s) : X \to X$, $(t,s) \in \Delta := \{(t,s) \in I \times I; s \leq t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t) \tag{2.1}$$

if

i) For any $x \in X$, the map $(t,s) \to \mathcal{U}(t,s)x$ is continuously differentiable and

a) $\mathcal{U}(t,t) = 0$, $t \in I$.

b) If $t \in I, x \in X$ then $\frac{\partial}{\partial t}\mathcal{U}(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)x|_{t=s} = -x$.

ii) If $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, the map $(t,s) \to \mathcal{U}(t,s)x$ is of class C^2 and

a) $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x \equiv A(t)\mathcal{U}(t,s)x$.

b) $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x \equiv \mathcal{U}(t,s)A(t)x$.

c) $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t,s)x|_{t=s} = 0$.

iii) If $(t,s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x$ and

a) $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x \equiv A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)x$ and the map $(t,s) \to A(t)\frac{\partial}{\partial s}\mathcal{U}(t,s)x$ is continuous.

b) $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x \equiv \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x$.

As an example for equation (2.1) one may consider the problem (e.g., [4])

$$\frac{\partial^2 z}{\partial t^2}(t,\tau) = \frac{\partial^2 z}{\partial \tau^2}(t,\tau) + a(t)\frac{\partial z}{\partial t}(t,\tau), \quad t \in [0,T], \tau \in [0,2\pi],$$

$$z(t,0) = z(t,\pi) = 0, \quad \frac{\partial z}{\partial \tau}(t,0) = \frac{\partial z}{\partial \tau}(t,2\pi),\ t \in [0,T],$$

where $a(.) : I \to \mathbf{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbf{R}, \mathbf{C})$ of 2π-periodic 2-integrable functions from $\mathbf{R}$ to $\mathbf{C}$, $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$ with domain $H^2(\mathbf{R}, \mathbf{C})$ the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbf{R}, \mathbf{C})$. It is well known that A_1 is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbf{Z}$ with associated eigenvectors $z_n(\tau) = \frac{1}{\sqrt{2\pi}} e^{in\tau}$, $n \in \mathbf{N}$. The set z_n, $n \in \mathbf{N}$ is an orthonormal basis of X. In particular, $A_1 z = \sum_{n\in\mathbf{Z}} -n^2 < z, z_n > z_n$, $z \in D(A_1)$. The cosine function is given by $C(t)z = \sum_{n\in\mathbf{Z}} \cos(nt) < z, z_n > z_n$ with the associated sine function $S(t)z = t < z, z_0 > z_0 + \sum_{n\in\mathbf{Z}^*} \frac{\sin(nt)}{n} < z, z_n > z_n$.

For $t \in I$ define the operator $A_2(t)z = a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R}, \mathbf{C})$. Set $A(t) = A_1 + A_2(t)$ and the problem may be written in the abstract form as in Definition 2.1.

Definition 2.2. A continuous mapping $x(.) \in C(I, X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I, X)$ such that

$$f(t) \in F(t, x(t)) \quad a.e.\ (I),$$

$$x(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)x_1 + \int_0^t \mathcal{U}(t,s)f(s)ds + \int_0^t \mathcal{U}(t,s)Bu(s)ds,\ t \in I.$$

Definition 2.3. Problem (1.1) is called *controllable* on the interval I if for every $x_0, x_1, y_1 \in X$ there exists a control $u \in L^2(I, U)$ and a mild solution of problem (1.1) such that $x(T) = y_1$.

In the sequel the following conditions are satisfied.

