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WEAK McCOY PROPERTY IN AMALGAMATED ALGEBRA ALONG IDEAL

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Abstract. Let $f : A \longrightarrow B$ be a ring homomorphism and J be an ideal of B. In this note, we investigate the transfer of the weak McCoy property to the amalgamation of A with B along J with respect to f (denoted by $A \bowtie^f J$) introduced and studied by D'Anna, Finocchiaro and Fontana in 2009. Our aim is to provide conditions under which $A \bowtie^f J$ is a left weak McCoy (resp. right weak McCoy, McCoy) ring. Our results enrich the literature with new families of left weak McCoy (resp. right weak McCoy, weak McCoy) rings.

1. Introduction and preliminaries

In this paper, all rings considered are associative with identity elements. Given a ring A, $A[X]$ denotes the polynomial ring with an indeterminate X over A. In [13], McCoy proved in 1942 that if two polynomials annihilate each other over a commutative ring, then each polynomial has a non-zero annihilator in the base ring. Following a suggestion by T.Y Lam, the second author made the following definition. A ring A is said to be right McCoy (resp. left McCoy) if for each pair of non-zero polynomials $f(x), g(x) \in A[X]$ with $f(x)g(x) = 0$, then there exists a non-zero element $r \in A$ with $f(x)r = 0$ (resp. $rg(x) = 0$). A ring is McCoy if it both left and right McCoy. Thus N.H. McCoy's result states that commutative rings are McCoy. Note that there are many ways to generalize his theorem.
In [7], Armendariz proved that $a_ib_j = 0$ for all i, j whenever polynomials $f = \sum_{i=0}^{i=n} a_ix^i$ and $g = \sum_{i=0}^{i=m} b_ix^i$ over a reduced ring satisfy $fg = 0$. In [15], Rege and Shhawchharia called the ring (not necessarily reduced) satisfying this property Armendariz ring. Thus, Armendariz rings are a generalization of reduced rings. Also, a ring A is called weak Armendariz if $a_ib_j \in nil(A)$ for all i, j whenever polynomials $f = \sum_{i=0}^{i=n} a_ix^i$ and $g = \sum_{i=0}^{i=m} b_ix^i$ satisfying $fg = 0$.
The foolowing diagram shows all implicaions among these properties.
Armendariz ring $\Longrightarrow$ weak Armendariz ring $\Longrightarrow$ weak McCoy ring $\Longrightarrow$ left weak McCoy ring.
Also Semicommutative ring $\Longrightarrow$ weak McCoy ring $\Longrightarrow$ left weak McCoy ring.
Let A and B be two rings with identity elements, J be an ideal of B and let

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$f : A \to B$ be a ring homomorphism. *The amalgamation of A and B along J with respect to f* is the subring of $A \times B$ defined by:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

This construction is a generalization of the so called *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [2, 4, 5]). Also, many other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([3, Examples 2.5 and 2.6]). In this paper, we investigate the transfer of the left weak McCoy (resp. right weak McCoy, resp. weak McCoy) property to the amalgamation. Following our results a class of original examples of left weak McCoy (resp. right weak McCoy, resp. weak McCoy) rings are given.

2. Main results

The main result of this section, is the study of the transfer of right weak McCoy property in $A \bowtie^f J$ and the investigation of the conditions under which the amalgamation is a right weak McCoy ring. Similarly, we have the same result for left weak McCoy rings and thus weak McCoy rings.
We start by giving definition of weak McCoy ring.

Definition 2.1. let A be an associative ring with identity. We say that a ring A is right weak McCoy whnever $f(x) = a_0 + a_1x + \cdots + a_nx^n$, $g(x) = b_0 + b_1x + \cdots + b_mx^m \in A[x] - \{0\}$ satisfies $f(x)g(x) = 0$, then $sb_j \in nil(A)$ for some $s \in A - \{0\}$. We defined left weak McCoy ring similarly. If a ring is both left and right weak McCoy, we say that the ring is weak McCoy.

