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REAL FORMS OF STRANGE LIE SUPERALGEBRAS AND THE FROBENIUS-SCHUR METHOD

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Abstract. This article provides a classification of real forms of strange Lie superalgebras by applying the Frobenius-Schur method of representation theory.

1. Introduction

Classical Lie superalgebras are finite dimensional complex simple Lie superalgebras $L = L_{\bar{0}} + L_{\bar{1}}$ in which $L_{\bar{0}}$ are reductive Lie algebras acting reducibly on $L_{\bar{1}}$ [3, §2]. Ignoring Lie algebras (i.e. assuming $L_{\bar{1}} \neq 0$), they are divided into *contragredient* (infinite series A, B, C, D and exceptional $D(2,1;\alpha), F(4), G(3)$) and *strange* (infinite series P, Q) [3, p.44]. For the strange Lie superalgebras, we adopt the following notations:

$$\begin{aligned} n \geq 3, \quad & \mathbf{P} = P(n-1)\ ,\ \mathbf{P} = \mathbf{P}_{\bar{0}} + \mathbf{P}_{\bar{1}} = \mathfrak{sl}(n,\mathbb{C}) + S(n) + \wedge(n), \\ & \mathbf{Q} = Q(n-1)\ ,\ \mathbf{Q} = \mathbf{Q}_{\bar{0}} + \mathbf{Q}_{\bar{1}} = \mathfrak{sl}(n,\mathbb{C}) + \mathfrak{sl}(n,\mathbb{C}), \end{aligned}$$

where $S(n)$ and $\wedge(n)$ are respectively the complex $n \times n$ symmetric and skew-symmetric matrices.

A real form of L is a real subalgebra $\mathfrak{g} \subset L$ such that $L = \mathfrak{g} \oplus i\mathfrak{g}$. There are earlier works on the classification of real forms [3, p.89][5, p.691][6]. However, the list in [3] is incomplete (it misses real form with $\mathfrak{g}_{\bar{0}} = \mathfrak{su}^*(n) \subset \mathbf{P}_{\bar{0}}$), and furthermore [3][5] do not explain the absence of real form with $\mathfrak{g}_{\bar{0}} = \mathfrak{su}(p,q) \subset \mathbf{P}_{\bar{0}}$. Here $\mathfrak{su}^*(n)$ is defined only for n even, and $\mathfrak{su}^*(n) \cong \mathfrak{sl}(\frac{n}{2},\mathbb{H})$. The above classifications adopt careful case-by-case computations, and it is desirable to have other conceptual and systematic approach. More recently, the notions of Cartan involutions and Vogan diagrams of semisimple Lie algebras [4] have been generalized to Cartan automorphisms and Vogan superdiagrams, and they provide classifications of real forms of contragredient Lie superalgebras [2, Thms.1.1,1.3]. However, these objects rely on invariant nondegenerate bilinear super-symmetric forms, which do not exist on strange Lie superalgebras, and so they cannot be used for their real forms (see Remark 3.6). In this article, we introduce the Frobenius-Schur method of representation theory as another way to classify the real forms of strange Lie superalgebras. In particular, we prove the following theorem, which is our main result.

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Theorem 1.1. *Every real form of $L_{\bar{0}}$ extends to a real form of L, except for $\mathfrak{su}(p,q) \subset \mathbf{P}_{\bar{0}}$. The extension is unique up to isomorphism.*

2. Real forms

In this and the next sections, we prove Theorem 1.1, which classifies the real forms of strange Lie superalgebras. We first introduce several notions of real forms as follows.

Definition 2.1. Let $\oplus$ denote vector space direct sum.
(a) A real form of a complex Lie algebra or superalgebra L is a real subalgebra $\mathfrak{g}$ such that $L = \mathfrak{g} \oplus i\mathfrak{g}$;
(b) A real form of a complex Lie group G is a real Lie subgroup H such that $\mathrm{Lie}(H)$ is a real form of $\mathrm{Lie}(G)$;
(c) If ξ is a real Lie group or real Lie algebra, and V is a complex ξ-representation, a real form of (ξ, V) is a real subspace $W \subset V$ such that $V = W \oplus iW$ and $\xi \cdot W \subset W$.