Hypothesis. i) $F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty bounded closed convex values, for any $x \in X$, $F(., x)$ is strong measurable, for any $t \in I$, $F(t, .)$ is upper semicontinuous and for any $x(.) \in C(I, X)$ the set

$$S_{F,x} := \{h(.) \in L^1(I, X); \quad h(t) \in F(t, x(t)) \quad a.e.\ (I)\} \tag{2.2}$$

is nonempty.

ii) $B : U \to X$ is a linear operator, the linear operator $W : L^2(I, U) \to X$ defined by

$$Wu = \int_0^T \mathcal{U}(T,s)Bu(s)ds$$

is invertible with its inverse taking values in $L^2(I, U)/ker(W)$ and there exists $M_1, M_2 > 0$ such that

$$|B| \le M_1, \quad |W^{-1}| \le M_2.$$

iii) There exist $M \ge 1, M_0 \ge 0$ such that $|\mathcal{U}(t,s)|_{B(X)} \le M$, $|\frac{\partial}{\partial s}\mathcal{U}(t,s)| \le M_0$, for all $(t,s) \in \Delta$.

iv) There exist $q(.) \in L^1(I, (0,\infty))$ and $\psi : (0,\infty) \to (0,\infty)$ continuous and increasing that satisfy $M\int_0^T q(s)ds < \int_a^\infty \frac{1}{\psi(u)}du$, such that

$$|F(t,x)| \le q(t)\psi(|x|) \quad a.e.\ (I), \quad \forall x \in X,$$

where $a = |x_0|M_0 + |x_1|M + k_0$ and $k_0 := MM_1M_2[|y_1| + |x_0|M_0 + |x_1|M_1 + M\int_0^T q(s)\psi(s)ds]$.

Remark 2.4. Several remarks are in order.

i) If $dimX < \infty$, then $S_{F,x} \neq \emptyset$ for any $x(.) \in C(I, X)$ (e.g., [7]).

ii) If $dimX = \infty$, then $S_{F,x} \neq \emptyset$ if and only if the map $Y : I \to \mathbf{R}$ defined by

$$Y(t) := \inf\{|v|; \quad v \in F(t, x)\}$$

belongs to $L^1(I, \mathbf{R})$ (e.g., [3]).

iii) Hypothesis iv) is satisfied if, for example, $F(., .)$ satisfies a classical linear growth condition

$$|F(t, x)| \leq q(t)(1 + |x|) \quad q(.) \in L^1(I, X), t \in I, x \in X.$$

We recall two results that are essential in the proof of our result. The first one is proved in [7] and the second one is proved in [8].

Lemma 2.5. *Let X be a Banach space, let F be a set-valued map that satisfy Hypothesis i) and let $\Gamma : L^1(I, X) \to C(I, X)$ be a linear continuous mapping.*

Then the set-valued map $\Gamma \circ S_F : C(I, X) \to \mathcal{P}(C(I, X))$ defined by

$$(\Gamma \circ S_F)(x) = \Gamma(S_{F,x})$$

has closed graph in $C(I, X) \times C(I, X)$.

Lemma 2.6. *Let X be a Banach space and let $N : X \to \mathcal{P}(X)$ be a set-valued map with nonempty bounded closed convex values and completely upper semicontinuous.*

If the set

$$\Omega := \{x \in X; \quad \text{there exists } \lambda > 1 \text{ such that } \lambda x \in N(x)\}$$

is bounded, then $N(.)$ has a fixed point.

3. Main result

We are ready now to present our main result.

Theorem 3.1. *Assume that Hypothesis is satisfied.*

Then, problem (1.1) is controllable on I.

Proof. Taking into account Hypothesis ii) we define for $x(.) \in C(I, X)$ the control

$$u_x(t) = W^{-1}[y_1 + \frac{\partial}{\partial s}\mathcal{U}(T, 0)x_0 - \mathcal{U}(T, 0)x_1 - \int_0^T \mathcal{U}(T, s)f(s)ds](t),$$

where $f \in S_{F,x}$ and $S_{F,x}$ is defined in (2.2).

Consider the set-valued map $N : C(I, X) \to \mathcal{P}(C(I, X))$ defined by

$$N(x) := \{h(.) \in C(I, X); \quad h(t) = -\frac{\partial}{\partial s}\mathcal{U}(t, 0)x_0 + \mathcal{U}(t, 0)x_1 +$$

$$\int_0^t \mathcal{U}(t, s)f(s)ds + \int_0^t \mathcal{U}(t, s)(Bu_x)(s)ds, ; \quad f(.) \in S_{F,x}\}.$$

Obviously, the fixed points of $N(.)$ are solutions of problem (1.1). Therefore, we show that $N(.)$ is completely upper semicontinuous with non empty bounded closed convex values.