In the following proposition, we characterize when the amalgamtion of rings is semicommutative.

Proposition 2.2. *Let A, B two rings, $f : A \longrightarrow B$ a ring homomorphism, and J be a proper ideal of B.*

1. *If A and $f(A) + J$ are semicommutatives rings. Then so is $A \bowtie^f J$.*
2. *Assume that $f^{-1}(J) = (0)$, then A and $f(A) + J$ are semicommutatives rings if and only $A \bowtie^f J$ is.*

Proof.
1. Let $(a, f(a) + j_1)$ and $(b, f(b) + j_2) \in A \bowtie^f J$ such that $(a, f(a) + j_1)(b, f(b) + j_2) = (0, 0)$, then $ab = 0$ and $(f(a) + j_1)(f(b) + j_2) = 0$. Since A and $f(A) + J$ are both semicommutatives rings, then $aAb = 0$ and $(f(a) + j_1)(f(A) + J)(f(b) + j_2) = 0$. Thus for every $(r, f(r) + j) \in A \bowtie^f J$ we have $arb = 0$ and $(f(a) + j_1)(f(r) + j)(f(b) + j_2) = 0$ which implies that $(a, f(a) + j_1)(r, f(r) + j)(b, f(b) + j_2) = (0, 0)$. Consequently $A \bowtie^f J$ is semicommutative.
2. For the first implication, follows from 1. Conversely, let $a, b \in A$ such that $ab = 0$, since $(a, f(a))(b, f(b) = (0, 0)$ and $A \bowtie^f J$ is semicommutative then $aAb = 0$. Thus A is semicommutative.
   Let $f(a) + j_1$ and $f(b) + j_2$ such that $(f(a) + j_1)(f(b) + j_2) = f(ab) + t = 0$ where $t \in J$ then $ab \in f^{-1}(J) = 0$. Hence $(a, f(a) + j_1)(b, f(b) + j_2) =$

$(0,0)$. since $A \bowtie^f J$ is semicommutative ring, then for every $(r, f(r)+j) \in A \bowtie^f J$ we have $(f(a)+j_1)(f(r)+j)(f(b)+j_2) = 0$. Thus $(f(a)+j_1)(f(A)+J)(f(b)+j_2) = 0$. Consequently $f(A)+J$ is semicommutative. As desired.

□

Now, we annonce the main result for the right weak McCoy property in amalgamted algebra. Notice that, we have the same result for left weak McCoy and thus for weak McCoy rings:

Theorem 2.3. *Let A, B two rings, $f : A \longrightarrow B$ a ring homomorphism, and J be a proper ideal of B.*

1. *Suppose that $f^{-1}(J) = (0)$. $A \bowtie^f J$ is a right weak McCoy if and only $f(A)+J$ is a right weak McCoy ring.*
2. *Suppose that $f^{-1}(J) \subset nil(A)$. If $f(A)+J$ is a right weak McCoy ring then $A \bowtie^f J$ is a right weak McCoy ring.*
3. *Suppose that $J \subset nil(B)$. If A is a right weak McCoy ring, then $A \bowtie^f J$ is a right weak McCoy ring.*
4. *Assume that $J \cap S \neq \emptyset$ where S is the set of regular central elements of B. If $f(A)+J$ is a right weak McCoy ring, then $A \bowtie^f J$ is a right weak McCoy ring.*
5. *Assume that f is injective*
   5.1. *If $J \subset f(A)$ and $A \bowtie^f J$ is a right weak McCoy ring, then so is A.*
   5.2. *If $f(A) \cap J = (0)$, then $A \bowtie^f J$ is a right weak McCoy ring if and only if $f(A)+J$ is a right weak McCoy ring.*