Proposition 2.2. (Kac) [3, Prop.5.3.2] *$\mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is a real form of L if and only if $\mathfrak{g}_{\bar{1}}$ is a real form of $(\mathfrak{g}_{\bar{0}}, L_{\bar{1}})$. In this case $\mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is uniquely determined by $\mathfrak{g}_{\bar{0}}$ up to isomorphism.*

The following proposition classifies the real forms of $\mathbf{Q}$.

Proposition 2.3. *Any real form $\mathfrak{g}_{\bar{0}}$ of $\mathbf{Q}_{\bar{0}}$ extends to a real form $\mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ of $\mathbf{Q}$, where $\mathfrak{g}_{\bar{0}} \cong \mathfrak{g}_{\bar{1}}$.*

Proof: We have $\mathbf{Q}_{\bar{0}} \cong \mathbf{Q}_{\bar{1}} \cong \mathfrak{sl}(n,\mathbb{C})$, and the adjoint $\mathbf{Q}_{\bar{0}}$-representation on $\mathbf{Q}_{\bar{1}}$ is just the usual adjoint representation of $\mathfrak{sl}(n,\mathbb{C})$ [3, p.33]. So for any real form $\mathfrak{g}_{\bar{0}}$ of $\mathbf{Q}_{\bar{0}}$, we can always take $\mathfrak{g}_{\bar{1}} \cong \mathfrak{g}_{\bar{0}}$ to be a real form of $(\mathfrak{g}_{\bar{0}}, \mathbf{Q}_{\bar{1}})$. Then by Proposition 2.2, $\mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ is a real form of $\mathbf{Q}$. $\square$

For the rest of this section and the next section, we work on $\mathbf{P}$. Let $(\cdot)^t$ be the transpose of matrix. Let

$$\begin{aligned} &\mathbf{P} = \mathbf{P}_{\bar{0}} + \mathbf{P}_{\bar{1}} = \mathfrak{sl}(n,\mathbb{C}) + S(n) + \wedge(n) \ , \\ &S(n) = \{n \times n \text{ complex symmetric matrices}\} \ , \\ &\wedge(n) = \{n \times n \text{ complex skew-symmetric matrices}\}. \end{aligned}$$

Here $\mathbf{P}_{\bar{0}} = \mathfrak{sl}(n,\mathbb{C})$, while $S(n)$ and $\wedge(n)$ are the irreducible components of the adjoint $\mathbf{P}_{\bar{0}}$-representation on $\mathbf{P}_{\bar{1}}$ given by

$$\begin{aligned} &\tau : \mathfrak{sl}(n,\mathbb{C}) \longrightarrow \mathrm{End}(S(n)) \ ; \ \tau_A B = AB + BA^t, \\ &\pi : \mathfrak{sl}(n,\mathbb{C}) \longrightarrow \mathrm{End}(\wedge(n)) \ ; \ \pi_A C = -A^t C - CA. \end{aligned} \tag{2.1}$$

The $\mathbf{P}_{\bar{0}}$-representations (2.1) integrate to group representations

$$\begin{aligned} &\tau : \mathrm{SL}(n,\mathbb{C}) \longrightarrow \mathrm{Aut}(S(n)) \ ; \ \tau_A B = ABA^t, \\ &\pi : \mathrm{SL}(n,\mathbb{C}) \longrightarrow \mathrm{Aut}(\wedge(n)) \ ; \ \pi_A C = (A^t)^{-1} C A^{-1}. \end{aligned} \tag{2.2}$$

We can thus formulate the problem in terms of Lie groups. Namely for a real form G of $\mathrm{SL}(n,\mathbb{C})$, we consider the real forms of complex G-representations.