First, we show that $N(x)$ is convex for any $x \in C(I,X)$.

If $h_1, h_2 \in N(x)$ then there exist $f_1, f_2 \in S_{F,x}$ such that for any $t \in I$ one has

$$h_1(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)x_1 + \int_0^t \mathcal{U}(t,s)f_1(s)ds + \int_0^t \mathcal{U}(t,s)(Bu_x)(s)ds,$$

$$h_2(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)x_1 + \int_0^t \mathcal{U}(t,s)f_2(s)ds + \int_0^t \mathcal{U}(t,s)(Bu_x)(s)ds.$$

Let $0 \le \alpha \le 1$. Then for any $t \in I$ we have

$$(\alpha h_1 + (1-\alpha)h_2)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)x_1 +$$
$$\int_0^t \mathcal{U}(t,s)[\alpha f_1(s) + (1-\alpha)f_2(s)]ds + \int_0^t \mathcal{U}(t,s)(Bu_x)(s)ds.$$

The values of $F(.,.)$ are convex, thus $S_{F,x}$ is a convex set and hence $\alpha h_1 + (1-\alpha)h_2 \in N(x)$.

Secondly, we show that $N(.)$ is bounded on bounded sets of $C(I,X)$.

It is enough to prove that there exists $l \ge 0$ such that for any $h \in N(x)$, $x \in B_r := \{x \in C(I,X); |x|_C \le r\}$ one has $|h|_c \le l$, where as usual $|x|_C := \sup_{t\in I}|x(t)|$.

If $h \in N(x)$ there exists $f \in S_{F,x}$ such that, for $t \in I$,

$$h(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)x_1 + \int_0^t \mathcal{U}(t,s)f(s)ds + \int_0^t \mathcal{U}(t,s)(Bu_x)(s)ds.$$

From Hypothesis one may write, for any $t \in I$

$$\begin{aligned}&|h(t)| \le M_0|x_0| + M|x_1| + |\textstyle\int_0^t \mathcal{U}(t,s)(Bu_x)(s)ds| + |\int_0^t \mathcal{U}(t,s)f(s)ds|\\ &\le M_0|x_0| + M|x_1| + MM_1M_2(|y_1| + M_0|x_0| + M|x_1| +\\ &M\textstyle\int_0^T q(s)ds \cdot \sup_{t\in I}\psi(|x(t)|)) + M\int_0^T q(s)ds\cdot \sup_{t\in I}\psi(|x(t)|) \le M_0|x_0| +\\ &M|x_1| + MTM_1M_2(|y_1| + M_0|x_0| + M|x_1| + M|q|_1\psi(r)) + M|q|_1\psi(r) =: l\end{aligned}$$

We show next that $N(.)$ maps bounded sets from $C(I,X)$ into equicontinuous sets of $C(I,X)$.

Let $t_1, t_2 \in I$, $t_1 < t_2$. For any $x \in B_r$ and $h \in N(x)$ there exists $f \in S_{F,x}$ such that

$$h(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)x_1 + \int_0^t \mathcal{U}(t,s)f(s)ds + \int_0^t \mathcal{U}(t,s)(Bu_x)(s)ds, \quad t \in I.$$