Proof. (1) $\Rightarrow$) Let $f_B(x) = \sum_{i=0}^{n}(f(a_i)+j_i)x^i$ and $g_B(x) = \sum_{j=0}^{m}(f(b_j)+k_j)x^j$ such that $f_B g_B = 0$. Then for $k \in \{0,\dots,n+m\}$, we have $\sum_{k=0}^{k=n+m} f(\sum_{i+j=k} a_i b_j) + t = f(\sum_{k=0}^{k=n+m}(\sum_{i+j=k} a_i b_j)) + t = 0$ where $t \in J$.
Hence $\sum_{k=0}^{k=n+m}(\sum_{i+j=k} a_i b_j) \in f^{-1}(J) = \{0\}$. Thus for $F(x) = \sum_{i=0}^{n}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{m}(b_j, f(b_j)+k_j)x^j \in (A \bowtie^f J)[x]$, we have $F(x)G(x) = 0$. Since $A \bowtie^f J$ is right weak McCoy then there exist $(r, f(r)+l) \in (A \bowtie^f J) - \{(0,0)\}$ such that $(r, f(r)+l)(b_j, f(b_j)+k_j) \in nil(A \bowtie^f J)$ for all j.
If $f(r)+l \neq 0$ we are done. Else, $f(r)+l = 0$. In this case $r \in f^{-1}(J) = \{0\}$ which implies that $(r, f(r)+l) = (0,0)$ which is contradiction. In conclusion $f(A)+J$ is a right weak McCoy.
$\Leftarrow$)
Let $F(x) = \sum_{i=0}^{n}(a_i, f(a_i)+j_i)x^i$ and $G(x) = \sum_{j=0}^{m}(b_j, f(b_j)+k_j)x^j$ in $(A \bowtie^f J)[x]$ such that $F(x)G(x) = 0$.
Set $f_B(x) = \sum_{i=0}^{n} f(a_i)x^i$ and $g_B(x) = \sum_{j=0}^{m} f(b_j)x^j$. Then we have $f_B g_B = 0$. Hence there exist $(f(s)+t) \in (f(A)+J) - \{0\}$ such that $(f(s)+t)(f(b_j)+k_j) = (f(sb_j)+l) \in nil(f(A)+J)$ for all j, where $l \in J$. Then there exist $n_j \in \mathbb{N}$ such that $(f(sb_j)+l)^{n_j} = f((sb_j)^{n_j}) + c = 0$ where $c \in J$. Which implies that $(sb_j)^{n_j} \in f^{-1}(J) = \{0\}$. Hence $(s, f(s)+t)(b_j, f(b_j)+k_j) \in nil(A \bowtie^f J)$. As desired.

(2) Let $F(x) = \sum_{i=0}^{n}(a_i, f(a_i) + j_i)x^i$ and $G(x) = \sum_{j=0}^{m}(b_j, f(b_j) + k_j)x^j$ in $A \bowtie^f J[x]$ such that $F(x)G(x) = 0$.
Set $f_B(x) = \sum_{i=0}^{n}(f(a_i) + j_i)x^i$, $g_B(x) = \sum_{j=0}^{m}(f(b_j) + k_j)x^j$ in $(f(A) + J)[x]$. Then $F(x)G(x) = 0$ implies that $f_B(x)g_B(x) = 0$, since $f(A) + J$ is a right weak McCoy which in turn implies that, there existe $(f(r) + t) \in (f(A)+J)-\{0\}$ such that $(f(r)+t)(f(b_j)+k_j) = f(rb_j)+l \in nil(f(A)+J)$, where $l \in J$ for every j. Consider $(r, f(r) + t) \in (A \bowtie^f J)$.
We get $(rb_j) \in f^{-1}(J) \subset nil(A)$. Wich implies that $(r, f(r)+t)(b_j, f(b_j)+k_j) \in nil(A \bowtie^f J)$ for every j. Consequently $A \bowtie^f J$ is a right weak McCoy.