Let E be a complex G-representation. Let $\overline{E}$ be the complex vector space with the same underlying real vector space, but with the new complex multiplication $(z,e) \mapsto \bar{z} \cdot e$ for all $z \in \mathbb{C}$ and $e \in E$. Let $\mathrm{Hom}_G(E, \overline{E})$ denote all $\mathbb{C}$-linear maps $E \longrightarrow \overline{E}$ which intertwine with the G-action. If $\rho \in \mathrm{Hom}_G(E, \overline{E})$, we obtain $\overline{\rho} \in \mathrm{Hom}_G(\overline{E}, E)$ by the same formula $\overline{\rho}(e) = \rho(e)$. We have the identification

$$\{\text{real forms } X \text{ of } (G,E)\} \longleftrightarrow \{\rho \in \mathrm{Hom}_G(E, \overline{E})\ ;\ \overline{\rho} \cdot \rho = 1\}. \tag{2.3}$$

Here X leads to ρ by $\rho(e+if) = \rho(e-if)$ for all $e, f \in X$; and conversely ρ leads to X by $X = \{e \in E\ ;\ \rho(e) = e\}$.

The real form $\mathfrak{g}$ of $\mathbf{P}$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{su}^*(n)$ is missing in [3, p.88]. We construct it here.

Proposition 2.4. $\mathbf{P}_{\bar{0}}$ *has a real form* $\mathfrak{g}$ *with* $\mathfrak{g}_{\bar{0}} = \mathfrak{su}^*(n)$.

Proof: Let $n = 2p$. Let

$$V = \left\{ \begin{pmatrix} r & -\overline{s} \\ s & \overline{r} \end{pmatrix} \in S(n) \right\}, \ W = \left\{ \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix} \in \wedge(n) \right\},$$

where r, s, u, v are $p \times p$ matrices. The conditions imposed by $S(n)$ and $\wedge(n)$ automatically imply that $r = r^t$, $s^t = -\overline{s}$, $u = -u^t$, $v^t = \overline{v}$. Then $\mathfrak{g} = \mathfrak{su}^*(n) + V + W$ is a real form of $\mathbf{P}$. This can be checked directly, or by the following: Define $\rho : \mathbf{P} \longrightarrow \mathbf{P}$ by $\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{P}_{\bar{0}}$, $S(n)$ or $\wedge(n)$. Then $\rho \in \mathrm{Hom}(\mathbf{P}, \overline{\mathbf{P}})$ and $\overline{\rho} \cdot \rho = 1$. So by (2.3), $\mathfrak{g} = \{A \in \mathbf{P}\ ;\ \rho(A) = A\}$ is a real form of $\mathbf{P}$. This proves the proposition. $\square$

The real form of $\mathbf{P}$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(n, \mathbb{R})$ is given by all the real matrices of $\mathbf{P}$. So together with Propositions 2.3 and 2.4, we obtain all the real forms needed for Theorem 1.1. It remains to show that $\mathbf{P}$ has no real form with $\mathfrak{g}_{\bar{0}} = \mathfrak{su}(p,q)$, and we shall do this in the next section.

3. $\mathrm{SU}(p,q)$

In this section, we show that $\mathbf{P}$ has no real form $\mathfrak{g}$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{su}(p,q)$, and complete the proof of Theorem 1.1. We first provide some general properties of group representations.

Proposition 3.1. *Let* G *be a real semisimple Lie group and* $V \oplus W$ *a finite dimensional complex* G*-representation, where* V *and* W *are irreducible and* $\dim V \neq \dim W$. *Then* $(G, V \oplus W)$ *has a real form if and only if both* (G,V) *and* (G,W) *have real forms.*

Proof: Let $E = V \oplus W$. Then $\overline{E} = \overline{V} \oplus \overline{W}$. Write $\rho \in \mathrm{Hom}_G(E, \overline{E})$ as a matrix

$$\rho = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\ ; \quad \begin{array}{ll} A \in \mathrm{Hom}_G(V, \overline{V}), & B \in \mathrm{Hom}_G(W, \overline{V}), \\ C \in \mathrm{Hom}_G(V, \overline{W}), & D \in \mathrm{Hom}_G(W, \overline{W}). \end{array} \tag{3.1}$$