One has

$$|h(t_2) - h(t_1)| \le |\frac{\partial}{\partial s}\mathcal{U}(t_2,0)x_0 - \frac{\partial}{\partial s}\mathcal{U}(t_1,0)x_0| + |\mathcal{U}(t_2,0)x_1 - \mathcal{U}(t_1,0)x_1| +$$
$$|\int_0^{t_2}[\mathcal{U}(t_2,s) - \mathcal{U}(t_1,s)]BW^{-1}[y_1 + \frac{\partial}{\partial s}\mathcal{U}(T,0)x_0 - \mathcal{U}(T,0)x_1 - \int_0^T \mathcal{U}(T,v)f(v)dv]$$
$$(s)ds| + |\int_{t_1}^{t_2}\mathcal{U}(t_1,s)BW^{-1}[y_1 + \frac{\partial}{\partial s}\mathcal{U}(T,0)x_0 - \mathcal{U}(T,0)x_1 - \int_0^T \mathcal{U}(T,v)f(v)dv](s)$$

$$ds + |\int_0^{t_2} |\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s)|f(s)ds| + |\int_{t_1}^{t_2} \mathcal{U}(t_1, s)f(s)ds| \leq$$

$$|\frac{\partial}{\partial s}\mathcal{U}(t_2, 0)x_0 - \frac{\partial}{\partial s}\mathcal{U}(t_1, 0)x_0| + |\mathcal{U}(t_2, 0)x_1 - \mathcal{U}(t_1, 0)x_1| +$$

$$\int_0^{t_2} |\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s)|M_1M_2[|y_1| + M_0|x_0| + M|x_1| + M|q|_1\psi(r)]ds$$

$$+ \int_{t_1}^{t_2} |\mathcal{U}(t_1, s)|M_1M_2[|y_1| + M_0|x_0| + M|x_1| + M|q|_1\psi(r)](s)ds +$$

$$\int_0^{t_2} |\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s)|q(s)\psi(r)ds + M(t_2 - t_1)|q|_1.$$

The right hand side in the above inequality does not depend on x and tends to 0 when $t_2 - t_1 \to 0$.

From the last two steps of the proof and from Arzela-Ascoli's theorem we deduce that $N(.)$ is completely continuous.

We show next that $N(.)$ has a closed graph.

Let $x_n \to x^*$, $h_n \in N(x_n)$ and $h_n \to h^*$. We prove that $h^* \in N(x^*)$.

Since $h_n \in N(x_n)$, there exists $f_n \in S_{F,x_n}$ such that

$$h_n(t) = -\frac{\partial}{\partial s}\mathcal{U}(t, 0)x_0 + \mathcal{U}(t, 0)x_1 + \int_0^t \mathcal{U}(t, s)f_n(s)ds + \int_0^t \mathcal{U}(t, s)(Bu_{x_n})(s)ds,$$

where

$$u_{x_n}(t) = W^{-1}[y_1 + \frac{\partial}{\partial s}\mathcal{U}(T, 0)x_0 - \mathcal{U}(T, 0)x_1 - \int_0^T \mathcal{U}(T, s)f_n(s)ds](t).$$

We must prove that there exists $f^* \in S_{F,x^*}$ such that

$$h^*(t) = -\frac{\partial}{\partial s}\mathcal{U}(t, 0)x_0 + \mathcal{U}(t, 0)x_1 + \int_0^t \mathcal{U}(t, s)f^*(s)ds + \int_0^t \mathcal{U}(t, s)(Bu_{x^*})(s)ds,$$

where

$$u_{x^*}(t) = W^{-1}[y_1 + \frac{\partial}{\partial s}\mathcal{U}(T, 0)x_0 - \mathcal{U}(T, 0)x_1 - \int_0^T \mathcal{U}(T, s)f^*(s)ds](t).$$

We define

$$\overline{u}_x(t) = W^{-1}[y_1 + \frac{\partial}{\partial s}\mathcal{U}(T, 0)x_0 - \mathcal{U}(T, 0)x_1].$$

Obviously, if $n \to \infty$, we have

$$|(h_n(t) + \frac{\partial}{\partial s}\mathcal{U}(t, 0)x_0 - \mathcal{U}(t, 0)x_1 - \int_0^t \mathcal{U}(t, s)(B\overline{u}_{x_n})(s)ds)$$

$$-(h^*(t) + \frac{\partial}{\partial s}\mathcal{U}(t, 0)x_0 - \mathcal{U}(t, 0)x_1 - \int_0^t \mathcal{U}(t, s)(B\overline{u}_{x^*})(s)ds)|_C \to 0.$$

Define $\Gamma : L^1(I, X) \to C(I, X)$ by

$$(\Gamma f)(t) := \int_0^t \mathcal{U}(t, s)[BW^{-1}(\int_0^T \mathcal{U}(T, \tau)f(\tau)d\tau)(s) + f(s)]ds.$$

Since
$$|\Gamma f|_C \le M(MM_1M_2+1)|f|_1,$$
Γ is linear and continuous.