(3) Let $F(x) = \sum_{i=0}^{n}(a_i, f(a_i) + j_i)x^i$ and $G(x) = \sum_{j=0}^{m}(b_j, f(b_j) + k_j)x^j$ in $A \bowtie^f J[x]$ such that $F(x)G(x) = 0$.
Set $f_A(x) = \sum_{i=0}^{n} a_i x^i$ and $g_A(x) = \sum_{j=0}^{m} b_j x^j$ we have $f_A(x)g_A(x) = 0$. Since A is a right weak McCoy, there exist $r \in A - \{0\}$ such that $rb_j\ nil(A)$ for all j. Consider $(r, f(r)) \in A \bowtie^f J$ then $(r, f(r))(b_j, f(b_j) + k_j) = (rb_j, f(rb_j) + t)$ for all j, where $t \in J$. Since $rb_j \in nil(A)$ and $J \subset nil(B)$ then $f(rb_j) + l \in nil(f(A) + J)$. Consequently we have $(r, f(r)(b_j, f(b_j) + k_j)) \in nil(A \bowtie^f J)$. Hence $A \bowtie^f J$ is a right weak McCoy.

(4) Let S be the set of regular central elements of B, and suppose that $J \cap S \neq \varnothing$.
Suppose that $f(A) + J$ is a right weak McCoy ring, we show that $A \bowtie^f J$ is .
Let $F(x) = \sum_{i=0}^{n}(a_i, f(a_i) + j_i)x^i$ and $G(x) = \sum_{j=0}^{m}(b_j, f(b_j) + k_j)x^j$ in $A \bowtie^f J[x]$ such that $F(x)G(x) = 0$.
Set $f_B(x) = \sum_{i=0}^{n}(f(a_i) + j_i)x^i$, $g_B(x) = \sum_{j=0}^{m}(f(b_j) + k_j)x^j$ in $(f(A) + J)[x]$ and $f_A(x) = \sum_{i=0}^{i=n} a_i x^i$, $g_A(x) = \sum_{j=0}^{j=m} b_j x^j$ in $A[x]$. $F(x)G(x) = 0$ implies that $f_A(x)g_A(x) = 0$ and $f_B(x)g_B(x) = 0$.
$f_B(x)g_B(x) = 0$ and $(f(A) + J)$ is a right weak McCoy ring implies that there exist non zero element $(f(r) + l) \in f(A) + J$ such that $(f(r) + l)(f(b_j) + k_j) \in nil(f(A) + J)$ for all j. Thus $e(f(r) + l)(f(b_j) + j_i) \in nil(f(A) + J)$ and so $(0, e(f(r) + l))(b_j, f(b_j) + k_j) \in nil(A \bowtie^f J)$ for any $e \in S \cap J$ and for all j. Hence $A \bowtie^f J$ is a right weak McCoy ring.

(5) Assume that f is injective.
   5.1 Suppose that $J \subset f(A)$. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ in $A[x]$ such that $f(x)g(x) = 0$.
   Set $F(x) = \sum_{i=0}^{n}(a_i, f(a_i))x^i$ and $G(x) = \sum_{i=0}^{n}(b_i, f(b_j))x^j$.

$$\begin{aligned} F(x)G(x) &= \sum_{k=0}^{k=n+m} \left(\sum_{i+j=k} (a_i b_j, f(a_i b_j)) \right) x^k \\ &= \sum_{k=0}^{k=n+m} \left(\sum_{i+j=k} a_i b_j, \sum_{i+j=k} f(a_i b_j) \right) x^k \\ &= \sum_{k=0}^{k=n+m} \left(\sum_{i+j=k} a_i b_j, f\left(\sum_{i+j=k} a_i b_j \right) \right) x^k \end{aligned}$$

Hence $F(x)G(x) = 0$.
Since $A \bowtie^f J$ is a right weak McCoy then there exist $(r, f(r)+t) \in (A \bowtie^f J) - \{(0,0)\})$ such that $(r, f(r)+t)(b_j, f(b_j)) \in \mathrm{Nil}(A \bowtie^f J)$ for all j.
If $r \neq 0$ we are done.
If $r = 0$ then $f(r) = 0$ which implies that $f(s) = t \neq 0$ where $s \in J$, and so $s \neq 0$. Then $f(s)f(b_j) = f(sb_j) \in nil(f(A))$. Since f is injective then $sb_j \in nil(A)$. Consequently A is right weak McCoy.