Since V and W are irreducible G-subrepresentations and $\dim V \neq \dim W$, by Schur's lemma, $B = C = 0$ in (3.1). It leads to

$$\overline{\rho} \cdot \rho = \begin{pmatrix} \overline{A}A & 0 \\ 0 & \overline{D}D \end{pmatrix}. \tag{3.2}$$

Recall that (2.3) identifies the real forms of (G, E) with mappings in $\mathrm{Hom}_G(E, \overline{E})$ of order 2. We have

(G, E) has a real form
$\iff$ there exists $\rho \in \mathrm{Hom}_G(E, \overline{E})$, $\overline{\rho} \cdot \rho = 1$ by (2.3)
$\iff$ there exist $A \in \mathrm{Hom}_G(V, \overline{V}), D \in \mathrm{Hom}_G(W, \overline{W})$, $\overline{A}A = 1, \overline{D}D = 1$ by (3.2)
$\iff$ (G, V) and (G, W) have real forms by (2.3).

This proves Proposition 3.1. □

Let G be a connected real semisimple Lie group, and V a finite dimensional complex irreducible G-representation. Let $G = KAN$ be an Iwasawa decomposition [4, Thm.6.46], where N corresponds to the positive root spaces of A. The A-representation on V has a highest weight space given by the N-invariant vectors $V^N \subset V$. Let M be the centralizer of A in K. Then M normalizes N, so M preserves V^N.

Proposition 3.2. *Let G be a connected real semisimple Lie group, and V a finite dimensional complex irreducible G-representation. The following are equivalent.*
(a) *(G, V) has a real form;*
(b) *(M, V^N) has a real form.*

Proof: Suppose that (G, V) has a real form. By (2.3), we identify it with $\rho \in \mathrm{Hom}_G(V, \overline{V})$, where $\overline{\rho} \cdot \rho = 1$. Since ρ intertwines with the MAN-action, the restriction of ρ to V^N is an element of $\mathrm{Hom}_M(V^N, \overline{V^N})$. By (2.3), this can be identified with a real form of (M, V^N). We have shown that (a) implies (b).

Let the lower case Gothic letters denote the Lie algebras of Lie groups, for example $\mathrm{Lie}(G) = \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. Let $\overline{\mathfrak{n}}$ denote the negative root spaces of A. Then $\mathfrak{g} = \overline{\mathfrak{n}} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$.

Conversely, suppose that (M, V^N) has a real form W. We first claim that

$$\overline{\mathfrak{n}} \cdot W \cap i\overline{\mathfrak{n}} \cdot W = 0. \tag{3.3}$$

Since A acts on V^N as real scalars, it preserves W, and so preserves $\overline{\mathfrak{n}} \cdot W$ as well. Therefore, it suffices to check (3.3) on the A-weight vectors of $\overline{\mathfrak{n}} \cdot W$. Suppose otherwise, namely there exists a weight vector $0 \neq \xi \cdot v = i\eta \cdot w$ for some $\xi, \eta \in \overline{\mathfrak{n}}$ and $v, w \in W$. Since v, w are highest weight vectors, we have $\xi, \eta \in \mathfrak{g}_\alpha$ for some negative root $\alpha \in \mathfrak{a}^*$. Then there exists $\gamma \in \mathfrak{g}_{-\alpha}$ such that

$$0 \neq \gamma \cdot (\xi \cdot v) = i\gamma \cdot (\eta \cdot w) \in V^N. \tag{3.4}$$