With Lemma 2.5 we deduce that $\Gamma \circ S_F$ has closed graph. Moreover, one has
$$h_n(t)+\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0-\mathcal{U}(t,0)x_1-\int_0^t \mathcal{U}(t,s)(B\overline{u}_{x_n})(s)ds \in \Gamma(S_{F,x_n}).$$

Since $x_n \to x^*$, we infer that
$$h^*(t)+\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0-\mathcal{U}(t,0)x_1-\int_0^t \mathcal{U}(t,s)(B\overline{u}_{x^*})(s)ds =$$
$$=\int_0^t \mathcal{U}(t,s)[BW^{-1}(\int_0^T S(T,\tau)f^*(\tau)d\tau)(s)+f^*(s)]ds.$$
with $f^* \in S_{F,x^*}$.

Finally, it remains to prove that the set
$$\Omega := \{x \in X; \quad \text{there exists } \lambda > 1 \text{ such that } \lambda x \in N(x)\}$$
is bounded.

Consider $x \in \Omega$. Therefore, there exists $\lambda > 1$ such that $\lambda x \in N(x)$. Hence, there exists $f \in S_{F,x}$ such that, for $t \in I$,
$$x(t) = -\lambda^{-1}\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0+\lambda^{-1}\mathcal{U}(t,0)x_1+\lambda^{-1}\int_0^t \mathcal{U}(t,s)BW^{-1}[y_1-$$
$$\frac{\partial}{\partial s}\mathcal{U}(T,0)x_0-\mathcal{U}(T,0)x_1-\int_0^T \mathcal{U}(T,s)f(v)dv](s)ds+\lambda^{-1}\int_0^t \mathcal{U}(t,s)f(s)ds.$$

Taking into account Hypothesis for any $t \in I$ we have
$$\begin{array}{l} |x(t)| \le M_0|x_0|+M|x_1|+|\int_0^t \mathcal{U}(t,s)(Bu_x)(s)ds|+|\int_0^t \mathcal{U}(t,s)f(s)ds| \\ \le M_0|x_0|+M|x_1|+MM_1M_2(|y_1|+M_0|x_0|+M|x_1|+ \\ M\int_0^T q(s)\psi(|x(s)|)ds)+M\int_0^t q(s)\psi(|x(s)|)ds \end{array}$$

Denote $\mu := \sup\{|x(t)|;\ t \in I\}$ and $v(t) := M_0|x_0|+M|x_1|+MM_1M_2(|y_1|+M_0|x_0|+M|x_1|+M\int_0^T q(s)\psi(s)ds)+M\int_0^t q(s)\psi(\mu)ds$. We have, for $t \in I$,
$$\mu \le v(t), \quad v(0) = M_0|x_0|+M|x_1|+k_0 = a, \quad v'(t) = Mq(t)\psi(\mu).$$

Since ψ is increasing we get
$$v'(t) \le Mq(t)\psi(v(t)), \quad t \in I.$$

So, for any $t \in I$,
$$\int_{v(0)}^{v(t)} \frac{1}{\psi(u)}du \le M\int_0^T q(s)ds \le \int_{v(0)}^{\infty}\frac{1}{\psi(u)}du.$$

From the last inequality there exists $d \ge 0$, depending only on q and ψ such that $v(t) \le d$ for any $t \in I$; thus $|x|_C \le d$, i.e., Ω is bounded.

We apply Lemma 2.6 on $C(I,X)$ and we obtain that $N(.)$ has a fixed point. $\square$

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