5.2 Suppose that $f(A) \cap J = (0)$.
In this case $A \bowtie^f J \simeq f(A) + J$ and the conclusion follows.

□

Corollary 2.4. *Let (A, B) be a pair of rings, $f : A \longrightarrow B$ a ring homomorphism, and J be a proper ideal of B.*

1. *Assume that J is semicommutative. If A is weak Armendariz then $A \bowtie^f J$ is right weak McCoy.*
2. *Assume that $f^{-1}(J)$ is semicommutative. If $f(A)+J$ is weak Armendariz then $A \bowtie^f J$ is right weak McCoy.*

Proof. (1) Follows from [14][Theorem 4.1(6)] and the fact that weak Armendariz ring is right weak McCoy.
(2) Follows from [14][Theorem 4.1(7)] and the fact that weak Armendariz ring is right weak McCoy.

□

Proposition 2.5. *If $nil(A) \trianglelefteq A$ and $J \subset nil(B)$, then $A \bowtie^f J$ is weak McCoy.*

Proof. Since $nil(A) \trianglelefteq A$ then A is weak Armendariz, which implies that $A \bowtie^f J$ is weak Armendariz by [14][Theorem 4 (4)]. Wich in turn implies that $A \bowtie^f J$ is weak McCoy. □

Corollary 2.6. *If A is semicommutative and $J \subset nil(B)$, then $A \bowtie^f J$ is weak McCoy.*

Proof. Immidiately from the proposition above and the fact that $nil(A) \trianglelefteq A$ since A is semicommutative [12][Lemma 3.1]. □

Corollary 2.7. *If A and $f(A)+J$ are semicommutatives rings, then $A \bowtie^f J$ is weak McCoy.*

Proof. Immidiately from 2.2 since $A \bowtie^f J$ is a semicommutative ring. □

Now, we construct a new class of right McCoy rings.

Example 2.8. Let R be a semicommutative ring. We consider $R[x]/(x^n)$ where n is a positive integer such that $n \geq 2$. Let $f : R[x]/(x^n) \longrightarrow R$ such that, $f(\overline{P(x)}) = P(0)$. Let $J = (eRe)$ be an ideal of R generated by e where e is a nilpotent element of R. Then:

(1) $R[x]/(x^n)$ is a right weak McCoy.
(2) $R[x]/(x^n) \bowtie^f J$ is a right weak McCoy ring.

Proof. (1) Clear since $R[x]/(x^n)$ is weak Armendariz ring by [12][Theorem 3.9].
(2) Follows from 2.3 since $J \subset nil(R) = (0)$.

□

Example 2.9. Let A be a semicommutative ring, $f : A \longrightarrow A[X]$ the canonical surjection and $J = (X)$.
Note that $f^{-1}(J) = \{0\}$.

(1) $f(A) + J$ is right weak McCoy.
(2) $A \bowtie^f J$ is a right weak McCoy ring.

Proof. (1) Clear since $f(A) + J = A + (X) = A[X]$ is a weak Armendariz then right weak McCoy.
(2) Follows from 2.3 since $f^{-1}(J) = (0)$ and $f(A) + J$ is right weak McCoy.

□

Example 2.10. Let A be a semicommutative ring, $f : A \longrightarrow A[X]$ the canonical injection, and $J = (X)$ the ideal of $A[X]$ generated by X. Let $S = \{X\}$. Note that X is a regular central element of $A[X]$. Then:

(1) $f(A) + J = A + (X) = A[X]$ is a right weak McCoy ring.
(2) $A \bowtie^f J$ is a right McCoy ring.

Proof. (1) Clear since $f(A)+J = A[X]$ is a weak Armendariz ring by [12][Theorem 3.8] hence right McCoy ring which implies that $A[X]$ is right weak McCoy ring .
(2) Follows from 2.3 since $J \cap S \neq \emptyset$.

□

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