We can rewrite $\gamma \cdot (\xi \cdot v)$ of (3.4) as $[\gamma, \xi] \cdot v + \xi \cdot (\gamma \cdot v)$. Since γ is a positive root vector and v is a highest weight vector, $\gamma \cdot v = 0$. Hence $\gamma \cdot (\xi \cdot v) = [\gamma, \xi] \cdot v$. Similarly, $i\gamma \cdot (\eta \cdot w) = i[\gamma, \eta] \cdot w$. So (3.4) becomes

$$0 \neq [\gamma, \xi] \cdot v = i[\gamma, \eta] \cdot w \in V^N. \tag{3.5}$$

Note that $[\gamma,\xi],[\gamma,\eta]\in\mathfrak{m}+\mathfrak{a}$, and A acts as real scalars. Hence $[\gamma,\xi]\cdot v\in\mathfrak{m}\cdot W$ and $i[\gamma,\eta]\cdot w\in i\mathfrak{m}\cdot W$. But W is a real form of (M,V^N), so $\mathfrak{m}\cdot W\cap i\mathfrak{m}\cdot W=0$. This contradicts (3.5). This proves (3.3) as claimed.

Since M preserves W, A acts as real scalars and $\mathfrak{n}$ annihilates V^N, it follows that $(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})\cdot W\subset W$. Together with (3.3) and $\mathfrak{g}=\overline{\mathfrak{n}}+\mathfrak{m}+\mathfrak{a}+\mathfrak{n}$, we get $\mathfrak{g}\cdot W\cap i\mathfrak{g}\cdot W=0$. Since V is irreducible and $\mathfrak{g}\cdot W+i\mathfrak{g}\cdot W$ is a complex subrepresentation of V, it follows that $\mathfrak{g}\cdot W+i\mathfrak{g}\cdot W=V$. Hence $\mathfrak{g}\cdot W$ is a real form of $(\mathfrak{g},V)$. We have shown that (b) implies (a). This proves the proposition.□

Let $p+q=n$ and $I_{p,q}=\mathrm{diag}(I_p,-I_q)$. Let $(\cdot)^*$ denote conjugate transpose. Recall that

$$\mathrm{SU}(p,q)=\{A\in\mathrm{SL}(n,\mathbb{C})\ ;\ AI_{p,q}A^*=I_{p,q}\}.$$

We shall restrict τ of (2.2) to $\mathrm{SU}(p,q)$ and show that $(\mathrm{SU}(p,q),S(n)+\wedge(n))$ has no real form.

We divide the discussions into two cases, the compact group $\mathrm{SU}(n)$ and the noncompact group $\mathrm{SU}(p,q)$ for positive p and q. We start with $\mathrm{SU}(n)$.

Proposition 3.3. *For $n\geq 3$, $(\mathrm{SU}(n),S(n))$ has no real form.*

Proof: Let T be the maximal torus of $\mathrm{SU}(n)$ given by the diagonal matrices. In the following discussion, we allow both $j=k$ and $j\neq k$. Let $C=\mathrm{diag}(e^{i\theta_1},...,e^{i\theta_n})\in T$, where $\prod_1^n e^{i\theta_j}=1$. By (2.2),

$$\tau_C(E_{j,k}+E_{k,j})=C(E_{j,k}+E_{k,j})C=e^{i\theta_j}e^{i\theta_k}(E_{j,k}+E_{k,j}). \tag{3.6}$$

Suppose that $S(n)$ has an $\mathrm{SU}(n)$-invariant bilinear form $(\cdot,\cdot)$. By (3.6),

$$(E_{1,1},E_{j,k}+E_{k,j})=(\tau_C E_{1,1},\tau_C(E_{j,k}+E_{k,j}))=e^{2i\theta_1}e^{i\theta_j}e^{i\theta_k}(E_{1,1},E_{j,k}+E_{k,j}). \tag{3.7}$$

For all $j,k=1,...,n$, we can find $C=\mathrm{diag}(e^{i\theta_1},...,e^{i\theta_n})\in T$ such that $e^{2i\theta_1}e^{i\theta_j}e^{i\theta_k}\neq 1$, hence (3.7) implies that $(E_{1,1},E_{j,k}+E_{k,j})=0$. Since $\{E_{j,k}+E_{k,j}\}_{j,k}$ generate $S(n)$, we conclude that $S(n)$ has no nondegenerate $\mathrm{SU}(n)$-invariant bilinear form.

Normalize the Haar measure dg so that $\mathrm{SU}(n)$ has volume 1. Let $\chi:\mathrm{SU}(n)\longrightarrow\mathbb{C}$ be the character of the $\mathrm{SU}(n)$-representation on $S(n)$. Its Frobenius-Schur indicator is defined as

$$\int_{\mathrm{SU}(n)}\chi(g^2)\,dg\in\{1,0,-1\}. \tag{3.8}$$

Since $S(n)$ has no nondegenerate $\mathrm{SU}(n)$-invariant bilinear form, it implies that (3.8) has value 0, which further implies that $(\mathrm{SU}(n),S(n))$ has no real form (real form exists if and only if (3.8) has value 1) [1, Prop.45.1,Thm.45.1]. This proves the proposition. □

Proposition 3.3 is not valid for $n=2$. See Remark 3.5 for more explanations.

Next we consider the noncompact group $\mathrm{SU}(p,q)$, where p,q are positive integers and $3\leq p+q=n$. Relabel if necessary, we assume that $p\leq q$. We first introduce some notations.

For matrix groups G of 1×1 or 2×2 matrices, we describe some imbeddings $G_{i,j}\subset\mathrm{GL}(n,\mathbb{C})$, where $i<j$. If G contains 1×1 matrices (especially $\mathrm{U}(1)$), define $G_{i,j}$ by putting G at the (i,i) and (j,j) entries of $\mathrm{GL}(n,\mathbb{C})$. If G contains

2×2 matrices, define $G_{i,j}$ by putting the $(1,1),(1,2),(2,1),(2,2)$ entries of G at the $(i,i),(i,j),(j,i),(j,j)$ entries of $\mathrm{GL}(n,\mathbb{C})$ respectively. In both cases, write 1 at the remaining diagonal entries, and 0 at the remaining non-diagonal entries. For example,

$$\mathrm{U}(1)_{2,3}=\left\{\begin{pmatrix}1&0&0\\0&e^{i\theta}&0\\0&0&e^{i\theta}\end{pmatrix}\right\},\ \mathrm{SU}(2)_{1,3}=\left\{\begin{pmatrix}\alpha&0&-\overline{\beta}\\0&1&0\\\beta&0&\overline{\alpha}\end{pmatrix}\ ;\ \alpha\overline{\alpha}+\beta\overline{\beta}=1\right\}. \tag{3.9}$$

Let

$$a(t)=\begin{pmatrix}\cosh t&\sinh t\\\sinh t&\cosh t\end{pmatrix},\ X=\begin{pmatrix}1&1\\1&1\end{pmatrix},\ Y=\begin{pmatrix}1&-1\\-1&1\end{pmatrix},\ H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

One checks that

$$a(t)Xa(t)=e^{2t}X\ ,\ a(t)Ya(t)=e^{-2t}Y\ ;\ a(t)Ha(t)=H. \tag{3.10}$$

Let $V=\{a(t)\ ;\ t\in\mathbb{R}\}$. We now put V,X,Y,H inside the $n\times n$ matrices. By the notation of (3.9), let

$$A=V_{1,p+1}\times\ldots\times V_{p,2p}\subset\mathrm{SU}(p,q). \tag{3.11}$$

For all $i=1,\ldots,p$, define $X_{i,p+i},Y_{i,p+i},H_{i,p+i}\in S(n)$ similarly as (3.9), but with 0 at the irrelevant diagonal entries because we are dealing with vector space. For instance $Y_{1,p+1}=E_{1,1}-E_{1,p+1}-E_{p+1,1}+E_{p+1,p+1}$. Then

$$S(n)=\sum_1^p\mathbb{C}(X_{i,p+i},Y_{i,p+i},H_{i,p+i})+Z,$$

where Z are the symmetric matrices with zero entries in the upper left $2p\times 2p$ block.

As multiplicative groups, $V\cong\mathbb{R}^+$ by $a(t)\mapsto e^t$. Hence $A\cong(\mathbb{R}^+)^p$ and $\mathfrak{a}\cong\mathbb{R}^p$. Define $\alpha_1,\ldots,\alpha_p\in\mathfrak{a}^*$ by letting $\alpha_i:\mathbb{R}^p\longrightarrow\mathbb{R}$ be the projection onto the i-th component. Let A act on $S(n)$ by (2.2). By (3.10), $\mathbb{C}(X_{i,p+i})$ (resp. $\mathbb{C}(Y_{i,p+i})$, $\mathbb{C}(H_{i,p+i})+Z$) are weight spaces with weights $2\alpha_i$ (resp. $-2\alpha_i$, 0). In particular,

$$\mathbb{C}(X_{1,p+1})\text{ is the highest weight space of }A\text{ with weight }2\alpha_1. \tag{3.12}$$

Although τ of (2.2) is given as an $\mathrm{SL}(n,\mathbb{C})$-representation, we can treat it as a $\mathrm{GL}(n,\mathbb{C})$-representation with the same formula. In this way we can work on $\mathrm{U}(p,q)$ tentatively (simpler notation), then restrict the result to $\mathrm{SU}(p,q)$.

Let $\mathrm{U}(p,q)=KAN$ be the Iwasawa decomposition. Since N corresponds to the positive root spaces of A, by (3.12),

$$S(n)^N=\mathbb{C}(X_{1,p+1}). \tag{3.13}$$

We have $K=\mathrm{U}(p)\times\mathrm{U}(q)$. Let M be the centralizer of A in K. By (3.11),

$$M=\mathrm{U}(1)_{1,p+1}\times\ldots\times\mathrm{U}(1)_{p,2p}\times\mathrm{U}(q-p). \tag{3.14}$$

Here $\mathrm{U}(q-p)$ is the natural imbedding into the lower right corner of $\mathrm{U}(p,q)$. Note that $\mathrm{U}(1)_{1,p+1}\subset M$ acts nontrivially on $S(n)^N$ as rotation by $e^{2i\theta}$.

Proposition 3.4. *Let $1\le p\le q$ and $3\le p+q=n$. Then $(\mathrm{SU}(p,q),S(n))$ has no real form.*

Proof: From $\mathrm{U}(p,q) = KAN$, we have $\mathrm{SU}(p,q) = K'AN$ where $K' = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$. Here $\mathrm{S}(\cdot)$ denotes determinant 1. Let M' be the centralizer of A in K'. By (3.14),

$$M' = \mathrm{S}(\mathrm{U}(1)_{1,p+1} \times ... \times \mathrm{U}(1)_{p,2p} \times \mathrm{U}(q-p)).$$

Since $1 \le p \le q$ and $3 \le p + q = n$, we have either $1 < p$ or $p < q$ (or both). Construct a circle group $C \subset M'$ by

$$C = \begin{cases} \mathrm{S}(\mathrm{U}(1)_{1,p+1} \times \mathrm{U}(1)_{2,p+2}) & \text{if } 1 < p, \\ \{\mathrm{diag}\,(e^{i\theta}, e^{i\theta}, 1, ..., 1, e^{-2i\theta})\ ;\ \theta \in \mathbb{R}\} & \text{if } 1 = p. \end{cases} \tag{3.15}$$

Note that if $1 = p$, then $p < q$, hence the last entry $e^{-2i\theta}$ in (3.15) lies in $\mathrm{U}(q-p)$. By (3.13) and (3.15), C acts nontrivially on the complex 1-dimensional space $S(n)^N$ as rotation by $e^{2i\theta}$. Hence $(C, S(n)^N)$ has no real form. Since C is a subgroup of M', $(M', S(n)^N)$ has no real form. By Proposition 3.2, $(\mathrm{SU}(p,q), S(n))$ has no real form. □

Remark 3.5. Real forms of $(\mathrm{SU}(2), S(2))$ and $(\mathrm{SU}(1,1), S(2))$.

Propositions 3.3 and 3.4 say that for $n \ge 3$, $(\mathrm{SU}(n), S(n))$ and $(\mathrm{SU}(p,q), S(n))$ do not have real form. Let us see what happens when $n = 2$. Since $\dim S(2) = \dim \mathfrak{sl}(2,\mathbb{C}) = 3$, we have $S(2) \cong \mathfrak{sl}(2,\mathbb{C})$ as $\mathrm{SL}(2,\mathbb{C})$-modules because the dimension uniquely determines irreducible $\mathrm{SL}(2,\mathbb{C})$-modules. Consequently $(\mathrm{SU}(2), S(2))$ and $(\mathrm{SU}(1,1), S(2))$ have real forms given respectively by $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ inside $\mathfrak{sl}(2,\mathbb{C})$. Hence Propositions 3.3 and 3.4 are not valid for $n = 2$, and we can check how their proofs fail. For $\mathrm{SU}(2)$, we always have $e^{i\theta_2} = e^{-i\theta_1}$, so (3.7) does not imply that $(E_{1,1}, E_{2,2}) = 0$. For $\mathrm{SU}(1,1)$, the construction (3.15) of $C \subset M'$ fails because M' is too small (merely 2×2 matrices). However, since $P(n-1)$ is defined only for $n \ge 3$ [3, p.32], we can ignore $(\mathrm{SU}(2), S(2))$ and $(\mathrm{SU}(1,1), S(2))$.

Proof of Theorem 1.1:

By Proposition 2.3, every real form of $\mathfrak{sl}(n,\mathbb{C})$ extends to a real form of $\mathbf{Q}$.

Up to isomorphism, the real forms of $\mathbf{P}_{\bar{0}} = \mathfrak{sl}(n,\mathbb{C})$ are $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{su}(p,q)$ and $\mathfrak{su}^*(n)$ [4, Thm.6.105]. The real form of $\mathbf{P}$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(n,\mathbb{R})$ is given by all the real matrices in $\mathbf{P}$. The real form of $\mathbf{P}$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{su}^*(n)$ is constructed in Proposition 2.4. By Proposition 2.2, these real forms $\mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$ are determined by $\mathfrak{g}_{\bar{0}}$.

By Propositions 3.3 and 3.4, $(\mathrm{SU}(p,q), S(n))$ has no real form. Since $\dim S(n) \neq \dim \wedge(n)$, by Proposition 3.1, it implies that $(\mathrm{SU}(p,q), S(n) + \wedge(n))$ has no real form. Hence $\mathbf{P}$ has no real form with $\mathfrak{g}_{\bar{0}} = \mathfrak{su}(p,q)$. □

Remark 3.6. Comparison with contragredient Lie superalgebras.

For contragredient Lie superalgebras L, the Killing form of $L_{\bar{0}}$ extends to a nondegenerate supersymmetric bilinear form B on L [3, §2.5]. Hence the Cartan involutions and Vogan diagrams of $L_{\bar{0}}$ [4] extend accordingly to provide systematic classifications of real forms of L by *Cartan automorphisms* [2, Thm.1.1] and *Vogan* *superdiagrams* [2, Thm.1.3]. Namely, a Cartan automorphism $\theta : L \longrightarrow L$ corresponds to a real form $\mathfrak{g} \subset L$ by the condition that θ stabilizes $\mathfrak{g}$, and $-B(x, \theta y)$ is an inner product on $\mathfrak{g}$. The eigenvalues of θ are ± 1 and $\pm i$, and the Vogan superdiagram reveals these eigenvalues on its vertices. But strange Lie superlgebras have no nondegenerate supersymmetric bilinear form [3, p.50], so there is no compatible way to extend the Cartan involutions and Vogan diagrams to the odd part. In fact for $\mathbf{P}$, the Cartan involution and Vogan diagram of $\mathfrak{sl}(n, \mathbb{R})$ or $\mathfrak{su}^*(n)$ do not extend to $\mathbf{P}$.

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